Notre Dame Journal of Formal Logic
Volume XXI, Number 1, January 1980
NDJFAM

ON A TABLEAU RULE FOR IDENTITY

KATHLEEN JOHNSON WU

Lambert and van Fraassen, [1], Part II, developed a system of Fitch-style intelim rules for a logic of general terms with identity, valid for the empty domain. For the cases in which the domain is assumed non-empty, a special rule **VQE** is provided. A system of tableau rules, similar to those of Beth [2], is also presented. Each of these rules is derivable within the system of intelim, including a special rule **VU** if **VQE** is made available. The tableau rule **I** for identity[1] is the following [1], p. 119:

$$\mathbf{I}\quad \frac{\mathsf{E}\,(X\,\&\ x = y\ \&\ Y)}{\mathsf{E}\,(y/x)(X\ \&\ Y)}\quad .$$

In [1], Part III, both the system of intelim rules and the system of tableau rules are extended to deal with singular terms, without the standard assumption that each term denotes a single individual. In this extension, rule **I** is generalized by the replacement of x and y with t and t_1, respectively, and t with or without subscripts is used to refer to individual variables and individual constants alike. Individual constants are, of course, used to abbreviate singular terms.

The purposes of this paper are: (1) to show that rule **I** is applicable (i) in its initial form, only when the domain is non-empty and (ii) in its generalized form, only when each individual constant denotes a single individual and the domain has no more than one member; and (2) to propose new tableau rules for identity not subject to these limitations.

1 Tableau rules are used to construct tableau sequences. Through their application, an initial statement is progressively transformed into a multiple disjunction, each disjunct of which is a multiple conjunction. A sequence is said to terminate if, at the end, each disjunct has among its conjuncts a statement and its negate or a statement of the form $t \neq t$. Supposedly, a sequence terminates if and only if the initial statement is logically false. In order for the technique to be entirely reliable, the following criteria must be met: (1) sequence construction is mechanical, (2) each tableau rule is reversible, and (3) every applicable rule is eventually applied.

Received June 27, 1977; Revised September 16, 1977

It is not difficult to see why a tableau rule must be reversible, that is, must be valid even though the order of the statements is reversed. This ensures that each line of the sequence is equivalent to the first. If this were not the case, a sequence might fail to terminate, although the initial statement was, in fact, logically false. For the same reason, every rule that is applicable must be applied.

Rule **I**, in its initial form, is reversible if an assumption is made that the domain is non-empty. This can be easily shown by deriving the reverse of the rule within the system of intelim plus **VQE**. However, if the assumption that the domain is non-empty is dropped, the rule in question is not reversible. To see this, let X and Y both denote blanks and let the metalogical symbols x and y both denote the variable x. If rule **I** were reversible, the following statement would be derivable from no assumptions: $(\mathrm{E}x)(x = x)$. But it cannot be in a logic valid for the empty domain.

Rule **I**, as generalized to deal with singular terms, is reversible if it is assumed that each individual constant denotes a single individual and, further, that the domain has no more than one member. This can be easily shown by deriving the reverse of the rule within the system of intelim plus two special rules, one allowing for the introduction at any point of the line $(\mathrm{E}x)(t = x)$ and the other, of the line $(x)(y)(x = y)$. However, if these assumptions are dropped, the rule in question is not reversible. To see that the first assumption is needed, let X and Y both denote blanks, t denote a, where a abbreviates "Pegasus" or any singular term you please, and t_1 denotes the variable x. If the rule were reversible, then the following statement would be derivable from no assumptions: $(\mathrm{E}x)(a = x)$. But such a statement clearly cannot be in a logic in which individual constants are not assumed to denote. To see that the second assumption is needed, let X and Y denote blanks and let t and t_1 denote any two distinct individual constants. If the rule were reversible, then $t = t_1$ would be derivable from no assumptions. But such a statement cannot be in a logic valid for domains with more than one member.

2 What, then, would serve as an acceptable rule? It must be reversible, but it must also be such that, in the construction of a tableau sequence, it does not prevent every applicable rule from being eventually applied. Together with one that "shuffles" the alternates [1], p. 121, the following proviso is intended to ensure that the latter criterion is satisfied.

> PROVISO: the letter X in the statement of the tableau rules must stand for a formula which does not have any conjunct of form $\neg\neg A$, or $\neg(A \,\&\, B)$, or $(A \vee B)$, or $\neg(A \vee B)$, or $(x)A$, or $\neg(x)A$, or $x = y$. [1], p. 121

It is natural to assume that this proviso is also intended to hold for any conjunct of form $t = t_1$.

In search of a new rule not subject to the limitations on rule **I**, it might seem reasonable to restrict rule **I** to only those cases in which X and Y do not both stand for blanks. This restriction would, however, not make the rule reversible. As long as it is possible that no variable occurs free in $(y/x)(X \& Y)$ and there is no statement among the conjuncts of X or Y to the effect that something exists, an inference from a statement which makes no claim that something exists to one that does would be justified. If this possibility is blocked by a restriction, however, the initial form of rule **I** is reversible. But, given the proviso quoted above, whenever the unrestricted rule would be applicable, but the restricted rule would not, no rule could be applied to any conjunct of Y. Thus, if for no other reason, the restriction cannot be allowed because the third criterion for the construction of a tableau sequence would not be met, that is, every applicable rule would not eventually be applied.

Another, more promising approach is to alter the structure of rule **I** to ensure that both statements have initial existential quantifiers. The following rule does that:

$$\begin{array}{|l} \underline{\mathsf{E}(X \& x = y \& Y)} \\ \mathsf{E}(y/x)(X \& x = y \& Y) \end{array} \quad .$$

It is clearly reversible; however, given the proviso quoted above, it would never allow application of any rule to any conjunct of Y. The following rule, on the other hand, is derivable in intelim without benefit of **VQE**, is reversible, and, patterned after the rule **U1**, seems unlikely to have any difficulties with the provisos[2]:

$$\begin{array}{|l} \underline{\mathsf{E}(X \& x = y \& Y)} \\ \mathsf{E}(y/x)(X \& Y \& x = y) \end{array} \quad .$$

One final question remains: If this rule were generalized to deal with singular terms by replacing x and y by t and t_1, respectively, would it be reversible? It would be only if—as in the case of the generalized form of rule **I**—it were assumed that each individual constant denotes a single individual and the domain has no more than one member. To see that the first assumption is needed, let X and Y both denote blanks, t denote a and t_1 denote the variable x. If the rule were reversible, then $(\mathsf{E}x)(a = x)$ would be derivable from $(\mathsf{E}x)(x = x)$. To see that the second assumption is needed, let X and Y both denote blanks, t denote a and t_1 denote b.[3] If the rule were reversible, then $a = b$ would be derivable from $b = b$. One final adjustment, however, gives us the following rule:

$$\begin{array}{|l} \underline{\mathsf{E}(X \& t = t_1 \& Y)} \\ \mathsf{E}((t_1/t)(X \& Y) \& t = t_1). \end{array}$$

That this rule is not subject to the limitations on rule **I** can be readily seen from the fact that it as well as its reverse can be derived within the system of intelim without benefit of **VQE** or any other special rule.

NOTES

1. $\mathsf{E}(A)$ abbreviates $(\mathsf{E}x_1)\ldots(\mathsf{E}x_n)A$, where A is a statement and $x_1, \ldots, x_n$ are all the variables free in A. $\mathsf{E}(A)$ is A if no variables are free in A. The metalogical symbols X and Y refer to

statements or blanks; x and y refer to individual variables; and $(y/x)A$ refers to the result of substituting y for x throughout A', where z is a variable that does not occur in A and A' is the result of relettering y to z any part of A in which y is bound and x is free.

2. The rule **U1** is the following:

$$\frac{\mathsf{E}(X \,\&\, (x)A \,\&\, Y)}{\mathsf{E}(X \,\&\, (y_1/x)A \,\&\, \ldots \,\&\, (y_n/x)A \,\&\, Y \,\&\, (x)A),}$$

where $y_1, \ldots, y_n$ are the variables free in X & $(x)A$ & Y.

3. X and Y are taken to denote blanks in this case as in earlier ones to simplify the counterexample. Other counterexamples where X and Y are taken to denote statements could have been used equally as well; i.e., X and Y need not denote blanks in order for counterexamples to be generated in any of the cases considered.

REFERENCES

[1] Lambert, K., and B. C. van Fraassen, *Derivation and Counterexample*, Dickenson Publishing Co., Encino, California, 1972.

[2] Beth, E., *Formal Methods*, D. Reidel Publishing Co., Dordrecht, Holland, 1962.

The University of Alabama
University, Alabama