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SOME REMARKS ABOUT THE FAMILY $\mathcal{K}$ OF MODAL SYSTEMS

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1 *Introduction* $\mathcal{K}$-systems can be defined as follows:

D1. S is a $\mathcal{K}$-system $=_{df}$ S is deductively equivalent to some system S′ such that:

a. S′ is an extension of the propositional calculus (PC)
b. S′ is governed by the following rules of inference: the rule of uniform substitution, *modus ponens* (MP), and RL($\vdash\alpha \Rightarrow \vdash L\alpha$)
c. The following formulas are axioms of S′:

$$\left.\begin{array}{ll} \text{A1.} & Lp \supset p \\ \text{A2.} & L(p \supset q) \supset (Lp \supset Lq) \\ \text{A3.} & Lp \supset LLp \end{array}\right\} \text{ S4-axioms}$$

A4. $LMp \supset MLp$

d. Every other axiom of S′ is an S5-thesis
e. S5 is not contained in S′.

The following formulas have been used in the construction of $\mathcal{K}$-systems:

G1. $MLp \supset LMp$
D2. $L(Lp \supset Lq) \vee L(Lq \supset Lp)$
J1. $L(L(p \supset Lp) \supset p) \supset p$
H1. $p \supset L(Mp \supset p)$
F1. $L(Lp \supset q) \vee (MLq \supset p)$
R1. $p \supset (MLp \supset Lp)$.

As far as I know, only nine nonequivalent $\mathcal{K}$-systems have been so far described (see [2], [4], [7], [11], and [12]):

K1 = {S4; A4}
K2 = {S4; A4; G1}
K3 = {S4; A4; D2}
K1.1 = {S4; J1}
K2.1 = {S4; J1; G1}
K3.1 = {S4; J1; D2}
K1.2 = {S4; H1}

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K3.2 = {S4; A4; F1}
K4 = {S4; A4; R1}.

A system S is a proper part of a system S′ (symbolically: S′ → S) iff S is deducible from S′ but S′ is not deducible from S. Sobociński [12] has proved that the following proper-part relations hold between the known $\mathcal{K}$-systems:

$$\begin{tikzcd}
\mathrm{K1.2} \arrow[dddd] & & & \\
& \mathrm{K4} \arrow[ul] \arrow[r] \arrow[d] & \mathrm{K3.2} \arrow[d] & \\
& \mathrm{K3.1} \arrow[r] \arrow[d] & \mathrm{K3} \arrow[d] & \\
& \mathrm{K2.1} \arrow[r] \arrow[dl] & \mathrm{K2} \arrow[dr] & \\
\mathrm{K1.1} \arrow[rrr] & & & \mathrm{K1}
\end{tikzcd}$$

Diagram A

Using the following S5-theses:

L1. $MLp \supset L(Mq \,.\, {\sim} q \supset MLp)$
L2. $p \supset L(Mq \,.\, {\sim} q \supset Mp)$
M1. $p \supset (LMp \vee MLp \vee L(Mp \supset p))$,

we can construct five new $\mathcal{K}$-systems:

KL = {S4; A4; L1}
KB = {S4; A4; L2}
K1.1.1 = {S4; A4; M1}[1]
K2.1.1 = {S4; A4; M1; G1}
K3.1.1 = {S4; A4; M1; D2}.

That KL-K3.1.1 are $\mathcal{K}$-systems can easily be shown. Since they are extensions of S4 and have A4 as an axiom, conditions a, b, and c of D1 are satisfied. This applies even to condition d, since L1, L2, and M1 are S5-theses. (In the field of S4 they follow trivially from the Brouwerian axiom: $p \supset LMp$.) That S5 is not contained in KL-K3.1.1 (condition e) will be proved later. In the second section, I will show that the proper-part relations holding between KL-K3.1.1 and other $\mathcal{K}$-systems can be represented in Diagram B.

I will now present Kripke-type semantics for all the $\mathcal{K}$-systems in Diagram B. Δ is a Kripke-type semantics iff Δ is a class of ordered triples $\langle W, R, V\rangle$ ("models") such that W is a nonempty set (of "possible worlds"), R is a dyadic relation defined over the members of W ("accessibility relation"), and V is a valuation function satisfying the standard conditions (see, for example, [3], p. 73). I will define different semantics by imposing different conditions on R.

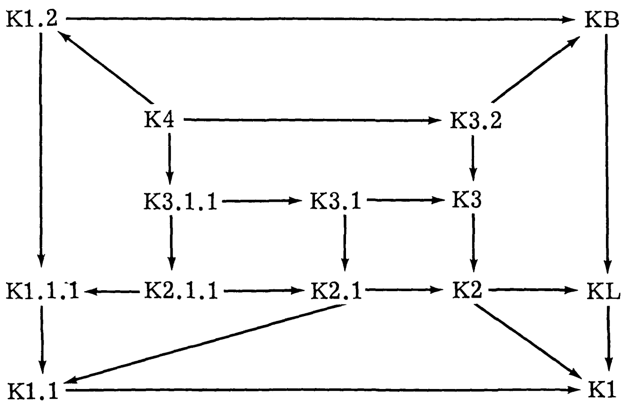

Diagram B

Relevant conditions on R: w_i, w_j, . . . are members of W. Universal quantifiers are omitted whenever it is possible. We first introduce some definitions:

D2. $O = \{w_i | (\exists w_j)(w_iRw_j \cdot w_i \neq w_j)\}$ (O is a set of "open" worlds)

D3. $C = W - O$ (C is a set of "closed" worlds)

D4. $B_{w_i} = \{K | K$ is a smallest set of worlds such that: (a) $w_i \in K$, and (b) for every w_j, if w_iRw_j and, for every $w_k \in K$, w_jRw_k or w_kRw_j, then $w_j \in K.\}$ (B_{w_i} is a set of "branches" with respect to w_i.)

Using these conventions we can define:

Refl.	w_iRw_i	reflexivity
Trans.	$w_iRw_j \cdot w_jRw_k \supset w_iRw_k$	transitivity
Fin.	$(\exists w_j)(w_iRw_j \cdot w_j \in C)$	finiteness
Conv.	$w_iRw_j \cdot w_iRw_k \supset (\exists w_l)(w_jRw_l \cdot w_kRw_l)$	convergence
Conn.	$w_iRw_j \cdot w_iRw_k \supset (w_jRw_k \vee w_kRw_j)$	connectedness
Antisymm.	$w_iRw_j \cdot w_jRw_i \supset w_i = w_j$	antisymmetry
BrFin.	$K \in B_{w_i} \supset \mathrm{Card}(K) < \aleph_0$	branch-finiteness
Short.	$w_iRw_j \cdot w_jRw_k \cdot w_i \neq w_j \cdot w_j \neq w_k \supset w_k \in C$	shortness
StrShort.	$w_iRw_j \cdot w_i \neq w_j \supset w_j \in C$	strict shortness[2]
ClConn.	$w_iRw_j \cdot w_iRw_k \cdot w_j \in O \cdot w_k \in C \supset w_jRw_k$	closed-world connectedness
OSymm.	$w_iRw_j \cdot w_j \in O \supset w_jRw_i$	open-world symmetry

Every *K*-semantics satisfies the following R-conditions: *Refl., Trans.,* and *Fin.* Additional conditions are:

K1	none
K2	Conv.
K3	Conn.
K1.1	Antisymm. and BrFin.
K2.1	Antisymm., BrFin., and Conv.

K3.1	Antisymm., BrFin., and Conn.
K1.1.1	Short.
K2.1.1	Short. and Conv.
K3.1.1	Short, and Conn.
K1.2	StrShort.
KL	ClConn.
KB	OSymm.
K3.2	OSymm. and Conv. (or Conn.)
K4	StrShort and Conv. (or Conn.)

Informal interpretation of some $\mathcal{K}$-semantics: Every $\mathcal{K}$-ms (model structure) can be treated as a set of worlds such that: (a) every world is accessible to itself, (b) the accessibility relation is transitive, and (c) from every world some closed world is accessible. This is, by the way, an exhaustive characteristic of a K1-ms.

A K1.1.1-ms is a set of worlds such that to every open world accessible to some other open world only closed worlds are accessible. A K3.1.1-ms is a chain of worlds of type ≤ 3. A K1.2-ms is a set of worlds such that to every open world only this world itself and closed worlds are accessible. If we additionally stipulate that the number of these closed worlds equals 1, we get a K4-ms.

A KL-ms is a set of worlds such that every closed world is accessible to every open world. We get a KB-ms by an additional stipulation that the accessibility relation between the open worlds is symmetric.

From a KB-ms results a K3.2-ms if we limit the number of the accessible closed worlds to 1. (If we, instead, stipulate that there is only one open world, we get a K1.2-ms. Both stipulations taken together give us a K4-ms.)

It is easy to show that, for every system K_i represented in Diagram B, and for every formula α, if α is a thesis of K_i then α is K_i-valid. (We have only to prove that our K_i-semantics validates all the axioms of K_i and that the rules of inference are K_i-validity-preserving.) We can now complete our proof that KL-K3.1.1 are $\mathcal{K}$-systems. We prove that S5 is contained in neither of these systems by constructing a falsifying KL (KB, K1.1.1, K2.1.1, K3.1.1)-model for the Brouwerian axiom:

$$W = \{w_1, w_2\}, R = \{\langle w_1, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w_2\rangle\}, V(p, w_1) = 1, V(p, w_2) = 0.$$

It is an equally easy job (but a more tedious one) to prove the following result:

T1 *Every K_i-semantics is minimal with respect to Diagram* B.

A K_i-semantics is minimal with respect to Diagram B $=_{df}$. For every system K_j represented in Diagram B such that $K_i \neq K_j$ and $K_i \not\rightarrow K_j$, there is a formula, α, such that α is an axiom of K_j and α is K_i-invalid (i.e., there exists a K_i-model which falsifies α).

It has been proved by Segerberg [8] and Zeman [14] that K1, K2, K3, K1.1, K3.1, and K4 are complete for their respective semantics. In Section **3** I give the completeness proofs for K1.2, KB, and K3.2.[3]

It is known (see McKinsey [4]) that: (a) K1 has exactly 10 (distinct) modalities: -, M, L, LM, ML and their negations.[4] Since the formula: $LM\alpha \equiv ML\alpha$, is a K2-thesis and K2 contains K1, we get the result that (b) K2 has at most 8 modalities. We can easily prove that (c) K4 has at least 8 modalities by constructing falsifying K4-models for the following equivalences: $\alpha \equiv M\alpha$, $\alpha \equiv L\alpha$, $\alpha \equiv LM\alpha$, $M\alpha \equiv L\alpha$, $M\alpha \equiv LM\alpha$, and $L\alpha \equiv LM\alpha$. In the same way we prove that: (d) K1.2 has at least 10 modalities. Both proofs are omitted here.

Results (b) and (c) imply that (e) K2, K4 and all systems between K2 and K4 have exactly 8 distinct modalities. Since K1.2 contains K1, (a) and (d) imply that: (f) K1.2 has the same number of modalities as K1. This applies also to all systems between K1.2 and K1. We get therefore the following result:

T2. A. *All systems contained in* K1.2 *and containing* K1 *have exactly 10 distinct modalities.* B. *All systems contained in* K4 *and containing* K2 *have exactly 8 distinct modalities.*

However, T2 ought to be qualified. It holds only for systems which have $\sim$ and L (or M) as primitive operators. It is, of course, possible to construct $\mathcal{K}$-systems with other modal primitives. In Section **4** I present two such systems, K1.2.G and K4.G, which are deductively equivalent to K1.2 and K4, respectively. I shall prove that K1.2.G (K4.G) has 6 (4) distinct modalities.

2 *Position of* KL-K3.1.1 *in the family* $\mathcal{K}$

T3 *Diagram* B *adequately represents the logical relations between* $\mathcal{K}$*-systems.*

Proof: As we know, Sobociński has proved that Diagram A is correct. Given this result, it remains only to prove the following lemmas:

1. K3.1.1 is properly contained in K4.
2. K2.1.1 is properly contained in K3.1.1 (and therefore even in K4).
3. K1.1.1 is properly contained in K1.2 and K2.1.1 (and therefore even in K3.1.1 and K4).
4. K1.1 is properly contained in K1.1.1 (and therefore even in K2.1.1 and K3.1.1).
5. K2.1 is properly contained in K2.1.1 (and therefore even in K3.1.1)
6. K3.1 is properly contained in K3.1.1.
7. KB is properly contained in K3.2 and K1.2 (and therefore even in K4).
8. KL is properly contained in KB and K2 (and therefore even in K2.1, K2.1.1, K3, K3.1, K3.1.1, K3.2, K1.2, and K4).
9. K1 is properly contained in KL (and therefore even in KB).
10. KL is independent from K1.1 and K1.1.1.[5]
11. KB is independent from K2, K3, K1.1, K2.1, K3.1, K1.1.1, K2.1.1, and K3.1.1.
12. K1.1.1 is independent from KL, KB, K2, K3, K2.1, K3.1, and K3.2.
13. K2.1.1 is independent from KB, K3, K3.1, K3.2, and K1.2.
14. K3.1.1 is independent from KB, K3.2, and K1.2.

It is easy to ascertain that in order to prove 1-14 it is sufficient to show that the following hold:

a. M1 is a K1.2-thesis.
b. J1 is a K1.1.1-thesis.
c. L2 is a: (1) K3.2- and (2) K1.2-thesis.
d. L1 is a: (1) KB- and (2) K2-thesis.
e. There exist falsifying K1.1.1-models for: (1) L1 and (2) G1.
f. There exist falsifying K3.1.1-models for: (1) L2 and (2) H1.
g. There exist falsifying: (1) K3.2- and (2) K3.1-models for M1.
h. There exists a falsifying KB-model for J1.
i. There exists a falsifying K1.2-model for G1.
j. There exists a falsifying K2.1.1-model for D2.

Given that Diagram A is correct, the logical relations between a-j and 1-14 are as follows:

a & f2	$\Rightarrow$ 1
j	$\Rightarrow$ 2[6]
a & f2 & i	$\Rightarrow$ 3
b & g2	$\Rightarrow$ 4 & 5 & 6
c1 & c2 & h & i	$\Rightarrow$ 7
b & c2 & d1 & d2 & f1 & i	$\Rightarrow$ 8
e1	$\Rightarrow$ 9
b & d1 & e1 & h	$\Rightarrow$ 10
b & c2 & f1 & h & i	$\Rightarrow$ 11
c1 & d1 & e1 & e2 & g1 & g2	$\Rightarrow$ 12
c1 & f1 & f2 & g1 & g2 & i & j	$\Rightarrow$ 13
c1 & f1 & f2 & g1 & g2	$\Rightarrow$ 14.

Proof of a:

(1) $p \supset L(Mp \supset p)$ M1
(2) $p \supset (LMp \vee MLp \vee L(Mp \supset p))$ (1), PC

Proof of b:

(1) $p \supset (LMp \vee MLp \vee L(Mp \supset p))$ M1
(2) $LMp \supset MLp$ A4
(3) $p \supset (MLp \vee L(Mp \supset p))$ (1), (2), PC
(4) $p \supset (MLp \vee M(L(Mp \supset p) \cdot p))$
(3), PC, $\vdash_T p \supset Mp[p/L(Mp \supset p) \cdot p]$, PC
(5) $MLp \supset M(L(Mp \supset p) \cdot p)$ T[7]
(6) $p \supset M(L(Mp \supset p) \cdot p)$ (4), (5), PC
(7) $\sim p \supset M(L(M \sim p \supset \sim p) \cdot \sim p)$ (6) $[p/\sim p]$
(8) $\sim p \supset M(L(p \supset Lp) \cdot \sim p)$ (7), $\vdash_T (M \sim p \supset \sim p) \equiv (p \supset Lp) \times$ Eq[8]
(9) $\sim p \supset \sim L(L(p \supset Lp) \supset p)$
(8), $\vdash_T M(p \cdot \sim q) \equiv \sim L(p \supset q)[p/L(p \supset Lp), q/p] \times$ Eq.
(10) $L(L(p \supset Lp) \supset p) \supset p$ (9) $\times$ Transp.

Proof of c1:

(1) $\sim(MLq \supset \sim p) \supset L(L \sim p \supset q)$	F1 $[p/\sim p]$, PC
(2) $MLq \cdot p \supset L(L \sim p \supset q)$	(1), PC
(3) $p \supset LM(Mq \cdot \sim q \supset Mp)$	T
(4) $p \supset ML(Mq \cdot \sim q \supset Mp) \cdot p$	(3), A4 $[p/Mq \cdot \sim q \supset Mp]$, PC
(5) $p \supset L(L \sim p \supset (Mq \cdot \sim q \supset Mp))$	(4), (2) $[q/Mq \cdot \sim q \supset Mp]$ × Syll.
(6) $L(L \sim p \supset (Mq \cdot \sim q \supset Mp)) \supset L(Mq \cdot \sim q \supset Mp)$	T
(7) $p \supset L(Mq \cdot \sim q \supset Mp)$	(5), (6) × Syll.

Proof of (c2):

(1) $p \supset ((MLq \vee q) \cdot \sim q \supset p) \cdot LM((MLq \vee q) \cdot \sim q \supset p)$	T
(2) $p \supset (LMp \supset Lp)$	H1, A2 $[p/Mp, q/p]$ × Syll.
(3) $p \cdot LMp \supset Lp$	(2) × Imp.
(4) $Lp \supset p \cdot LMp$	T
(5) $Lp \equiv p \cdot LMp$	(3), (4), PC
(6) $\sim L \sim p \equiv p \vee \sim LM \sim p$	(5) $[p/\sim p]$, PC
(7) $Mp \equiv p \vee MLp$	(6), Def M, $\vdash_{\mathrm{T}} \sim LM \sim p \equiv MLp$
(8) $p \supset L((MLq \vee q) \cdot \sim q \supset p)$	(1), (5) $[p/(MLq \vee q) \cdot \sim q \supset p]$ × Syll.
(9) $p \supset L(Mq \cdot \sim q \supset p)$	(8), (7) $[p/q]$ × Eq
(10) $L(Mq \cdot \sim q \supset p) \supset L(Mq \cdot \sim q \supset Mp)$	T
(11) $p \supset L(Mq \cdot \sim q \supset Mp)$	(9), (10) × Syll.

Proof of (d1):

(1) $MLp \supset L(Mq \cdot \sim q \supset MMLp)$	L2 $[p/MLp]$
(2) $MLp \supset L(Mq \cdot \sim q \supset MLp)$	(1), $\vdash_{\mathrm{S4}} MMLp \equiv MLp$ × Syll.

Proof of (d2):

(1) $MLp \supset (Mq \cdot \sim q \supset MLp)$	PC
(2) $LMp \supset (Mq \cdot \sim q \supset MLp)$	A4, (1) × Syll.
(3) $LLMp \supset L(Mq \cdot \sim q \supset MLp)$	
	(2) × RL, A2 $[p/LMp, q/Mq \cdot \sim q \supset MLp]$ × MP
(4) $LMp \supset L(Mq \cdot \sim q \supset MLp)$	(3), $\vdash_{\mathrm{S4}} LLp \equiv Lp$ $[p/Mp]$ × Eq
(5) $MLp \supset L(Mq \cdot \sim q \supset MLp)$	G1, (4) × Syll.

We prove e-j by constructing the appropriate falsifying models:

e: $W = \{w_1, w_2, w_3, w_4\}$, $R = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle, \langle w_3, w_4\rangle, \langle w_1, w_4\rangle\}$,[9]
$V(p, w_2) = V(q, w_4) = 1$, $V(q, w_3) = V(p, w_4) = 0$.

f1: $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2\rangle, \langle w_2, w_3\rangle, \langle w_1, w_3\rangle\}$,
$V(p, w_1) = 1$, $V(p, w_2) = V(q, w_2) = V(p, w_3) = 0$, $V(q, w_4) = 1$.

f2: $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2\rangle, \langle w_2, w_3\rangle, \langle w_1, w_3\rangle\}$,
$V(p, w_1) = V(p, w_3) = 1$, $V(p, w_2) = 0$.

g1: $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle, \langle w_1, w_3\rangle, \langle w_2, w_3\rangle\}$,
$V(p, w_1) = 1$, $V(p, w_2) = V(p, w_3) = 0$.

g2: $W = \{w_1, w_2, w_3, w_4\}$, $R = \{\langle w_1, w_2\rangle, \langle w_2, w_3\rangle, \langle w_3, w_4\rangle, \langle w_1, w_3\rangle, \langle w_1, w_4\rangle$
$\langle w_2, w_4\rangle\}$, $V(p, w_1) = V(p, w_3) = 1$, $V(p, w_2) = V(p, w_4) = 0$.

h: $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle, \langle w_1, w_3\rangle, \langle w_2, w_3\rangle\}$,
$V(p, w_1) = 0$, $V(p, w_2) = V(p, w_3) = 1$.

i: $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle\}$, $V(p, w_2) = 1$, $V(p, w_3) = 0$.

j: $W = \{w_1, w_2, w_3, w_4\}$, $R = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle, \langle w_1, w_4\rangle, \langle w_2, w_4\rangle, \langle w_3, w_4\rangle\}$,
$V(p, w_2) = V(p, w_4) = V(q, w_3) = V(q, w_4) = 1$,
$V(q, w_2) = V(p, w_3) = 0$.

This completes our proof of T3.

3 *Completeness* I will give now Henkin-type proofs that K1.2, KB, and K3.2 are complete for the given semantics. I will show, namely, that for any formula, α, if α is consistent with respect to K1.2 (KB, K3.2), we can construct a verifying K1.2 (KB, K3.2)-model for α (or, equivalently, that every K1.2 (KB, K3.2)-valid formula is a thesis of K1.2 (KB, K3.2). My proofs will strongly rely on those given in [3], pp. 149-159, for T and S4.

The following lemmas have been proved there for any system S which is an extension of PC (and therefore for all $\mathcal{K}$-systems). Let Γ_i be a maximal consistent set relative to S[10] and let α and β be wffs of S.

Lemma 1 *α and $\sim\alpha$ are not both in Γ_i.*
Lemma 2 *Either $\alpha \in \Gamma_i$ or $\sim\alpha \in \Gamma_i$. If $\vdash_S \alpha$, $\alpha \in \Gamma_i$.*
Lemma 3 *If $\alpha \in \Gamma_i$ and either $(\alpha \supset \beta) \in \Gamma_i$ or $\vdash_S (\alpha \supset \beta)$, $\beta \in \Gamma_i$.*

These lemmas and even other results proved in [3] will be used in what follows.

A. K1.2 Let α be a formula consistent relative to K1.2. Beginning with $\Gamma_{10} = \{\alpha\}$, we construct an initial maximal consistent set Γ_1 in the same way as it has been done in [3]. We construct further maximal consistent sets according to the following plan:

For every already constructed Γ_i and for every formula $M\beta \in \Gamma_i$, such that $\sim\beta \in \Gamma_i$, we define a set $\Gamma_{j0} = \{\beta, \gamma_1, \ldots, \gamma_n, \gamma_{n+1}, \ldots\}$, such that, for every γ_i, $\gamma_i \in \Gamma_{j0}$ iff $L\gamma_i \in \Gamma_i$. In [3] a proof is given that, for every system S which contains T, Γ_i's consistency relative to S entails Γ_{j0}'s consistency relative to S. It follows that Γ_{j0} is consistent with respect to K1.2. Starting with Γ_{j0} we construct a maximal consistent set Γ_j by the usual methods.

For every Γ_i and Γ_j, if Γ_j is formed from Γ_i in the described way, Γ_j will be called a "subordinate" of Γ_i. If Γ_i is an ancestor of Γ_j with respect to the relation of subordination, we shall say that Γ_j is a "subordinate*" of Γ_i.

We define Γ as the smallest set whose members are Γ_1 and all its subordinates*.

Further definitions:

$O =_{df}$ a subset of Γ such that, for every $\Gamma_i \in \Gamma$, $\Gamma_i \in O$ iff there is some wff, δ, such that $(M\delta \cdot \sim\delta) \in \Gamma_i$ (or, equivalently, iff there is some $\Gamma_j \in \Gamma$, such that Γ_j is a subordinate of Γ_i).

$C =_{df} \Gamma - O$.

Lemma 4 *Every subordinate of* Γ, *belongs to* C.

(It follows from Lemma 4 (by the definition of C and the construction of Γ) that every member of Γ is either identical with Γ_1 or is its subordinate.)

Proof (by reductio ad absurdum): Suppose that Γ_i is a subordinate of Γ_1 and $\Gamma_i \notin C$. Then there is some wff, δ, such that $(M\delta \cdot \sim \delta) \in \Gamma_i$. Besides, by construction of Γ_i, there is some wff, β, such that:

(a) $(M\beta \cdot \sim \beta) \in \Gamma_1$, and
(b) $\beta \in \Gamma_i$.

As we already know, the following formula is a K1.2-thesis:

$$p \supset L(Mq \cdot \sim q \supset p).$$

By substitution, we get:

(c) $\vdash_{\overline{K1.2}} \sim \beta \supset L(M\delta \cdot \sim \delta \supset \sim \beta)$.

By Lemma 3, (b) and (c) imply that

(d) $L(M\delta \cdot \sim \delta \supset \sim \beta) \in \Gamma_1$.

Therefore, by the construction of Γ_i,

(e) $(M\delta \cdot \sim \delta \supset \sim \beta) \in \Gamma_i$.

By hypothesis, $(M\delta \cdot \sim \delta) \in \Gamma_i$. Therefore, by Lemma 3, (2) implies that

(f) $\sim \beta \in \Gamma_i$.

But this is contradicted by (b) and Lemma 1.

Lemma 5 *For every* $\Gamma_i \in \Gamma$, *either* $\Gamma_i \in C$ *or there is some* Γ_j *such that* Γ_j *is a subordinate of* Γ_i *and* $\Gamma_j \in C$.

Lemma 5 follows trivially from Lemma 4.

We construct now a K1.2-model $\langle W, R, V\rangle$ on the basis of Γ. Let $W = \Gamma$. $\Gamma_i R \Gamma_j =_{df} \Gamma_i = \Gamma_j$ or Γ_j is a subordinate of Γ_i. This definition makes R a reflexive relation. Lemma 5 warrants that R is finite. By Lemma 4, R is strictly short and transitive. We define the valuation function V in a standard way.[12] $\langle W, R, V\rangle$ is then clearly a K1.2-model.

The completeness-theorem for K1.2:

T4. *If* W, R *and* V *are defined as above, then, for any wff* β *of* K1.2 *and for any* $\Gamma_i \in W$, $V(\beta, \Gamma_i) = 1$ *iff* $\beta \in \Gamma_i$. *Otherwise*, $V(\beta, \Gamma_i) = 0$.

T4 is proved by induction on the construction of wffs of K1.2 in exactly the same way as the analogous theorem for the system T (see [3], pp. 157f).

We conclude that $\langle W, R, V\rangle$ is a verifying K1.2-model for our initial formula α, since T4 guarantees that $V(\alpha, \Gamma_1) = 1$.

B. KB We construct Γ in the same way as before. The only difference is that we now make members of Γ maximal consistent relative to KB. Definitions of O and C remain unchanged.

Lemma 6 $\quad C \neq \emptyset$.

Proof (by reductio ad absurdum): Suppose that $C = \emptyset$. In such case, $\Gamma = 0$. We prove first that, if $\Gamma = 0$, then for every $\Gamma_i \in \Gamma$ and for every wff, α,

I. $(\alpha \supset LM\alpha) \in \Gamma_i$, and
II. $(ML \sim \alpha \supset \sim M\alpha) \in \Gamma_i$.

Proof of I: By Lemma 2, either $\alpha \in \Gamma_i$ or $\sim\alpha \in \Gamma_i$.

a. Suppose that $\sim\alpha \in \Gamma_i$. Then, by PC and Lemma 3, $(\alpha \supset LM\alpha) \in \Gamma_i$.
b. Suppose that:

(1) $\alpha \in \Gamma_i$.

In such case, $(\alpha \supset LM\alpha) \in \Gamma_i$ iff $LM\alpha \in \Gamma_i$. Suppose, then, that $LM\alpha \notin \Gamma_i$. By Lemma 2,

(2) $\sim LM\alpha \in \Gamma_i$.

Since the following formula:

(3) $\sim LM\alpha \supset M \sim M\alpha$,

is a thesis of T and therefore even of KB, we get (by Lemma 3) the following result:

(4) $M \sim M\alpha \in \Gamma_i$.

(1) and the KB-thesis: $\alpha \supset \sim\sim M\alpha$, imply, by Lemma 3, that

(5) $\sim\sim M\alpha \in \Gamma_i$.

(4) and (5) imply (by the construction of Γ) that there is some $\Gamma_j \in \Gamma$, such that Γ_j is a subordinate of Γ_i and

(6) $\sim M\alpha \in \Gamma_j$.

By hypothesis, $\Gamma_j \in 0$. In other words, there exists some wff, δ, such that

(7) $(M\delta \cdot \sim\delta) \in \Gamma_j$.

By substitution in L2, we get:

(8) $\vdash_{\overline{KB}} \alpha \supset L(M\delta \cdot \sim \delta \supset M\alpha)$.

(1) and (8) imply, by Lemma 3, that

(9) $L(M\delta \cdot \sim \delta \supset M\alpha) \in \Gamma_i$.

Therefore, by the construction of Γ_j,

(10) $(M\delta \cdot \sim \delta \supset M\alpha) \in \Gamma_j$.

(7) and (10) imply, by Lemma 3, that

(11) $M\alpha \in \Gamma_j$.

But this is contradicted by (6) and Lemma 1.

Proof of II:

(1)	$(M\alpha \supset LMM\alpha) \in \Gamma_i$	I$[\alpha/M\alpha]$
(2)	$\vdash_{\overline{KB}} (M\alpha \supset LMM\alpha) \supset (M\alpha \supset LM\alpha)$	provable in S4
(3)	$(M\alpha \supset LM\alpha) \in \Gamma_i$	(1), (2), Lemma 3
(4)	$\vdash_{\overline{KB}} (M\alpha \supset LM\alpha) \supset (ML \sim \alpha \supset \sim M\alpha)$	provable in T
(5)	$(ML \sim \alpha \supset \sim M\alpha) \in \Gamma_i$	(3), (4), Lemma 3

We prove now Lemma 6. If $\Gamma = 0$, then, for every $\Gamma_i \in \Gamma$, there exists some wff, δ, such that

(1) $M\delta \in \Gamma_i$ and

(2) $\sim\delta \in \Gamma_i$.

By substitution in I, we get:

(3) $(\sim\delta \supset LM \sim \delta) \in \Gamma_i$.

By Lemma 3, (2) and (3) imply that

(4) $LM \sim \delta \in \Gamma_i$.

By substitution in A4, we get:

(5) $\vdash_{\overline{KB}} LM \sim \delta \supset ML \sim \delta$.

By Lemma 3, (4) and (5) imply that

(6) $ML \sim \delta \in \Gamma_i$.

By the same lemma, it follows from (6) and II$[\alpha/\delta]$ that

(7) $\sim M\delta \in \Gamma_i$.

But this is contradicted by (1) and Lemma 1. We conclude that $\Gamma \neq 0$, or, equivalently, that $C \neq \emptyset$.

We construct now a KB-model $\langle W, R, V\rangle$ on the basis of Γ. W and V are defined as before.

$\Gamma_i R \Gamma_j =_{df}$ (1) $\Gamma_i = \Gamma_j$, or (2) Γ_i 0.

This definition guarantees that R is reflexive, transitive, and open-world symmetric. It also implies, given Lemma 6, that R is finite. We can therefore be assured that $\langle W, R, V\rangle$ is a KB-model.

Our proof of the completeness-theorem for KB is similar to the analogous induction proofs for T and S4, given in [3], pp. 157ff. The crucial difference is that now, in order to show that the theorem holds for L, we have to prove the following lemma:

(a) *For any* Γ_i *and* β, *if* $\Gamma_i \in 0$ *and* $L\beta \in \Gamma_i$, *then for any* Γ_j, $\beta \in \Gamma_j$.

Given (a), the definition of R, and the fact that the T-axiom $(L\beta \supset \beta)$ is a KB-thesis, we can easily prove that $V(L\beta, \Gamma_i) = 1$ if $L\beta \in \Gamma_i$. (As to the proof of the 'only if' part see [3], p. 158.)

By construction of Γ, we get:

(b) *For any* Γ_i *and* Γ_j, *there exists some* Γ_k *such that* Γ_i *and* Γ_j *are subordinates* of* Γ_k.

It is easy to see that (a) is implied by (b), given the following lemmas:

(c) *For any* $\Gamma_i \in 0$ *and* Γ_k, *if* Γ_i *is a subordinate* of* Γ_k *and* $L\beta \in \Gamma_i$, *then* $L\beta \in \Gamma_k$.
(d) *For any* Γ_j *and* Γ_k, *if* Γ_j *is a subordinate* of* Γ_k *and* $L\beta \in \Gamma_k$, *then* $\beta \in \Gamma_j$.

The proof of (d) is given in [3], p. 158. As for (c), we can prove it by induction on subordination if the following condition holds:

(e) *For any* $\Gamma_i \in 0$ *and* Γ_j, *if* Γ_i *is a subordinate of* Γ_j *and* $L\beta \in \Gamma_i$, $L\beta \in \Gamma_j$.

Proof of (e): We prove that, if $L\beta \notin \Gamma_j$, $L\beta \notin \Gamma_i$. Suppose that $L\beta \notin \Gamma_j$. In such a case, by Lemma 2:

(1) $\sim L\beta \in \Gamma_j$.

Since $\Gamma_i \in 0$, there is some wff, δ, such that

(2) $(M\delta \cdot \sim\delta) \in \Gamma_i$.

As we know,

(3) $\vdash_{\overline{\text{KB}}} \sim L\beta \supset L(M\delta \cdot \sim\delta \supset M \sim L\beta)$.

By Lemma 3, (1) and (3) imply that:

(4) $L(M\delta \cdot \sim\delta \supset M \sim L\beta) \in \Gamma_j$.

Therefore, by the construction of Γ_i;

(5) $(M\delta \cdot \sim\delta \supset M \sim L\beta) \in \Gamma_i$.

By Lemma 3, it follows from (2) and (5) that:

(6) $M \sim L\beta \in \Gamma_i$.

Since $\vdash_{\overline{\text{KB}}} M \sim L\beta \equiv \sim L\beta$, (6) is equivalent to:

(7) $\sim L\beta \in \Gamma_i$.

By Lemma 1, it follows from (7) that $L\beta \notin \Gamma_i$.

If we leave out the axiom A4 from KB, we get a new system, S4.B, which lies between S4 and S5. The S4.B-semantics can be defined by the following conditions on R: Refl., Trans., and OSymm. The completeness-proof for S4.B is nearly the same as the above proof for KB. The only difference is that Lemma 6 holds no longer.

C. K3.2 Our completeness-proof for K3.2 is exactly the same as for KB (as we know, KB is contained in K3.2), but now we also have to show that C has only one member. Since we already have shown that $C \neq \emptyset$, it remains to prove the following lemma:

Lemma 7 *For every* Γ_i, $\Gamma_j \in C$, $\Gamma_i = \Gamma_j$.

Given Lemma 6 and the fact that we define R as in the case of KB, Lemma 7 is equivalent to Conv. (and, what in this case amounts to the same thing, to Conn.).

Proof (by reductio ad absurdum): Suppose that Γ_i, $\Gamma_j \in C$ and $\Gamma_i \neq \Gamma_j$. Then there is some formula, α, such that:

(1) $\alpha \in \Gamma_i$

(2) $\sim\alpha \in \Gamma_j$.

Since $\Gamma_i \in C$, $(M \sim \alpha \cdot \alpha) \notin \Gamma_i$. Therefore, by Lemma 2:

(3) $\sim (M \sim \alpha \cdot \alpha) \in \Gamma_i$.

From (3) and $\vdash_{\mathrm{T}} \sim (M \sim \alpha \cdot \alpha) \supset (\alpha \supset L\alpha)$ we get, by Lemma 3, the following result:

(4) $(\alpha \supset L\alpha) \in \Gamma_i$.

(1) and (4) imply, by Lemma 3, that:

(5) $L\alpha \in \Gamma_i$.

In the same way we prove that:

(6) $L \sim \alpha \in \Gamma_j$.

By the construction of Γ, Γ_i and Γ_j are subordinates* of Γ_1. Therefore,

(7) $(ML\alpha \cdot ML \sim \alpha) \in \Gamma_1$.

Proof of (7): Suppose that (7) is false. Then, by Lemma 2: (a) $\sim (ML\alpha \cdot ML \sim \alpha) \in \Gamma_1$. (a) and $\vdash_{\mathrm{T}} \sim (ML\alpha \cdot ML \sim \alpha) \supset (L \sim L\alpha \vee L \sim L \sim \alpha)$ imply, by Lemma 3, that either: (b) $L \sim L\alpha \in \Gamma_1$ or (c) $L \sim L \sim \alpha \in \Gamma_1$. Suppose that (b) is true. Then, by the construction of Γ_i and the S4-axiom: $L\alpha \supset LL\alpha$, we get: $\sim L\alpha \in \Gamma_i$. But this is contradicted by (5) and Lemma 1. Suppose that (c) is true. Then, by the construction of Γ_j and the S4-axiom, we get the result: $\sim L \sim \alpha \in \Gamma_j$. But this is contradicted by (6) and Lemma 1. We conclude that neither (b) nor (c) is true. Therefore (a) is false and, consequently, (7) is true.

We shall show now that:

(8) $\vdash_{\overline{\mathrm{K3.2}}} \sim (ML\alpha \cdot ML \sim \alpha)$.

Proof of (8):

(a)	$L(Lp \supset \sim p) \vee (ML \sim p \supset p)$	F1 $[q/\sim p]$
(b)	$L(Lp \supset \sim p) \equiv \sim MLp$	T
(c)	$\sim MLp \vee (ML \sim p \supset p)$	(a), (b) × Eq.
(d)	$\sim p \supset (\sim MLp \vee \sim ML \sim p)$	(c), PC
(e)	$p \supset (\sim MLp \vee \sim ML \sim p)$	(d) $[p/\sim p]$, PC
(f)	$\sim MLp \vee \sim ML \sim p$	(d), (e), PC
(g)	$\sim (MLp \cdot ML \sim p)$	(f), PC
(h)	$\sim (ML\alpha \cdot ML \sim \alpha)$	(g) $[p/\alpha]$.

(8) implies, by Lemma 2, that

(9) $\sim (ML\alpha \cdot ML \sim \alpha) \in \Gamma_1$.

But, by Lemma 1, (9) is incompatible with (7). We conclude that Lemma 7 is valid since its negation leads to a contradiction. QED

4 K.G-*systems* We have already proved[13] that the following formula: $Lp \equiv p \cdot LMp$, is a thesis of K1.2 and—since K4 contains K1.2—even of K4.[14] We can therefore introduce a new modal operator, G, which, in the field of K1.2 and K4, is interdefinable with L:

$G\alpha =_{df} LM\alpha$
$L\alpha =_{df} \alpha \cdot G\alpha$ (or, equivalently,
$M\alpha =_{df} \alpha \vee \sim G \sim \alpha$).

This suggests that we can take G as a primitive and construct $\mathcal{K}$-systems equivalent to K1.2 and K4.

I. K4.G

Axioms:

GA0. All PC-valid formulas.
GA1. $G(p \supset Gp)$
GA2. $G(p \supset q) \supset (Gp \supset Gq)$
GA3. $G \sim p \supset \sim Gp$
GA4. $\sim Gp \supset G \sim p$

Rules:

RG. $\vdash \alpha \Rightarrow \vdash G\alpha$
Rule of substitution, Modus Ponens.

It has been proved by Thomas [13] that, given the above definitions of G and L, K4.G and K4 are equivalent systems.

II. K1.2.G K1.2.G is a system which we get from K4.G if we leave out Axiom GA4. I shall prove now that K1.2.G is equivalent to K1.2.

A. We prove first that all axioms and rules of K1.2 are derivable in K1.2.G. The following rules can be derived in K1.2.G.:

R1. $\vdash \alpha \supset \beta \Rightarrow \vdash G\alpha \supset G\beta$
R2. $\vdash \alpha \supset \beta, \vdash G\alpha \Rightarrow \vdash G\beta$.

Derivation of R1:

Given	:	(1) $\alpha \supset \beta$
(1) × RG	:	(2) $G(\alpha \supset \beta)$
(2), GA2 $[p/\alpha, q/\beta]$ × MP	:	(3) $G\alpha \supset G\beta$.

Derivation of R2:

Given	:	(1) $\alpha \supset \beta$
		(2) $G\alpha$
(1) × R1	:	(3) $G\alpha \supset G\beta$
(2), (3) × MP	:	(4) $G\beta$.

We can also prove:

T1 $Gp \cdot Gq \supset G(p \cdot q)$

Proof:

(1)	$p \supset (q \supset p \cdot q)$	PC
(2)	$Gp \supset (Gq \supset G(p \cdot q))$	(1) × R1, GA2 $[p/q, q/p \cdot q]$ × Syll.
(3)	$Gp \cdot Gq \supset G(p \cdot q)$	(2) × Imp.

A1 $Lp \supset p$

Proof:

(1)	$p \cdot Gp \supset p$	PC
(2)	$Lp \supset p$	(1), Def L

A2 $L(p \supset q) \supset (Lp \supset Lq)$

Proof:

(1)	$G(p \supset q) \cdot (p \supset q) \supset (Gp \supset Gp) \cdot (p \supset q)$	GA2, PC
(2)	$G(p \supset q) \cdot (p \supset q) \supset (Gp \cdot p \supset Gp \cdot p)$	(1), PC
(3)	$L(p \supset q) \supset (Lp \supset Lq)$	(2), Def L

A3 $Lp \supset LLp$

Proof:

(1)	$Gp \cdot p \supset GGp \cdot Gp \cdot Gp \cdot p$	GA1, GA2 $[q/Gp]$ × Syll., PC
(2)	$Gp \cdot p \supset G(Gp \cdot p) \cdot (Gp \cdot p)$	(1), T1 $[p/Gp, q/p]$, PC
(3)	$Lp \supset LLp$	(2), Def L

A4 $LMp \supset MLp$

Proof:

(1)	$(\sim p \supset G \sim p) \supset ((p \vee \sim G \sim p) \supset p)$	PC
(2)	$G(\sim p \supset G \sim p)$	GA1 $[p/\sim p]$
(3)	$G((p \vee \sim G \sim p) \supset p)$	(1), (2) × R2
(4)	$G(p \vee \sim G \sim p) \supset Gp$	(3), GA2 $[p/p \vee \sim G \sim p, q/p]$ × Syll.
(5)	$Gp \supset GGp$	GA1, GA2 $[q/Gp]$ × MP
(6)	$G(p \vee \sim G \sim p) \supset Gp \cdot GGp$	(4), (5), PC
(7)	$G(p \vee \sim G \sim p) \supset G(p \cdot Gp)$	(6), T1 $[q/Gp]$ × Syll.
(8)	$G(p \cdot Gp) \supset \sim G \sim (p \cdot Gp)$	GA3 $[q/p \cdot Gp]$ × Transp.
(9)	$G(p \vee \sim G \sim p) \supset \sim G \sim (p \cdot Gp)$	(7), (8) × Syll.
(10)	$G(p \vee \sim G \sim p) \cdot (p \vee \sim G \sim p) \supset \sim G \sim (p \cdot Gp) \vee (p \cdot Gp)$	(9), GA3 × Transp., PC
(11)	$LMp \supset MLp$	(10), Def L, Def M

H1 $p \supset L(Mp \supset p)$

Proof:

(1)	$G(\sim p \supset G \sim p)$	GA1 $[p/\sim p]$

(2)	$(\sim p \supset G \sim p) \supset ((\sim G \sim p \vee p) \supset p)$	PC
(3)	$G((\sim G \sim p \vee b) \supset p)$	(1), (2) × R2
(4)	$p \supset ((\sim G \sim p \vee p) \supset p) \cdot G((\sim G \sim p \vee p) \supset p)$	(3), PC
(5)	$p \supset L(Mp \supset p)$	(4), Def L, Def M

T2 $Gp \equiv LMp$

Proof:

(1)	$Gp \supset (p \vee \sim G \sim p)$	GA3 × Transp., PC
(2)	$p \supset (p \vee \sim G \sim p)$	PC
(3)	$Gp \supset G(p \vee \sim G \sim p)$	(2) × R1
(4)	$Gp \supset (p \vee \sim G \sim p) \cdot G(p \vee \sim G \sim p)$	(1), (3), PC
(5)	$(p \vee \sim G \sim p) \cdot G(p \vee \sim G \sim p) \supset Gp$	see step 4 in the proof of A4, PC
(6)	$Gp \equiv LMp$	(4), (5), PC, Def L, Def M

Derivation of RL ($\vdash \alpha \Rightarrow \vdash L\alpha$):

Given	:	(1) α
(1) × RG	:	(2) $G\alpha$
(1), (2)	:	(3) $\alpha \cdot G\alpha$
(3), Def L	:	(4) $L\alpha$

B. We prove now that K1.2 contains K1.2.G:

Proof of GA1:

(1)	$LM(p \supset LMp)$	T
(2)	$G(p \supset Gp)$	(1), Def G

Proof of GA2:

(1)	$ML(p \supset q) \cdot LMp \supset Mq$	S4
(2)	$LM(p \supset q) \cdot LMp \supset Mq$	(1), A4 $[p/p \supset q]$, PC
(3)	$L(LM(p \supset q) \cdot LMp) \supset LMq$	(2) × RL, A2 $[p/LM(p \supset q) \cdot LMp, q/Mq]$ × Syll.
(4)	$LM(p \supset q) \cdot LMp \supset LMq$	(3), $\vdash_{\overline{S4}} LM(p \supset q) \cdot LMp \supset L(LM(p \supset q) \cdot LMp)$ × Syll.
(5)	$LM(p \supset q) \supset (LMp \supset LMq)$	(4) × Exp.
(6)	$G(p \supset q) \supset (Gp \supset Gq)$	(5), Def G

Proof of GA3:

(1)	$LM \sim p \supset ML \sim p$	A4 $[p/\sim p]$
(2)	$LM \sim p \supset \sim LMp$	(1), $\vdash_{\overline{T}} ML \sim p \supset \sim LMp$ × Syll.
(3)	$G \sim p \supset \sim Gp$	(2), Def G

Derivation of RG:

Given	:	(1) α
T	:	(2) $\alpha \supset M\alpha$
(1), (2) × MP	:	(3) $M\alpha$
(3) × RL	:	(4) $LM\alpha$
(4), Def G	:	(5) $G\alpha$

III. The number of modalities in K1.2.G and K4.G We shall prove now that K1.2.G (K4.G) has precisely 6 (4) distinct modalities.

A. K1.2.G has at most 6 distinct modalities: (1) -, (2) G, (3) $\sim G \sim$, (4) $\sim$, (5) $\sim G$, and (6) $G\sim$.

Proof: First we prove that the following "reduction laws" are theorems of K1.2.G:

(a) $GGp \equiv Gp$
(b) $G \sim G \sim p \equiv Gp$
(c) $G \sim Gp \equiv G \sim p$
(d) $GG \sim p \equiv G \sim p$.

Proof of (a):

(1)	$Gp \supset GGp$	see step 5 in the proof of A4.
(2)	$Gp \supset \sim G \sim p$	GA3, PC
(3)	$(\sim G \sim p \supset p) \supset (Gp \supset p)$	(2), PC
(4)	$(\sim p \supset G \sim p) \supset (Gp \supset p)$	(3), PC
(5)	$G(\sim p \supset G \sim p)$	GA1 $[p/\sim p]$
(6)	$G(Gp \supset p)$	(4), (5) × R2
(7)	$GGp \supset Gp$	(6), GA2 $[p/Gp,\ q/p]$ × MP
(8)	$GGp \equiv Gp$	(1), (7), PC

Proof of (b):

(1)	$(G \sim p \supset \sim p) \supset (p \supset \sim G \sim p)$	PC
(2)	$G(G \sim p \supset \sim p)$	see step 6 in the above proof $[p/\sim p]$.
(3)	$G(p \supset \sim G \sim p)$	(1), (2) × R2
(4)	$Gp \supset G \sim G \sim p$	(3), GA2 $[q/\sim G \sim p]$ × MP
(5)	$(\sim p \supset G \sim p) \supset (\sim G \sim p \supset p)$	PC
(6)	$G(\sim p \supset G \sim p)$	GA1 $[p/\sim p]$
(7)	$G(\sim G \sim p \supset p)$	(5), (6) × R2
(8)	$G \sim G \sim p \supset Gp$	(7), GA2 $[p/\sim G \sim p,\ q/p]$ × MP
(9)	$G \sim G \sim p \equiv Gp$	(4), (8), PC

Proof of (c):

(1)	$Gp \equiv G \sim\sim p$	PC × R1
(2)	$\sim Gp \equiv \sim G \sim\sim p$	(1), PC
(3)	$G \sim Gp \equiv G \sim G \sim\sim p$	(2) × R1
(4)	$G \sim G \sim\sim p \equiv G \sim p$	(b) $[p/\sim p]$
(5)	$G \sim Gp \equiv G \sim p$	(3), (4) × Syll.

Proof of (d): by substitution in (a).

It is easy to ascertain that the addition of $\sim$ to any one of the modalities (1)-(6) gives us a modality which is equivalent to some already listed modality. We prove now that the addition of G has the same consequences.

If we add G to (1) or (4), we get, respectively, (2) and (6).

(a) implies that, if we add G to (2), we get an equivalent of (2).
(b) implies that, if we add G to (3), we get an equivalent of (2).
(c) implies that, if we add G to (5), we get an equivalent of (6).
(d) implies that, if we add G to (6), we get an equivalent of (6).
This completes our proof.

B. K4.G has at most 4 distinct modalities: -, G, and their negations.

Proof: Since K4.G contains K1.2.G, it cannot have more than 6 modalities. But it follows trivially from GA3 and GA4 that $\vdash_{\overline{K4.G}} Gp \equiv \sim G \sim p$ and $\vdash_{\overline{K4.G}} \sim Gp \equiv G \sim p$. Therefore the number of distinct modalities reduces to four.

C. K1.2.G (K4.G) has at least 6 (4) distinct modalities.

Proof: Since K1.2.G (K4.G) is deductively equivalent to K1.2 (K4), the K1.2 (K4)-semantics is adequate even for K1.2.G (K4.G). Therefore, in order to prove C, we should construct the falsifying K1.2 (K4)-models for all equivalences between p, Gp, $\sim G \sim p$, $\sim p$, $\sim Gp$, and $G \sim p$ (between p, Gp, $\sim p$, and $\sim Gp$). Since this proof is a purely mechanical task, I shall omit it here. But one thing remains to be done before we even can start constructing the falsifying models. We have to give such a truth-condition for G that the defunctional equivalences: $G\alpha \equiv LM\alpha$ and $L\alpha \equiv \alpha \cdot G\alpha$, will turn out to be valid.

VG. For any wff, α, and for any $w_i \in W$, $V(G\alpha, w_i) = 1$ iff for every w_j such that $w_i R w_j$ and $w_j \in C$, $V(\alpha, w_j) = 1$; otherwise $V(G\alpha, w_i) = 0$.

VG is a correct truth-condition for G iff, given VG, the following holds:

For any K1.2- or K4-model, $\langle W, R, V\rangle$, for any $w_i \in W$ and for any wff, α,

I. $V(G\alpha, w_i) = 1$ iff $V(LM\alpha, w_i) = 1$, and
II. $V(L\alpha, w_i) = 1$ iff $V(\alpha \cdot G\alpha, w_i) = 1$.

Proof of I: Since $\langle W, R, V\rangle$ is a K1.2- or K4-model:

(1) R is reflexive and strictly short.

(1) and the truth-condition for L imply that:

(2) $V(LM\alpha, w_i) = 1$ iff (a) $V(M\alpha, w_i) = 1$ and (b) $V(M\alpha, w_j) = 1$, for every w_j such that $w_i R w_j$ and $w_j \in C$.

The truth-condition for M and the definition of C entail the following equivalence:

(3) For every $w_j \in C$, $V(M\alpha, w_j) = 1$ iff $V(\alpha, w_j) = 1$.

From (2) and (3) we get:

(4) $V(LM\alpha, w_j) = 1$ iff (a) $V(M\alpha, w_j) = 1$ and (b) $V(\alpha, w_j) = 1$ for every w_j such that $w_i R w_j$ and $w_j \in C$.

(1) implies that:

(5) R is finite.

It follows from (5) and the truth-condition for M that the condition (a) in (4) is redundant. In other words, (4) can be shortened to

(6) $V(LM\alpha, w_i) = 1$ iff $V(\alpha, w_j) = 1$, for every w_j such that $w_i R w_j$ and $w_j \in C$.

Given VG, (6) is equivalent to I.

Proof of II: (1) and the truth-condition for L imply that

(7) $V(L\alpha, w_i) = 1$ iff $V(\alpha, w_i) = 1$ and $V(\alpha, w_j) = 1$, for every w_j such that $w_i R w_j$ and $w_j \in C$.

Given VG, and the truth-condition for conjunction, it follows immediately that (7) is equivalent to II. QED

NOTES

1. It is easy to prove that, in the field of S4, A4 and M1 taken together are equivalent to the following formula:

   M2. $p \supset (MLp \vee L(Mp \supset p))$.

   That A4 & M1 imply M2 and that M2 implies M1 is obvious. In Section 2 it is shown that M2 implies J1 in the field of T (see steps 3-10 in the proof of (b)). Since Sobociński [11] has proved that, given S4, A4 is deducible from J1, we can conclude that A4 is deducible even from M1.

2. StrShort. implies Fin. Given Trans., the same applies to Short.

3. This paper had already been written when K. Segerberg informed me that the completeness proofs for K1.2 and K3.2 can be found in his *Essay in Classical Modal Logic*, chapter II, section 7, Uppsala, 1971.

4. By a "modality" I shall mean here any unbroken sequence of zero or more monadic operators such that every operator in the sequence is either primitive or is equivalent to an unbroken sequence of primitive monadic operators.

5. Two systems are independent iff neither contains the other one.

6. j implies that K2.1.1 does not contain K3.1.1. That K3.1.1 contains K2.1.1 is, of course, trivial.

7. By "T", "S4", etc., I shall mean that the formula in question is a thesis of T (S4, etc.).

8. "Eq" stands for the "rule of substitution of proved equivalents" (derivable in every system which contains T).

9. For the sake of simplicity, I have omitted all identity-pairs belonging to R (R is, of course, a reflexive relation in every $\mathcal{K}$-model).

10. A *finite* set of wffs of S is *consistent* relative to S iff the negation of the conjunction of its members is not a thesis of S. An *infinite* set of wffs of S is *consistent* relative to S iff its every finite subset is consistent relative to S. A set of wffs of S, Γ_i, is *maximal consistent* relative to S iff: (1) Γ_i is consistent relative to S, and (2) for any wff of S, α, if $\alpha \notin \Gamma_i$, then $\Gamma_i \cup \{\alpha\}$ is inconsistent relative to S.

11. See Section 2, step 9 in the proof of (c2).

12. See, for example, [3], p. 157.

13. See Section 2, step 5 in the proof of (c2).

14. It can be easily shown that the formula in question is not provable in any other $\mathcal{K}$-system represented in Diagram B.

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