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A Weak Free Logic with the Existence Sign

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In [1] I constructed a semantics for the system *FLI** of quantification and identity theory having

(A0) A, *where A is a tautology*
(A2) $\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$
(A3) $A \supset \forall xA$
(A4) $a = a$
(A5) $a = b \supset (A \supset A^b//a)$
(A6) $\forall xA^x/a$, *where A is an axiom*

as axiom-schemata and

(R1) If A and $A \supset B$ are theorems, then B is a theorem

as rule of inference.[1] In the present paper, I want to extend this semantical analysis to the system *FLI*** which results from adding to *FLI** Leonard's weakened Principle of Specification

(A1″) $(\forall xA \;\&\; \mathrm{E}!a) \supset A^a/x$

(for which see [4]). Before doing this, however, I must spend a few words in motivating the enterprise.

As was pointed out in [1], in *FLI** the schema

(A1′) $(\forall xA \;\&\; \exists x(x = a)) \supset A^a/x$

is provable; hence, having in mind the definition of existence in terms of identity that free logicians have used at least since [3] and that we can express in the form

(1) $\mathrm{E}!a =_{df} \exists x(x = a)$, where x is the first individual variable in the alphabetical order,

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one might conclude that in *FLI** (A1″) is (the definitional abbreviation of) a theorem, and that *FLI** and *FLI*** are the same system after all. Things however are not so easy, as I will presently show.

In a standard free logic, the proviso qualifying x in (1) might be supplied just for the sake of definiteness (and as a matter of fact, it is often ignored), for in such a logic relettering of bound variables is a legitimate procedure of inference, and as a consequence x could be replaced in (1) by any other individual variable without ever altering the truth-value of the *definiens.* Not so in *FLI**: in this system

(2) $\exists x(x = a)$
(3) $\exists y(y = a)$
(4) $\exists z(z = a)$
... ...

(where $x, y, z, \ldots$ are all the (different) individual variables) are independent of one another,[2] hence some proviso like the above is absolutely necessary if we want (1) to be a good definition. And this means that

(a) what we *can* prove in *FLI** is not exactly (A1″) but rather something like

(A1‴) $(\forall xA \ \&\ \mathrm{E!}a) \supset A^a/x$, *where x is the first individual variable in the alphabetical order,*

which of course is a much weaker schema, and

(b) in *FLI**, (1) cannot be proposed as a reasonable definition of existence in terms of identity. For Leonard could have agreed on identifying being with being a value of a bound variable, but certainly would not have agreed (nor, I think, would anybody) on identifying being with being a value *of x* (or, for that matter, of any *specific* (bound) individual variable).

Thus, if we want the existence sign E! in *FLI**, it looks like we have no choice but to introduce it as a *primitive* sign and characterize it by some additional axiom-schemata. (A1″) itself is of course a very plausible candidate for this role, together with

(A7) $\forall x\mathrm{E!}x.$

The addition of both these schemata would give us a full-fledged free logic (in which, by the way, the biconditional analogue of (1) would be *provable,* for example, along the lines suggested in [2]), but there are at least two reasons for also considering the system *FLI***. On the one hand, it turns out that (A1″) is not enough to justify relettering of bound variables, hence *FLI*** would offer us a chance to study the semantical behavior of E! in model-structures in which the bound variables are allowed to range over different domains. The second reason is of a historical nature. *FLI*** is quite similar to a system suggested by Routley in [5].[3] Trew [6] showed that Routley's system was *in*complete with respect to Routley's semantics. Nobody however has ever constructed a semantics for which Routley's system *is* (sound and) complete. What follows can be regarded as a decisive step in this direction.[4]

As the semantics that I am about to present is a modification of the one I defined in [1], I can avoid a number of repetitions by referring explicitly to that paper. With this qualification, the new semantics can be made precise by the following definitions.

(D1) = (D1) of [1].

(D2) In a weak free model-structure $\mathfrak{M} = \langle D, f \rangle$ a member d of D is an *intersection point* if and only if $d \in f(x)$ for every individual variable x.

(D3) The primary auxiliary valuation $V^*_{\mathfrak{M}}$ associated with a weak free model-structure $\mathfrak{M} = \langle D, f \rangle$ is the partial unary function W from the set of wffs to $\{T, F\}$ such that

(a), (b) = (D2)(a), (b) of [1], respectively
(c) if A is of the form E!a then $W(A) = T$ if $f(a)$ is an intersection point, and otherwise $W(A) = F$
(d) $W(A)$ is not defined if not by virtue of (a)-(c).

(D4)-(D9) = (D3)-(D8) of [1], respectively.

It is easy to see that this semantics validates (A1″), for if E!a is true then the denotation of a is an intersection point; hence it belongs to the domain of every individual variable x; hence it satisfies every wff which is satisfied by every value of x (for any individual variable x). And it is also easy to see that the semantics invalidates (A7); indeed, what the validity of (A7) would require is that, for every weak free model-structure $\mathfrak{M} = \langle D, f \rangle$ and every individual variable x, every member of $f(x)$ be an intersection point (which of course is not the case).

In view of the above remarks on the validity of (A1″), checking the soundness of *FLI*** with respect to the semantics in question is a trivial task. As to the completeness proof, it can be carried out as in [1] through the introduction of a suitable system of semantic tableaux. The rules of such a system (to be referred to in what follows as *STI***) are the rules (S1)-(S7) of the system *STI** defined in [1] together with the following

(S8) Every nonlast point of the form F & E!a & G has as its only successor F & $\neg\forall x\neg(x = a)$ & G & E!a, where x is the first individual variable in the alphabetical order such that $\neg\forall x\neg(x = a)$ does not occur as a conjunct either in F or in G.

It is easy to show that all theorems of *STI*** are theorems of *FLI***. The crucial step of course is the one for (S8), and for that step we can refer to the provability in *FLI*** of the schema

(5) $\quad$ E!$a \supset \neg\forall x\neg(x = a)$,

a simple consequence of (A0), (A4) and the instance

(6) $\quad$ $(\forall x\neg(x = a)$ & E!$a) \supset \neg(a = a)$

of (A1″). Thus the proof can be completed in the usual way, by showing that whenever the semantic tableau constructed in *STI*** for a wff A contains a nonclosing branch X, there are two (weak free) model-structures $\mathfrak{M}$ and $\mathfrak{M}'$

such that: (i) $\mathfrak{M}'$ is a completion of $\mathfrak{M}$ and (ii) $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(C) = T$ for every wff C occurring as a conjunct in X (hence in particular for the origin $\neg A$ of the tableau).

Suppose then that we have a nonclosing branch X in the semantic tableau for A. The model-structures $\mathfrak{M}$ and $\mathfrak{M}'$ can be defined exactly as in [1] by conditions (C1)-(C5) and (C1′)-(C4′), respectively. The proof of (i) above is trivial, and the proof of (ii) can be carried out by induction on the number of connectives and quantifiers occurring in C in the way suggested in [1]. The only interesting case, of course, is the one corresponding to the existence sign, for which it is convenient to extend the Closure Property Lemma by the addition of the clause

(**) *If* E!*a occurs as a conjunct in X, so does* $\neg\forall x\neg(x = a)$ *for every individual variable x.*

Once this is (easily) done, the proof of the relevant case may proceed as follows:

Case 3. C is of the form E!a. Then by (**) $\neg\forall x\neg(x = a)$ occurs in X for every individual variable x; hence by (C3), (C4), and the Closure Property Lemma $f(a) \in f(x)$ for every individual variable x; hence by (D2) $f(a)$ is an intersection point; hence by (D3)(c) $V^{*}_{\mathfrak{M}}(C) = T$, which is enough to assure that $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(C) = T$.

NOTES

1. As in [1], I will not spend any time in defining the syntax, which is standard. I will only point out that I am distinguishing between individual variables and individual constants, and that individual variables will only occur bound in the wffs of my language.

2. The construction of countermodels establishing these independence results is a simple matter of routine. If for example we want to show that (2) does not imply any of the (3), (4), . . ., it is enough to define a model-structure $\mathfrak{M} = \langle D, f\rangle$ such that $f(a) \in f(x)$ but $f(a) \notin f(y)$ for every individual variable y distinct from x.

3. The system that I am referring to is the one that bears to (Routley's) $=R^*$ the same relation that (his) FR^* bears to (his) R^*. The reason I must limit myself to say "quite similar", rather than "equivalent", is that instead of (A2)-(A3), Routley has an axiom-schema which (due to our particular treatment of individual variables) we can express in the form

   (A2′) $\forall x(A \supset B) \supset (A \supset \forall xB)$,

   but I suspect that in the present context (in contrast with what happens in standard logic) (A2′) is not enough to prove (A2). I hope to deal with this problem and related ones on another occasion.

4. "A decisive step", not the final step, for the reason explained in Note 3.

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