SIMPLIFYING THE AXIOMS OF THE PREDICATE CALCULUS

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1 *Introduction*   It is often useful, e.g., in algebraic research, to have the postulates of a formal system expressed in the simplest possible form— "simple" meaning here: with a minimum of "metamathematical" (i.e., English) comments. The aim of the present paper* is to "simplify" the system of Quine [6], amended to allow the use of free variables.[1]

Only three "metamathematical" notions will be used: closure, bound substitution, and free substitution. They will be denoted by special symbols.

(i) C is the closure in [6] [e.g., the closure of $R(x,y)$ is $\forall x \forall y R(x,y)$].

(ii) $\mathcal{B}_y^x$ means: substitution of y for every bound occurrence of x [for instance $\mathcal{B}_y^x(P(x) \wedge \forall x R(x,y))$ is $P(x) \wedge \forall y P(y,y)$].

(iii) $\mathcal{I}_y^x$ means: substitution of y for every free occurrence of x [for instance $\mathcal{I}_y^x(P(x) \wedge \forall x R(x,y))$ is $P(y) \wedge \forall x R(x,y)$].

$\mathcal{B}_y^x A = A$ means that x is not bound in A; $\mathcal{I}_y^x A = A$ means that x is not free in A.[2]

2 *The proposed system* A, B, etc., will denote formulas; x, y, etc., will denote individual variables; v_1, v_2, . . ., v_n will denote distinct individual variables, the natural order of the indices showing the *alphabetic order* of the variables.

In a formula such as CA, C denotes the string $\forall v_{i_1} \forall v_{i_2} \ldots$, where $v_{i_1}, v_{i_2}, \ldots$, are the variables which have at least one free occurrence in A, and with $i_1 < i_2 < \ldots$.

System I is:

(I1) $\vdash C((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$
(I2) $\vdash C(A \Rightarrow (B \Rightarrow A))$
(I3) $\vdash C((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$

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(I4) $\vdash C\,(\forall x \forall y A \Rightarrow \forall y \forall x A)$

(I5) $\vdash C\,(\forall x (A \Rightarrow B) \Rightarrow (\forall x A \Rightarrow \forall x B))$

(I6) $\vdash C\,(A \Rightarrow \forall x A)$, if $\mathcal{I}_y^x A = A$

(I7) $\vdash C\,(\forall x A \Rightarrow \mathcal{I}_y^x A)$, if $\mathcal{B}_z^y A = A$

(I8) $\vdash \forall x A \Rightarrow A$

(I9) $\dfrac{A,\ A \Rightarrow B}{B}$.

This system differs from that of Quine [6] in the following ways: by having I1-I3 instead of "if A is tautologous, then $\vdash CA$"; by the presence of I8, which allows the use of free variables; and by an important simplification in the "metamathematical" comment of I7. This simplification is the chief aim of this paper. The system of Quine [6] will be referred to as System II.

The new system differs also from the one of Quine's [7], because the latter system uses another definition of "closure" (anti-alphabetical instead of alphabetical order of quantifiers) and this change allowed (see Berry's [1]) suppression of the axiom similar to I4.[3] The system of [7] will be called System III.

System II, with the present notation, is:

(II1) *If A is tautologous, then* $\vdash CA$

(II2) $\vdash C\,(\forall x \forall y A \Rightarrow \forall y \forall x A)$

(II3) $\vdash C\,(\forall x (A \Rightarrow B) \Rightarrow (\forall x A \Rightarrow \forall x B))$

(II4) $\vdash C\,(A \Rightarrow \forall x A)$, *if* $\mathcal{I}_y^x A = A$

(II5) $\vdash C\,(\forall x A \Rightarrow \mathcal{I}_y^x A)$, *if, by the substitution* $\mathcal{I}_y^x$, *every free occurrence of x is replaced by a free occurrence of y.*

(II6) $\dfrac{A,\ A \Rightarrow B}{B}$.

The aim of this paper is to replace the long condition governing II5 by the simpler condition on I7.

3 *Methods*   It is well-known that System II, supplemented by axiom I8, is equivalent to the more classical systems for the predicate calculus (e.g., the well-known system given in Mendelson [3], p. 57), although this fact has seldom been put into print.

We will prove that every tautologous formula is a thesis of I. Then, we will prove that axioms II1-II5 are theses of I. The proof of equivalence will then be achieved. To prove the latter (and more difficult) point, the method we will use may be roughly stated as follows: Take a formula (the A in II5), replace bound variables by new variables (not appearing in A, neither bound nor free), apply I7, and restore the initial bound variables.

4 *Tautologous formulas*

Theorem 1 $\dfrac{C\,(A),\ C\,(A \Rightarrow B)}{C\,(B)}$.

(This is metatheorem *111 in [6], [7]. The proof does not use axiom II2 (i.e., my axiom I4).

Theorem 2   *If C is a thesis of the propositional calculus, then* $\vdash C(C)$.

(This is axiom II1, the first axiom in $\lfloor 6 \rfloor$, and also in $\lfloor 7 \rfloor$.)

Let us consider a formal deduction in the propositional calculus, and let us consider a parallel list of formulas in the System I, each formula beginning with "$C \ldots$". By Theorem 1, we can imitate in System I the formal propositional deduction step by step. The last formula of the new list will be just $C(C)$.

Definition: A *non-standard closure* of a formula of the predicate calculus, C, is a formula $\mathcal{P}C$ where the "prefix" $\mathcal{P}$ is $\forall x \forall y \forall z \ldots$ and where $x, y, z, \ldots$ are all the variables that have one or more free occurrence in C, these variables being arranged in any order. The "standard" closure, C, is the one in which these variables are in alphabetic order (from $\lfloor 6 \rfloor$). In $\lfloor 7 \rfloor$, Quine used another kind of closure in which the variables are arranged in anti-alphabetic order after (Berry $\lfloor 1 \rfloor$). Fitch $\lfloor 2 \rfloor$ used another kind of non-standard closure.

Theorem 3   *Every closure of a tautologous formula is a thesis of System* I.

Proof: By repeated uses of I4 and Theorem 2.

5 *Proof of the excluded axiom* II5

Theorem 4 (replacement theorem)   *If* $\vdash C(A \Longleftrightarrow A')$, *and B' is formed from B by putting A' for some occurrence of A, then* $\vdash C(B \Longleftrightarrow B')$.

This is theorem *123 of Quine $\lfloor 6 \rfloor$ and we may reproduce his proof (the whole of section I8), since this proof does not use II5.

Theorem 5   *If A contains no occurrence (bound or free) of y, then*

$$\vdash C(\forall x A \Longrightarrow \forall y \mathcal{I}^x_y A).$$

Proof: Axiom I6 can be applied, and then we have:

(1) $$\vdash C(\forall y \forall x A \Longrightarrow \mathcal{I}^x_y A)$$

according to I4, I5, and Theorem 3. Then

(2) $$\vdash C(\forall x A \Longleftrightarrow \forall y \forall x A)$$

according to I6. Formulas (1) and (2) give Theorem 5 from Theorem 3.

Theorem 6   *If A contains no occurrences (bound or free) of y, and no bound occurrence of x, then*

$$\vdash C(\forall x A \Longleftrightarrow \mathcal{B}^x_y \mathcal{I}^x_y \forall y A).$$

Proof: Using Theorem 5, and the fact that there is no bound occurrence of x:

$$\forall y \mathcal{I}^x_y A = \forall y \mathcal{B}^x_y A = \mathcal{B}^x_y \mathcal{I}^x_y \forall y A.$$

Theorem 7   *If A contains no occurrence of y and no bound occurrence of x, then*

$$\vdash C\,(\forall xA \Longleftrightarrow \mathcal{B}_y^x\mathcal{I}_y^x\forall xA).$$

Proof: According to Theorem 6, since

$$\mathcal{B}_y^x\mathcal{I}_y^x\mathcal{B}_x^y\mathcal{I}_x^y\forall xA = \forall yA$$

and then, using Theorem 6 (twice), and Theorem 2, we obtain Theorem 7.

Theorem 8   *If A contains no occurrence of y (even when it contains bound occurrences of x), then*

$$\vdash C\,(\forall xA \Longleftrightarrow \mathcal{B}_y^x\mathcal{I}_y^x\forall xA).$$

Proof: By induction, starting with the quantifiers $\forall x$, the scope of which is minimal, and then using Theorems 6, and 3. At each step, x is replaced by a new variable $(x', x'', x''', \ldots,$ etc.) that has no occurrence in the considered subformula. Afterward, we replace, in reverse order, . . ., x''', x'', x' by x.

Theorem 9   I15 *is provable.*

Proof: By a string of substitutions allowed by I6 and using Theorem 8. Let us suppose for instance that

$$A = P(x) \wedge \forall yQ(y).$$

(i)   We replace the bound occurrences of the variables other than x (in this case, y) by new variables using Theorems 8 and 4; in the example, we get

$$A' = P(x) \wedge \forall y'Q(y')$$

with

$$\vdash C\,(A \Longleftrightarrow A').$$

(ii)   We can now apply I7, and obtain

$$\vdash C\,(\forall x(P(x) \wedge \forall y'Q(y')) \Longrightarrow (P(y) \wedge \forall y'Q(y'))).$$

(iii) By Theorems 8 and 4 we go in reverse order from the new variable (in this case, y') to y, and we obtain

$$\vdash C\,(\forall x(P(x) \wedge \forall yQ(y)) \Longrightarrow (P(y) \wedge \forall yQ(y)))$$

which is an example of I15.

Step iii would not have been possible if the formula A did not satisfy the condition for application of I15. For instance, if A had been

$$\forall xR(x, x)$$

we would have obtained the invalid formula

$$C\,(\forall xR(x, y) \Longrightarrow R(y, y))$$

as a thesis.

6 *Remarks*   (i) The question of independence of postulates has not been examined, but I4 appears in so many proofs that it does not seem probable

that it could be proved by means of the other postulates. (ii) Only one kind of "closure" has been used; but it does not seem probable that the proofs could be fundamentally changed if another kind of closure had been used— for instance, that of [2] or that of [1] (see also [7]).

NOTES

1. The reason that the system of Quine [7] has not been considered will be apparent in what follows. The sign ⊢ has the usual meaning (showing theses), not the sense of Quine [6] or [7].

2. The sign "=" is metamathematical, being synonymous with the English "is". We do not consider the predicate calculus with (formal) equality.

3. Fitch [2], using another change in the notion of "closure", reached the same result.

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