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R-MINGLE AND BENEATH. EXTENSIONS OF THE ROUTLEY-MEYER SEMANTICS FOR R

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1 *Introduction** This note presupposes the notation, terminology, and results of [7]. There Routley and Meyer (in their section called "**R**-mingle and beyond. Extensions of the semantics") give a semantical postulate

p7 $0 < a \vee 0 < a^*$

to be added to their postulates for an r.m.s. ("relevant model structure") so as to get a *Mingle* r.m.s. They then prove (Theorem 5) that A is a theorem of **RM** (**R**-Mingle) iff A is valid in all Mingle r.m.s. The purpose of this note is to supply an alternative semantical postulate

Sem(1) $Rxya \Rightarrow x < a \vee y < a$.

This postulate has certain advantages over p7.[1] First it is more natural in that the characteristic axiom scheme for **RM**, $A \rightarrow (A \rightarrow A)$, is negation-free, whereas the specific mission of the $*$-operation in the Routley-Meyer semantics is to provide for the treatment of negation.[2] The second advantage is that Sem(1) generalizes in certain natural ways, as shall be shown, so as to provide semantical characterizations of certain natural subsystems of **R**.

2 *Semantics for* **RM** Recall that upon defining $A \circ B = \sim(A \rightarrow \sim B)$, we get a "consistency" connective that "imports" and "exports" (*cf.* [7]). This allows us to take the characteristic **RM** axiom in the form

Syn(1) $A \circ A \rightarrow A$.

Soundness Theorem *If* $\vdash_{\mathbf{RM}} A$, *then A is valid in all* r.m.s. *satisfying* Sem(1) (*for short, all* "rm.m.s.").

Proof: In view of Theorem 2 in [7], it suffices to verify that Syn(1) is valid

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in all rm.m.s. Consider an arbitrary such, $\langle 0,K,R,^*\rangle$, and let v be a valuation therein. Because of Theorem 1 of [7], it suffices to show that if $A \circ A$ is true on v at a point a, then so is A. But if the first, then $\exists x,y$ such that $Rxya$ and A is true on v at both x and y. But by Sem(1), either $x < a$ or $y < a$. In either case by Lemma 1 of [7] A is true on v at a.

Completeness Theorem *If A is valid in all* rm.m.s., *then* $\vdash_{\mathbf{RM}} A$.

Proof: We adapt the strategy in [7] for Theorem 5, the corresponding theorem relative to p7. It should be pointed out, however, that Lemma 15 is not strong enough as actually stated in [7] to support its direct citation in the proof of Theorem 5. However, it is said in [7] that Lemma 15 was originally proved in [6], and conveniently enough it was there stated in a subtly stronger way that suffices. Thus in [6] Theorem 3 asserts (translating into the jargon of [7]) that for any regular **R**-theory T_0 and for any formula A, if $A \notin T_0$ then there exists a prime regular **R**-theory T such that (a) $T_0 \subseteq T$ and (b) $A \notin T$. Fixing T_0 to be the set of theorems of **RM**, we are guaranteed for each non-theorem A of **RM** the existence of a prime **R**-theory T containing all the theorems of **RM** but not A. There are no further snags in the strategy presented in [7], for T may then be plugged in Lemmas 13 and 14 there so as to obtain the T-canonical r.m.s. $\langle 0_T{}', \mathcal{K}_T{}', R_T{}', {}^{*\prime}\rangle$ which fails to verify the given non-theorem A of **RM** on its T-canonical valuation v_T. It only remains to check then that $R_T{}'$ satisfies Sem(1).

Before we begin, recall that $\mathcal{K}_T{}'$ is the set of all prime T-theories. Putting primeness aside, the members of $\mathcal{K}_T{}'$ are sets of formulas closed under adjunction and T-entailment. In particular then the members a of $\mathcal{K}_T{}'$ are closed under **RM**-entailment, i.e., $A \in a$ and $\vdash_{\mathbf{RM}} A \rightarrow B$ imply $B \in a$. Let us simplify notation, calling $R_T{}'$ just "R" throughout this argument. Now if R fails to satisfy Sem(1), there are prime T-theories x, y, a so $Rxya$ and $x \nless a$ and $y \nless a$. These last two boil down by definitions (*cf.* Lemma 11 of [7]) to $x \not\subseteq a$ and $y \not\subseteq a$. There are then formulas X, Y so that $X \in x$, $Y \in y$, and yet $X, Y \notin a$. In the canonical r.m.s. $Rxya$ is defined so that $X \in x$, $Y \in y$ implies $X \circ Y \in a$ (for any formulas X, Y whatsoever). So $X \circ Y \in a$. But the following key theorem of **RM** will then allow us to infer that $X \vee Y \in a$ (since T contains all **RM** theorems and a is closed under T-entailment, (i.e., $A \in a$, $A \rightarrow B \in T \Rightarrow B \in a$):

Key $X \circ Y \rightarrow X \vee Y$.

But $X \vee Y \in a$ yields, since a is prime, that $X \in a$ or $Y \in a$, contradicting our choice of X and Y.

It only remains then to satisfy ourselves that Key is a theorem of **RM**. We sketch a derivation that generalizes nicely later on, citing besides Syn(1) only well-known and easily verified theorems and derived rules of **R**.

1.	$X \rightarrow X \vee Y$	Disjunction elimination
	$Y \rightarrow X \vee Y$	
2.	$X \circ Y \rightarrow (X \vee Y) \circ (X \vee Y)$	1, Monotonicity of $\circ$

3. $(X \vee Y) \circ (X \vee Y) \rightarrow X \vee Y$ Syn(1)
4. $X \circ Y \rightarrow X \vee Y$ 2,3, Transitivity

3 *Generalizations of the semantics* Define $A^1 = A$, and for each positive integer n, $A^{n+1} = A^n \circ A$. For each positive integer n, consider

Syn(n) $A^{n+1} \rightarrow A^n$.

Let **RM**(n) be **R** with the additional axiom scheme Syn(n) (**RM** is then **RM**(1)).[3]

The corresponding semantical postulates Sem(n) are easier to understand by illustration than by general specification. So we plop down a relatively formal general specification, and then proceed quickly to illustrations. We first need to introduce relations of "relative copossibility" of various degrees, following the lines of [7]. Set R^n for each natural number n as an n - 2 placed relation as follows:

$$R^0 x_1 a \Leftrightarrow R0x_1a,$$
$$R^1 x_1 x_2 a \Leftrightarrow Rx_1x_2a,$$

and for $n \geqslant 1$:

$$R^{n+1}x_1 \ldots x_{n+1}x_{n+2} \Leftrightarrow \exists y(R^n x_1 \ldots x_{n+1}y \;\&\; Ryx_{n+2}a)$$

Now for each positive integer n, set

Sem(n) $R^n x_1 \ldots x_{n+1}a \Rightarrow \bigvee_{y_1, \ldots, y_n \in \{x_1, \ldots, x_{n+1}\}} R^{n-1}y_1 \ldots y_{n+1}a$

Note that Sem(1) as first set down falls out as a special case. As further illustrations, consider the following (superscripts on variables abbreviate repetitions in an obvious way, so "Rx^2a" is shorthand for $Rxxa$, etc.):

Sem(2) $$R^2xyza \Rightarrow \begin{cases} Rxya \vee Rxza \vee Ryza \vee \\ Rx^2a \vee Ry^2a \vee Rz^2a \end{cases}$$

Sem(3) $$R^3xyzwa \Rightarrow \begin{cases} R^2xyza \vee R^2xywa \vee R^2xzwa \vee R^2yzwa \vee \\ R^2x^2ya \vee R^2x^2za \vee R^2x^2wa \vee \\ R^2xy^2a \vee R^2y^2za \vee R^2y^2wa \vee \\ R^2xz^2a \vee R^2yz^2a \vee R^2z^2wa \vee \\ R^2xw^2a \vee R^2yw^2a \vee R^2zw^2a \vee \\ R^2x^3a \vee Ry^3a \vee Rz^3a \vee Rw^3a \end{cases}$$

Define an rm(n).m.s. as an r.m.s. satisfying Sem(n).[4] We have, as generalizations of the Soundness and Completeness Theorems of the previous section for **RM** the

Soundness and Completeness Theorem for the Systems **RM**(n) *For each positive integer n,*

$$\vdash_{\mathbf{RM}(n)} A \Leftrightarrow A \textit{ is valid in all } \mathrm{rm}(n)\text{.m.s.}$$

Proof: We specialize to the case of n = 2, leaving it to the reader to detect that the pattern of moves can be lifted to the general case at an unjustified cost of notational complexity. Further, we make only those moves that generalize those made in the section previous for **RM**, leaving it to the reader to supply the same background as was given for **RM** as to why these moves suffice.

Soundness: If A^3 is true on v at a then there exist w, z such that $Rwza$, A^2 is true on v at w, and A at z. Chasing the point about A^2 down, we see that there exist x, y such that $Rxyw$ and A is true on v at both x and y. But $Rxyw$ and $Rwza$ assures R^2xyza, and then Sem(2) gives that at least two of x, y, and z bear R to a. Suppose, for sake of illustration, that $Rxya$. Then since A is true on v at both x and y, A^2 is true at a. Since A is true on v at all of x,y,z, the same argument would work in the two other cases.

Completeness: Suppose in the canonical r.m.s. that $Rxyza$ and yet

(1) not $Rxya$, because $X_1 \in x$, $Y_1 \in y$, $X_1 \circ Y_1 \notin a$;
(2) not $Rxza$, because $X_2 \in x$, $Z_2 \in z$, $X_2 \circ Z_2 \notin a$;
(3) not $Ryza$, because $Y_3 \in y$, $Z_3 \in z$, $Y_3 \circ Z_3 \notin a$;
(4) not Rx^2a, because $X_4' \in x$, $X_4'' \in x$, $X_4' \circ X_4'' \notin a$;
(5) not Ry^2a, because $Y_5' \in y$, $Y_5'' \in y$, $Y_5' \circ Y_5'' \notin a$;
(6) not Rz^2a, because $Z_6' \in z$, $Z_6'' \in z$, $Z_6' \circ Z_6'' \notin a$.

(We are utilizing obvious mnemonic conventions that allow one to handle the larger cases like **RM**(3) or even the general case without having to actually write out stuff like the above.)

Set

$$\begin{aligned} X &= X_1 \wedge X_2 \wedge X_4' \wedge X_4'', \\ Y &= Y_1 \wedge Y_3 \wedge Y_5' \wedge Y_5'', \\ Z &= Z_2 \wedge Z_3 \wedge Z_6' \wedge Z_6''. \end{aligned}$$

Since theories are closed under adjunction, $X \in x$, $Y \in y$, and $Z \in z$. But since $Rxyza$ and we are in the canonical R.M.S., $X \circ Y \circ Z \in a$.

Now for any formulas X, Y, Z whatsoever, the following may be shown to be a theorem of **RM**(2) (we begin to indicate $\circ$ by juxtaposition):

Key (2) $XYZ \to XY \vee XZ \vee YZ \vee X^2 \vee Y^2 \vee Z^2$.

Derivation sketch:

1.	$X \to X \vee Y \vee Z$	
	$Y \to X \vee Y \vee Z$	Disjunction introduction
	$Z \to X \vee Y \vee Z$	
2.	$X \circ Y \circ Z \to (X \vee Y \vee Z)^3$	1, Monotony of $\circ$
3.	$(X \vee Y \vee Z)^3 \to (X \vee Y \vee Z)^2$	Syn(2)
4.	$(X \vee Y \vee Z)^2 \to XY \vee XZ \vee YZ \vee X^2 \vee Y^2 \vee Z^2$	Distribution of $\circ$ over $\vee$
5.	$XYZ \to XY \vee XZ \vee YZ \vee X^2 \vee Y^2 \vee Z^2$	2,3,4 Transitivity

But a is closed under **RM**(2) entailment. So $XY \vee XZ \vee YZ \vee Z^2 \vee Y^2 \vee Z^2 \in a$, and, since a is prime, one of the disjuncts is in a. We can see this is impossible, choosing without loss of generality XY for illustration. First, we cite the easy fact that $\circ$ distributes over $\wedge$ in **R** in the direction we need, i.e., $A \circ (B \wedge C) \rightarrow AB \wedge AC$ is a theorem of **R** (and hence **RM**(2)). Repeated such distributions allow us to obtain

$$XY \rightarrow X_1Y_1 \wedge X_1Y_3 \wedge . . . X_4''Y_5''$$

as a theorem of **RM**(2). Hence by conjunction elimination $XY \rightarrow X_1Y_1$ is a theorem of **RM**(2). But since a is closed under **RM**(2) entailment and our illustrative case assumption is that $XY \in a$, we obtain $X_1Y_1 \in a$. This contradicts our assumption (1) above that $Rxya$ failed because (among other things) $X_1Y_1 \notin a$.

3 *Structure of the family of systems* **RM**(n) It is natural to ask how the various systems **RM**(n) are related to one another and to R. We begin with two easy observations. First, given positive integers m, n with $m \leqslant n$, **RM**(n) is a subsystem of **RM**(m). Thus if we have Syn(n) as axiom scheme we can derive Syn(n + 1) thusly:

1.	$A^{n+1} \rightarrow A^n$	Syn(n)
2.	$A \rightarrow A$	Self-implication
3.	$A^{n+1} \circ A \rightarrow A^n \circ A$	1,2, Monotonicity of $\circ$
4.	$A^{n+2} \rightarrow A^{n+1}$	3, Abbreviation.

Secondly, all of the systems **RM**(n) are distinct from **R**, as was shown in effect by Meyer [5] using a certain infinite matrix. By producing various finite versions of that matrix one can show that all of the various systems **RM**(n) are distinct from one another.[5]

Thus define for each positive integer n the matrix $\mathfrak{M}_n$ as follows: The elements of $\mathfrak{M}_n$ are the positive integers 1 through n, their negatives -1 through $-n$, 0, and ω. The only undesignated element is 0. The operations are defined exactly as on the infinite matrix except that multiplication and all of its cognate notions, e.g., division, used by Meyer are to be understood as "truncated" at n. More explicitly, let 'a', 'b', 'c' range over positive integers $\leqslant n$. Define:

(i) $a \times_n b = \min(a \times b, n)$;
(ii) a divides$_n$ b iff $\exists c(a \times_n c = b)$;
(iii) If a divides$_n$ b, $b /_n a$ = the greatest c such that $a \times_n c = b$.
(iv) the greatest common divisor$_n$ (a,b) = the greatest c such that c divides$_n$ both a and b.
(v) the least common multiple$_n$ (a,b) = the least c such that both a, b divide$_n$ c.
(vi) $a \times_n -b = -(a \times_n b)$

If the reader will take the trouble to rewrite clauses (1)-(7) of [5] by way of these truncations, he will have the definitions of the operations on $\mathfrak{M}_n$. In particular, tracing down definitions, when a, b are positive, $a \circ b = -(a \rightarrow -b)$ = (by (7iii) of [5]) $-(a \times_n -b)$ = (by (vi) above) $- -(a \times_n b) = a \times_n b$.

We also leave to the reader the laborious verification that each matrix $\mathfrak{M}_n$ satisfies all the axioms and rules of **R**.

Turning to the matter at issue, the distinctness of the systems, it will obviously suffice to show for each Syn(n) that it fails to be (schematically) valid in $\mathfrak{M}_{2^{n+1}}$ although Syn(n + 1) is valid in $\mathfrak{M}_{2^{n+1}}$. One can falsify $A^{n+1} \to A^n$ by assigning A the value 2. This assignment does the job, since the value of A^{n+1}, 2^{n+1}, fails to divide$_n$ the value of A^n, 2^n ("implication" is division$_n$). Not arguing the matter fully but getting to the nub, this same assignment is easily seen to be the best choice for falsifying Syn(n + 1), and yet it fails to do. Indeed because of truncation at 2^{n+1}, A^{n+2} and A^{n+1} both take on the value 2^{n+1}.

4 *Algebraic Modals for the Systems* **RM**(n) The appropriate algebraic models for the system **R** are DeMorgan monoids. These, briefly put, are residuated DeMorgan-lattice-ordered commutative monoids which are square increasing, i.e., where $\circ$ ("consistency") is the monoid operation, $a \leqslant a \circ a$ (= a^2, defining exponent notation in the usual way). As the square increasing postulate suggests, one can prove easily $\vdash_{\mathbf{R}} A \to A \circ A$. This means that $\vdash_{\mathbf{RM}} A \leftrightarrow A \circ A$, and in general $\vdash_{\mathbf{RM}(n)} A^n \leftrightarrow A^{n+1}$.

One can then prove that $\vdash_{\mathbf{RM}(n)} A$ iff A is valid in the class of "n-potent" DeMorgan monoids, i.e., those satisfying $a^n = a^{n+1}$. This is a simple mechanical matter of modifying the proof of the corresponding theorem for **R** and DeMorgan monoids (*cf.* [6]), since the Lindenbaum algebra of **RM**(n) is obviously n-potent by virtue of the equivalence of A^n and A^{n+1}. Another routine matter is the rewriting of the representation results of Routley and Meyer [7] for DeMorgan monoids in terms of r.m.s. so as to obtain corresponding representation results for n-potent DeMorgan monoids in terms of rm(n).m.s. The only thing needing verification is that the algebra of propositions determined by a rm(n).m.s. is n-potent, and this falls quickly out of Sem(n).

It seems proper to close this section by picking up the glove thrown down by Routley and Meyer [7, p. 223]. There they remark that imposing various postulates of finitude on the notion of a Mingle r.m.s. gives semantics for various proper extensions of **RM**. They then say they "leave to Dunn the question of whether we get them all that way." The answer, based on known results, is rather straightforwardly yes. Let us quickly sketch the proof, since it is fair to suppose that Routley and Meyer had in mind an answer based directly on their semantical methods, rather than the "old wine in new bottles" one we are about to give, which is based ultimately on algebraic methods.

Thus it is the result of [1] that each proper extension of **RM** has as a characteristic model some finite Sugihara algebra, and Sugihara algebras are easily seen to be prime DeMorgan monoids. This last outfits them for plugging into the construction of Collorary 9.1 of [7]. That construction yields an embedding of a prime DeMorgan monoid into an algebra of propositions determined by a certain corresponding r.m.s. whose points are the prime filters of the given DeMorgan monoid. One can straightforwardly argue that the corresponding r.m.s. is a Mingle r.m.s. (or an rm.m.s. for that matter). Also it is easy to see that a formula is valid in a given r.m.s. iff it is valid in the algebra of propositions determined by that r.m.s. And, of course, if the given prime DeMorgan monoid is finite, so is its set of prime filters and so its corresponding r.m.s. So it only remains to show that the embedding given by the construction is onto. The embedding maps a given element onto the set of prime filters having it as member. All Sugihara algebras are linear. This, together with the finiteness of the particular Sugihara algebras under consideration, gives us the coincidence of prime filters and principal filters. A given element a is then mapped to the set of principal filters determined by elements $x \leqslant a$. The question is then whether all propositions in the corresponding r.m.s. are of this form. It is easy to check that for prime filters P, Q, $P < Q$ in the corresponding r.m.s. iff $P \subseteq Q$. A proposition in the corresponding r.m.s. turns out then to be a set of prime filters closed upward under $\subseteq$. Because of the linearity and finiteness of the given Sugihara algebras, it is easy to see that any such proposition will contain a smallest prime filter P, and that the element a determining P as principal filter will be mapped onto the given proposition.

4 *Conjectures and exhortations* It is not unnatural to conjecture (or at least hope) that (1) **R** is the intersection of the family of systems **RM**(n), and (2) each **RM**(n) has the finite model property. The system **R** would itself then obviously have the finite model property, and hence be decidable by a well-known result of Harrop (*cf.* [3]).[6]

Besides such specific suggestions concerning study of the systems **RM**(n), it seems worthwhile to recommend in general study of the systems that extend **R**. The study of systems in a similar relation to the intuitionistic propositional calculus, often called "intermediate" or "superconstructive" logics, has been very fruitful (*cf.* [3]). The label "superrelevant" logics has some problems in that classical propositional calculus, with all its fallacies of relevance, is thereby "superrelevant." But the label "superconstructive" has survived similar problems. "Intermediate" is not specific enough as to between what, but one can always talk of "relevant intermediate logics" as opposed to "constructive intermediate logics" (at the price of once more having classical logic become both "relevant" and "constructive"). Whatever one calls the area, Meyer's pioneer work on **RM** in [4] is certainly seminal, and [1] and [2] suggest that **RM** is the **LC** of the *relevant* intermediate logics.

NOTES

1. This claim is by no means intended to negate other reasons for liking p7 given by Routley and Meyer [7].

2. To reinforce this point, the reader should compare the rather "indirect" verification of the characteristic **RM** axiom given in [7] (p. 221) using p7 with the routine verification using Sem(1) below.

3. Alternatively, one could take the characteristic axiom scheme of **RM**(n) as expressing a kind of "expansion." Setting $A \to^1 B = A \to B$ and $A \to^{n+1} B = A \to (A \to^n B)$, then $(A \to^n B) \to (A \to^{n+1} B)$ is deductively equivalent to Syn(n) (in the presence of the rules and axioms of R), as may easily be seen.

4. At the price of some "negative" strain on notation, we could have carried along the case $n = 0$. Thus defining $R^{-1}a \Leftrightarrow R00a$, Sem(0) becomes $R^0xa \Rightarrow R^{-1}a$. Since R^0aa is just p1 of [7], we would then have $R00a$ for any rm(0).m.s. This is just p7′ of [7, p. 223], and is shown there to give classical logic. Perhaps stretching a point and letting Syn(0) be $A \to \mathbf{t}$ (putting the constant conjunction of all truths, *cf.* [6], in place of a blank space), we also get classical logic, as is easily checked. We leave it to the interested reader to check that the argument for the Soundness and Completeness Theorem given immediately below could have been carried out relating Sem(0) to Syn(0) as well.

5. Using "trivial" in its accustomed mathematical sense, the following construction most likely is "trivially" implicit in Meyer's "Improved Decision Procedures for Pure Relevant Logics," draft portions of which were privately circulated January 1973.

6. The "base case" for (2), $n = 1$, was established by Meyer in [4] (*cf.* also [1]). Also it is worth pointing out that using the results of the last section it is easy to see that if **R** does have the finite model property, then (1) is true, basically because a finite DeMorgan monoid having n elements will trivially be n-potent. (There is a slight lacuna here, relating finite models of **R** in general to equivalent finite DeMorgan monoids. This is easily filled by "identifying" elements a, b in the given model when both $a \to b$ and $b \to a$ are designated, thereby obtaining a DeMorgan monoid.)

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