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CALCULEMUS

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In this paper I shall develop three methods of expressing propositions in algebraic notation along with purely computational tests for validity. The first two methods pertain to syllogistic arguments, the third to propositional logic. All seem to be of theoretical as well as pedagogical interest.

1 *The Additive Method* Traditional schematic formats for representing syllogisms have no doubt suggested some elementary mathematical operations to numerous logicians, but full mathematization has proved elusive. Fred Sommers [6], to my knowledge, was the first person to devise an adequate system whereby syllogisms can be treated as additions. It turns out that one can add up the premisses of a valid syllogism and the sum will be the uniquely correct conclusion. Sommers' system is, however, encumbered with several nonmathematical rules; and we lack a convincing explanation as to why we should expect his method to work. In reflecting on these issues, I was able not only to uncover how an additive system works and *why*, but also to produce a greatly simplified method.

The efficacy of my method depends on being able to capture mathematically certain traits of the syllogism and especially a complete set of three rules for determining validity employed by Wesley Salmon [4]. A syllogism has exactly two premisses and a conclusion, all of subject-predicate form. Furthermore, the three propositions always involve exactly three terms (which may be either subjects or predicates), one of which (the middle term) appears once in each premiss but never in the conclusion. Thus, in adding premisses it is necessary that the middle term drop out; otherwise it would appear in the sum representing the conclusion. One of Salmon's rules states that the middle term must be "distributed" (see [3] for a discussion of this concept) exactly once. This calls for some provision for mathematically differentiating a distributed occurrence of the middle term in one premiss from its mandated, undistributed occurrence in the other premiss. One way to accomplish this is to have the middle term be positive once and negative once.

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Another of Salmon's rules is that each of the other two terms (the end terms) must have the same distribution value in the conclusion that it had in the premiss containing it. This rule reinforces the idea of making + and - indicators of distribution value, because the end terms need to preserve this value in going from summand to sum. To highlight the distribution value of a term and because + is often omitted, let us make - the sign of a distributed term.

If we bear in mind that in a universal affirmative proposition, only the subject is distributed (hence negative), we are now in a position to test a basic syllogism:

First Premiss	All M is P	$-M+P$
Second Premiss	All S is M	$-S+M$
Conclusion	All S is P	$-S+P$.

Since the sum of the premisses is indeed the representation of the conclusion, the syllogism is valid.

Salmon's remaining rule is that the number of negative premisses in a syllogism must equal the number of negative conclusions to be valid. Since there is, of course, only one conclusion, this means that there is at most one negative premiss and some means of mathematically identifying negative propositions would be desirable. I chose to mark these propositions with a suffix of -1, because in the process of addition the sum will serve as a tally of negative premisses. The sum could not in this system properly represent an actual proposition with any suffix other than -1. Any sum with a different suffix would surely indicate a violation of the third rule. Here is an example of a syllogism that would have seemed valid without provision for markers:

Some M is not P	$+M-P-1$
Some S is not M	$+S-M-1$
Some S is not P	$+S-P-2$.

Not only is the invalidity easily spotted, but the reason for the invalidity is as well. Formally one might just say that the computed sum does not symbolize the anticipated conclusion, which calls for $+S-P-1$.

The four types of categorical sentence forms that can appear in a standard syllogism and their corresponding representations are

A	All S is P	$-S+P$
E	No S is P	$-S-P-1$
I	Some S is P	$+S+P$
O	Some S is not P	$+S-P-1$.

Now a standard syllogism which has three terms (each occurring twice) need present no problem if any single term is logically negative in each of its appearances. The matter is altogether different when a term and its negate (e.g., blue and nonblue) are both present; they must be counted as two separate terms. If this makes the total greater than three, one member of each offending pair must be transformed to the other by the processes of obversion or conversion—otherwise, the argument is not a standard syllogism and, as a result, is not amenable to the additive test.[1] So far there is no mathematical provision for negated terms; hence, obversion can not yet be accomplished algebraically. But, since the ordinary algebraic laws of associativity and commutativity apply, conversion (of E and I statements) is possible.

As a caution, we note that one must not infer from the fact that the algebraic sum of the premisses of a valid syllogism is equal to the conclusion that these premisses are logically equivalent to the conclusion. Nor can one confidently derive the original premisses from the conclusion by working backwards. The algebraic relation of premisses to conclusion is usually one-way, like implication. In fact, despite much effort, I was unable to discover within the additive scheme any representation of logical equivalence, say, of the sort arising from contraposition, that did not impose extremely artificial restrictions on the normal operations of addition and subtraction.

Despite (or perhaps because of) the paucity of algebraic operations permitted in the additive method, it is quite handy to use and easy to teach; for by means of a single rule (the sum of the premisses must equal the conclusion) one can prove the validity of the fifteen valid syllogisms sanctioned by Salmon's rules and more. Suppose we wished to bestow existential import on other than I and O propositions, thereby validating some nine additional syllogisms from traditional logic. We need only incorporate among the premisses a statement (of the existential import deemed to issue from a term T) written in the form 'Some T is T', which would be expressed algebraically as $+T+T$. For instance, to render AEO in the fourth figure valid, the formula $+S+S$ should be included among the premisses to assert the nonemptiness of the minor term (or subject of the conclusion). One can tell whether a syllogism is to be considered valid *only* on the traditional view by working with the given premisses and noting if an expression of the form $-2T$ occurs in the sum. Such a syllogism could be made valid with the supplementary premiss that there are T's, that is, $+T+T$. This maneuver suggests that the additive method is not restricted to syllogistic arguments with just two premisses, and indeed that is the case.

Sorites and every sort of polysyllogism (with any number of premisses) are similarly testable by this method, once the proper preparations are made. The number of different terms must equal the number of propositions—with due treatment of negates as before; the propositions must be in standard categorical form; no proposition should be duplicated; nor should a term appear other than once in each of two propositions; once these regularizations are accomplished, then provision for existential import may take place. It may occur that a sensible conclusion may derive from a nonstandard argument, but the above precautions should forestall hasty misjudgments. As for enthymemes, the missing proposition may be found—if in fact the argument is not hopelessly invalid—by simple addition (for a missing conclusion) or subtraction of the sum of the *given* premisses from the given conclusion (for a missing premiss).

While it is clear that regular syllogistic arguments can be treated quite readily, so can certain more shadowy, nonstandard ones. Consider this enthymeme, where we are given as premisses: No A is B, and Some B is not C. From their algebraic symbolization ($-A-B-1$, $+B-C-1$), it is apparent that their sum will contain -2, prima facie evidence that no standard syllogistic conclusion can validly be drawn as things now stand. However, once the first premiss is obverted to become 'All A is non-B' and the second contraposited to become 'Some non-C is not non-B', the new sum (of $-A+B'$ and $+C'-B'-1$) is $+C'-A-1$. The latter can be recovered in translation as 'Some non-C is not A'.

It would be highly desirable to obviate the necessity for obversions and conversions before calculation comes into play. Fortunately, such a method is at hand.

2 *The Multiplicative Method* One problem with the additive method was the sparsity of mathematical equipment in use. By switching to multiplication and its inverse, but still retaining signs, we have more matériel for symbolizing the various traits of syllogistic sentences. Interestingly enough, it was Sommers [5] who first produced such a method; yet it was again necessary to make what I believe are needed improvements as well as to supply the rationale for the maneuvers.

As before, in order to highlight distributed terms and have the middle terms cancel out, I decided to let a distributed term appear as an inverse or as a denominator, while undistributed terms appear as numerators. Thus, 'All S is P' is represented as $\left(\frac{1}{S}\right)P$ or more simply as $\frac{P}{S}$. Now it would seem natural to let a term prefixed by a minus sign represent a negated term; but unfortunately that leads to notational ambiguity between 'All non-P is S' and 'Some S is not P', for both would be symbolized as $-\frac{S}{P}$. A student of mine, Mike Eggleston, suggested using the mathematically negative inverse of a term letter to represent the term's logical negation, and that removes the ambiguity. This would indicate that 'All S is non-P' be symbolized as $\left(\frac{1}{S}\right)\frac{-1}{P}$, which could then represent its obverted form: 'No S is P'. Similarly, 'Some S is not P' becomes $S\left(\frac{-1}{P}\right)$. Yet trouble looms unless we further distinguish particular propositions (I and O) from universals (A and E). For without such a distinction, the premisses 'Some M is not P' and 'Some S is not M' would, when symbolized as $\left(M\left(\frac{-1}{P}\right)\right) \cdot \left(S\left(\frac{-1}{M}\right)\right)$ and multiplied, yield $\frac{S}{P}$, which represents the hardly derivable conclusion 'All P is S'. By affixing the curative coefficient 2 before all particular statements, such illicit conclusions are blocked. Just as the appearance of any marker other than -1 signalled an invalid syllogism in the additive method, so any coefficient other than 2 warns us of a multiplicative invalidity. This is seen in the reformulation of the above two premisses and their product: $\left(2M\left(\frac{-1}{P}\right)\right) \cdot \left(2S\left(\frac{-1}{M}\right)\right) = \frac{4S}{P}$. This use of the coefficient 2 bears further fruit, for now we have two types of denial which can be performed mathematically: one for negating just terms (the negative inverse mentioned before) and a similar, but separate way of negating whole propositions, namely, taking the negative inverse of one half the formula for the proposition. For instance, the contradictory of 'Some S is P' or $2SP$ is $\frac{-1}{SP}$, that is, 'No S is P'.

Here is the final multiplicative schedule expressed compactly:

A	All S is P	$\frac{P}{S}$
E	No S is P	$\frac{-1}{SP}$
I	Some S is P	$2SP$
O	Some S is not P	$\frac{-2S}{P}$.

As before, to bestow existential import involving a term T, supplement the premisses with a formula representing 'Some T is T', to wit, $2TT$ or $2T^2$.

The great advantage of the multiplicative over the additive system is that obversion and conversion (as well as all derivative operations) need not be performed in any but mathematical ways. For example, 'All S is P' is shown to be equivalent to 'No S is non-P', because the first is formulated as $\frac{P}{S}$ and the second as $\frac{-1}{S\left(\frac{-1}{P}\right)}$; the two formulas are, of course, equal. Thus a student does not even have to learn obversion, conversion, or contraposition, if he or she can manipulate fractions. Enthymemes and sorites are solvable by the multiplicative method in ways analogous to those prescribed for the additive method. The test for validity we have been using is that the product of the premisses must equal the conclusion.

As a last syllogistic example of this method, let us return to the non-standard enthymeme form 'No M is P' and 'Some M is not S'. If we multiply $\frac{-1}{MP}$ by $\frac{-2M}{S}$, we get $\frac{2}{PS}$, a form not appearing on the above schedule. What is to be done mathematically to regularize matters? Well, the 2 indicates that the conclusion (if salvageable) is particular, but every particular proposition is represented by a monomial with at least one term not in the denominator position. We can algebraically create a numerator by introducing two minus signs and taking the inverse of either or both terms to get $2\left(\frac{-1}{S}\right)\left(\frac{-1}{P}\right)$, $\frac{-2\left(\frac{-1}{S}\right)}{P}$, or $\frac{-2\left(\frac{-1}{P}\right)}{S}$. These translate into 'Some non-S is non-P', 'Some non-S is not P', and 'Some non-P is not S'. This result exhibits the greater perspicuity of the multiplicative method, in that all three conclusions are equally obtainable and also immediately seen to be equivalent.

A promising extension of this method would seem to be its application to the propositional calculus. Taking 'All S is P' as the guide for implication and the negative inverse for negation, one can derive the following schedule for common truth functional connectives:

Truth function	Standard Symbol	Multiplicative Symbol
not p	$\sim p$	$-1/p$
p and q	$p \cdot q$	pq
p or q	$p \vee q$	$-(pq)$
if p then q	$p \supset q$	q/p
any truth	t	1

The following results are rather nicely proved by multiplication:

Modus Ponens: $(((p \supset q) \cdot p) \supset q) \equiv t$ $\quad q/((q/p)p) = q/q = 1$

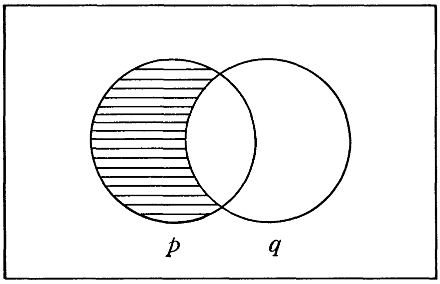
DeMorgan's Law: $\sim(\sim p \cdot \sim q) \equiv p \vee q$ $\quad -1/((-1/p)(-1/q)) = -(pq)$.

However, one could not prove by multiplication that $q \vee r$ also follows from $(p \supset q) \cdot p$; nor could a DeMorgan equivalence involving any odd number of letters be demonstrated. Even more embarrassing is that every biconditional $(p \supset q) \cdot (q \supset p)$ seems to come out as a tautology, since $(q/p)(p/q) = 1$.

These difficulties seemed irremediable, so I abandoned this method but happily discovered another system suggested by an algebraic analysis of Venn diagrams for propositions.

3 *The Boolean Form Method* This method was named for its use of Boole's Law ($x^2 = x$) and its use of the normal operations of an algebraic ring. Since a Boolean ring[2] need not have addition *and* subtraction, I prefer the distinguishing name 'form'.

Let us employ Venn diagrams to represent propositions and their interrelations. We can, as an hermeneutic aid, think of a circle as containing all states of affairs verifying the proposition it represents, while the rectangle contains all verifiers whatsoever. Now if we know that not-p is true, in other words that p is false, the p-circle will be devoid of all verifiers; hence, according to logical convention, it is shaded to show this emptiness. The natural algebraic description of this situation seems to be: the universe (represented as 1) less p-verifiers; the appropriate formula is $1-p$. Similarly, when we know that $p \supset q$ is true, the verifiers of p will all be contained within the circle of q-verifiers. That is to say, all p-verifiers are q-verifiers. The usual Venn diagram rendering for this is given below.



The shaded area is the p-circle except for that part which overlaps the q-circle; algebraically, this is p-pq. However, this area is what is absent from the universe; thus we write $1 - (p - pq)$ or $1 - p + pq$. Note that a Venn diagram could be constructed from a formula like this, which shows that the formula could be interpreted as a plan or instruction for making the pictorial representation of the proposition. All areas remaining unshaded but mentioned in the formula *could* have a verifier within, and their union *does* have at least one verifier according to the import of the proposition represented. The formula for the disjunction, $p \vee q$, which will be derived shortly, is $p - pq + q$. This means that if you remove the pq-area from the p-circle and then unite what remains with the entire q-circle (incidentally, thus restoring the pq-area), you arrive at that union of areas which (according to the proposition $p \vee q$) has at least one verifier. Had you grouped the terms like this: $(p - pq) + q$, it would perhaps more clearly have indicated a union of the left sector of the p-circle with the entire q-circle.

All other truth functions can be derived from those given for negation and implication. Thus, since $p \vee q$ is equivalent to $\sim p \supset q$, we can substitute $1 - p$ for p in our implication formula to get $1 - (1 - p) + (1 - p)q$, which equals the aforementioned formula for disjunction. The full schedule is given below.

Truth Function	Algebraic Representation
$\sim p$	$1 - p$
$p \supset q$	$1 - p + pq$
$p \vee q$	$p - pq + q$
$p \cdot q$	pq
$p \equiv q$[3]	$1 - p - q + 2pq$
any tautology	1
any contradiction	0

Notice that one could calculate the truth value (in terms of 1 and 0) of a function when given the truth values of the component atomic propositions (again, in terms of 1 and 0). For example, when p is true and q false, $p \equiv q$ is shown to be false because $1 - 1 - 1 + 2(1)(1) = 0$.

Now the algebraic expression for the biconditional calls for some comment. As an instruction for drawing a Venn diagram it would have been more usefully written as $1 - p + pq - q + pq$. Also, since in a Boolean ring there is no distinction between $+p$ and $-p$, the monomial $2pq$ would be discarded; because just as $p - p = 0$, so $p + p = 0 = 2p$. The algebraic representation would then be $1 - p - q$. When assigning the value truth to both p and q in $p \equiv q$, this last algebraic expression takes on the value -1, which, we then have to remind ourselves, in a Boolean ring is equal to +1. Thus, in teaching a student unfamiliar with these properties of Boolean rings or in trying to program these formulas, considerable difficulties could arise, should there be any departure from what I call the Boolean Form Method. Even greater problems could result from dispensing with both + and - in favor of a sign for symmetric difference, say, $\oplus$.

We recall that our effort to provide a purely multiplicative scheme for the propositional calculus failed in the case where the conclusion formula contained a letter not in the premiss formulas. By showing how such a problem is handled in the present (BFM) scheme we would be doing a double service: getting a feel for how validity is tested and assuring ourselves that this problem is not carried over to plague us again. The approach is to demonstrate that $q \vee r$ can in fact be deduced from $p \supset q$ and p by computing the algebraic formula for $[(p \supset q) \cdot p] \supset (q \vee r)$ as equal to 1. This could be done by substituting all (eight, in this case) combinations of 1's and 0's in the algebraic formula (a method suitable for computers) or by directly proving that the formula is identical to 1. The conjunction of the premisses, $(p \supset q) \cdot p$ is written initially as $(1 - p + pq)p$. When terms are multiplied, we get $p - p^2 + p^2q$, which by Boole's law becomes $p - p + pq$ and finally just pq. The conclusion $q \vee r$ is written as $q - qr + r$. The implication of the conclusion by the premisses can now be written as $1 - pq + pq(q - qr + r)$, which becomes $1 - pq + pq^2 + pqr - pq^2r$. Upon application of Boole's law, the exponents disappear, and the expression is easily seen to be equal to 1. Thus we have a decision algorithm for the propositional calculus: 1 - product of all the premisses + premiss product $\cdot$ conclusion is identical to 1, if valid.

A shorter procedure immediately suggests itself. Let P be the product of the premisses and C the conclusion; instead of working with the equation $1 - P + PC = 1$, use the equation $P(1 - C) = 0$, which is obtained by subtracting 1 from each side and factoring. This new equation embodies the principle of indirect proof, for it can be interpreted as saying that the conjunction of the premisses with the denial of the conclusion is false.

Employing this shorter rule, we can readily see that any argument, whose validity can be proved by some established, classical axiomatic system of the propositional calculus, can also be validated by BFM. The following relative completeness proof employs the general strategy of Canty [1], which Copi had turned to account in behalf of his own rules [2].

Suppose some argument, $p_1, \ldots, p_n \therefore q$, can be proved valid in a deductively complete and consistent system, like R.S. (Rosser's System, upon which Copi relies). By n applications of the deduction theorem, we have first: $p_1, \ldots, p_{n-1} \vdash (p_n \supset q)$ and finally $\vdash p_1 \supset (p_2 \supset (\ldots (p_n \supset q) \ldots))$. Repeated use of R.S. Theorem 25 (exportation) and then the corollary to Metatheorem IV (substitution) yields $\vdash (p_1 \cdot p_2 \cdot \ldots \cdot p_n) \supset q$. Let us abbreviate this last result as $P \supset C$. By analyticity of R.S., $P \supset C$ is a tautology; hence, $P \cdot \sim C$ is a contradiction, i.e., $P \cdot \sim C$ has the same truth value as $R \cdot \sim R$. Now in BFM, $R \cdot \sim R$ becomes $R(1 - R)$, a product which is clearly equal to 0. Therefore, $P(1 - C) = 0$ as well. But in BFM, this last formula is how we show an argument, $P \therefore C$ to be valid; thus $p_1, \ldots, p_n \therefore q$ is also valid by the rules of BFM.

Furthermore, BFM is consistent; for we can never deduce from true premisses, $p_1, \ldots, p_n$, a *false* conclusion q. Given that each premiss is true, each of their values is 1; consequently, the product of them all is also

1. Thus is BFM tells us that $(p_1p_2 \ldots p_n)(1 - q) = 0$, then $1(1 - q) = 0$. Whence, $q = 1$, and q is true.

As an historical note, BFM is literally faithful to Leibniz's desideratum that when the validity of an argument is in dispute, the antagonists can say: *calculemus*, let us calculate. Once the propositions are mathematized, one proceeds by routine calculation. Leibniz would only have lamented Church's proof of the undecidability of the predicate calculus.

NOTES

1. Here is a valid syllogism where such a pair (consisting of a term and its negate) would seem inoffensive: no New Yorker is a Bostonian; some non-Bostonians are New Yorkers; therefore, some non-Bostonian is not a Bostonian.

2. I am grateful to Prof. William Margolis for supplying information on Boolean rings.

3. To show formulas p and q equivalent, it is not really necessary to show that $1 - p - q + 2pq = 1$; it suffices to show that the mathematical formulas for p and q are always numerically equal. For instance, to show that $r \cdot s$ is equivalent to $\sim(\sim r \vee \sim s)$, one need only show that $rs = 1 - ((1 - r) - (1 - r)(1 - s) + (1 - s))$. This procedure avails, for if any biconditional $p \equiv q$ is true, then $1 - p - q + 2pq = 1$; whence we derive $2pq = p + q$. In BFM, the (numerical) truth values for p and q can only be 0 or 1; on the other hand, this last equation is only satisfied by integral values when p and q are simultaneously both 0 or both 1. This means that the equation is logically satisfied only when p and q are equivalent in truth value.

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