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Nominalization and Scott's Domains II

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1 Introduction In Turner [17] we developed a semantics for nominalized predicates within the general framework of Montague Grammar. We offered an extension of Montague's *PTQ* which sanctioned the occurrence of nominalized verb-phrases and sentences. The actual semantics was furnished by the semantic domains of Scott's theory of computation. One issue of some importance was given scant attention in that presentation, namely, the role of a comprehension schema in the theory. In this paper we investigate this issue in some detail. Our more general objective is to examine the logical foundations of the enterprise in more depth than was possible in the earlier paper. In particular, we here provide a more general model-theoretic setting for the analysis of nominalization.

First, however, we must say a few words about the process of nominalization itself. In this paper we shall be exclusively concerned with nominalized predicates, and by the term 'nominalization' we shall mean any process which transforms a predicate or predicate phrase into a noun or noun phrase. For example, 'feminine' is transformed into 'femininity', 'divine' into 'divinity', and 'obscene' into 'obscenity'. Following Cocchiarella [8] I shall call these derivative nouns 'abstract singular terms'. Of course, the phenomenon of nominalization is not restricted to such instances of morphological nominalization. Consider the following pairs of sentences:

1. (a) The book is brown
   (b) *Brown* is a colour

2. (a) The cup is gold
   (b) *Gold* is an element

3. (a) Tammy and Toby are students
   (b) *Students* are numerous

4. (a) John is honest
   (b) *Honesty* is a hindrance

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5. (a) Susan runs
   (b) *Running* is a waste of energy

6. (a) Peter jogs
   (b) *To jog* is silly

7. (a) Peter left with a man with one eye
   (b) *To leave with a man with one eye* is short-sighted

8. (a) Jill is not nice
   (b) *To be not nice* is not nice.

In sentence 1(a) the phrase *is brown* functions semantically as a predicate: it attributes a certain property to the book. In 1(b), however, the word *Brown* seems to occupy a subject position in the sentence and some property is attributed to it. Similarly, in 2(a) the phrase *is gold* functions semantically as a predicate whereas in 2(b) the mass noun *Gold* is said to have a certain property. Sentence 3(a) states that Tammy and Toby have the property of being students; in 3(b) the denotation of the bare-plural *students* is ascribed a certain property. In 5(a) the intransitive verb *runs* is used to ascribe a certain property to Susan; in 5(b) the gerundive *running* becomes the subject and some property is said to hold of its denotation. Sentence pair 6 illustrates a similar phenomenon with the infinitive form of the verb. Sentence 8(b) is, presumably, an instance of self-application. In each of the sentence pairs 1 through 8, a term which is semantically a *predicate* is transformed into something which is semantically an *object*—and some property is ascribed to it. There are important linguistic differences between these various instances of nominalization but I will not be concerned with such here. Our present concern relates only to the existence of such abstract singular terms and the issues they raise for any semantic theory.

To gain some insight to the problems raised by such singular terms we adopt the following assumptions. Let E be the domain of objects and P the domain of one-place predicates which is to be interpreted as some class of functions from E to B (the domain of truth-values).

(1) $P = [E \to B]$.

The idea is that phrases which function semantically as objects should receive their denotations from E and those which function semantically as predicates should receive theirs from P. This leaves us to deal with the denotations of abstract singular terms; from where are nominalized forms, such as those occurring in 1(b)–8(b), to receive their denotations? For the sake of argument, let us introduce a new domain, PC, of predicate correlates, which will serve this purpose: each nominalized form is to get its denotation from PC. Now, the process of nominalization, exemplified by 1–8, seems to induce a function from predicate phrases, which receive their denotations in P, to phrases which get assigned their denotations in PC. A little reflection on our examples should convince the reader that the following assumptions are reasonable:

(I) Two predicate phrases with the same denotation give rise to nominalized forms with the same denotation

(II) Two predicate phrases with distinct denotations give rise to nominalized forms with distinct denotations.

The first assumption guarantees the existence of a function F: $P \rightarrow PC$ and the second ensures that F is injective or one-to-one. We are not assuming that nominalized forms, as singular terms, necessarily refer to the same properties or forms designated by these predicates in their primary linguistic role (i.e., in predicate positions); we are only assuming that there is a one-to-one correspondence between the two. In other words:

(2) $P \simeq PC$.

This brings us to our final point. Predicate phrases take their denotations from $P = [E \rightarrow B]$, so that any argument to any element of P must be an element of E. However, the (b) cases of 1–8 are examples where elements of PC serve as arguments to elements of P. In other words, PC must be a subset of E.

(3) $PC \subseteq E$.

The upshot of all this is clear: to provide a model-theoretic semantics for nominalized predicates we need to find domains E, PC, and P which satisfy (1), (2), and (3). This is a hard task given that a constraining objective is to provide nontrivial models.

We cannot, for example, simply take P to be all the functions from E to B without falling into inconsistency. The class of functions P must not be 'too big'. But it must be 'quite big'. In this regard, consider sentence pair 7. This is an example where a rather complex predicate phrase gets nominalized: in 7(b) the whole quantifier phrase is subject to nominalization. So that, whatever other attributes the domain P is to possess, it must facilitate the nominalization of complex predicates such as those exemplified in sentence pair 7. The domain P must be 'big' enough to include the denotations of such complex predicates.

To state this constraint more precisely we need to come clean about the formal language of our theory. We shall in fact employ a second-order language due to Cocchiarella [7] which, in keeping with our enterprise, allows relation symbols to occur in both subject and predicate positions. The basic symbols of the language (L) include a denumerable number of individual variables ($x_0, x_1, x_2, x_3, \ldots$) and individual constants ($c_0, c_1, c_2, c_3, \ldots$) together with a denumerable number of relation variables ($X_0^k, X_1^k, X_2^k, X_3^k, \ldots$), and a denumerable number of relation constants ($R_0^k, R_1^k, R_2^k, R_3^k, \ldots$), for each $k \geq 0$ (where there is no danger of confusion we shall drop the subscripts and superscripts). We shall use $M_0, \ldots, M_n, \ldots$ to range over both individual and relation variables. The basic symbols of L also include the logical constants $\sim$, $\vee$, &, and the quantifiers $\forall$, $\exists$. The atomic well-formed formulas of L are of the form

$$K(t_0, \ldots, t_{n-1})$$

where K is an n-place relation constant or relation variable and the t_i are individual variables/constants or relation variables/constants. More complex formulas can be obtained by conjunction and negation and two types of quantification corresponding to our two types of variables: if A is already a well-formed formula so are $\forall xA$ and $\forall X^n A$. Other well-formed formulas are obtained by definition in the normal way.

We are now in a position to state the constraint concerning the nominalization of the complex predicates more precisely. We want to be able to nominalize all the wffs of L. This leads us to postulate the following schema of comprehension: for each wff, A, of L, with free variables $M_0, \ldots M_{n-1}$,

$$(\exists X)(\forall M_0, \ldots, M_{n-1})(A \leftrightarrow X(M_0, \ldots, M_{n-1})) \ .$$

Unfortunately, where classical logic is employed, such a strong schema leads to Russell's paradox. To see this one has only to observe that the wff $\sim X(X)$ has to denote an element of P and, consequently, via nominalization, its denotation can be applied to itself. Obviously, something must give: either we weaken the principle of comprehension by restricting its domain of authority to a suitable class of wff, or we embrace some form of nonstandard logic. Cocchiarella chooses the first option and restricts the domain to those wff which he terms 'homogeneously stratified'. In keeping with the approach developed in Turner [17] our intention is to explore the latter alternative.

2 A semantic theory of nominalized predicates Our objective is to develop a semantic theory of nominalized predicates in which partial predicates play a central role. It is a common move in logic to blame the derivability of the paradoxes on reasoning with meaningless statements of one form or another. Bochvar's logic, for example, was developed to cope with the semantical paradoxes in just this way. In the case of the logical paradoxes several authors (e.g., Brady [4], Gilmore [11]) have developed set theories in which the membership relation is partial. Much in the same spirit we shall employ partial predicates in our analysis of nominalization.

By a partial k-ary predicate on a set E is meant a partial function from E^k to $\{0, 1\}$. Employing the symbol 'u' for undefined ($1 \neq u$, $0 \neq u$) each such predicate can be identified with a function from E^k to $\{1, 0, u\}$, and indeed it is this representation we shall employ in the sequel. The central semantic notion of our theory is the following.

Definition 2.1 *A partial frame M* consists of a triple $\langle E, R, F\rangle$ where E is a nonempty set, $R = \bigcup R_n$ (where for each $n \geq 0$, R_n is a nonempty set of n-ary partial predicates on E), and F is an injective function from R to E.

The function F reflects the first of our general constraints on a semantic theory of nominalized predicates: the function F associates with each predicate a corresponding object or predicate correlate; the range of the function F is exactly the set, PC, of predicate correlates.

The semantic definition of L, with respect to such a frame, is much the same as that of Cocchiarella; the difference is to be located in the form of the underlying logic supported. Here we shall utilise Kleene's strong three-valued connectives. We employ the notation

$$[A]_g^M$$

to represent the value of the wff A with respect to the assignment function g and partial frame M (where assignment functions send individual variables to elements of E and n-place relation variables to elements of R_n, $(n \geq 0)$). We also assume that each individual constant c denotes an element c' of E and each n-place relation constant R denotes an element R' of R_n. The actual definition of $[A]_g^M$ is then given by recursion as follows[1]:

(1) $[K(t_0, \ldots, t_{n-1})]_g^M = Val(K, g)(Val'(t_0, g), \ldots, Val'(t_{n-1}, g))$

(2) $[A \;\&\; B]_g^M \quad = [A]_g^M \wedge [B]_g^M$

(3) $[\sim A]_g^M \quad = \neg [A]_g^M$

(4) $[\forall x A]_g^M \quad = \bigwedge_{e \in E} [A]_{g(e/x)}^M$

(5) $[\forall X^k A]_g^M \quad = \bigwedge_{r \in R_k} [A]_{g(r/X^k)}^M$

where

(i) $Val(K, g) = K'$, if K is a constant symbol and $g(K)$ if K is a variable and $Val'(K, g) = F(Val(K, g))$

(ii) $\wedge$, $\neg$, $\bigwedge$ are Kleene's strong three-valued connectives given below

(iii) $g(e/x)$ is that assignment function identical to g except perhaps on the variable x, where it returns the value e.

Kleene's strong three-valued connectives

$\wedge$	1	0	u
1	1	0	u
0	0	0	0
u	u	0	u

$\neg$	
1	0
0	1
u	u

$$\bigwedge_{i \in I} p_i = \begin{cases} 1 \text{ if each } p_i = 1 \\ 0 \text{ if some } p_i = 0 \\ u \text{ otherwise} \end{cases}$$

The meanings of the other connectives can be derived via the standard definitions but, for future reference, we provide the truth-tables for Klenne's conditional and biconditional.

$\rightarrow$	1	0	u
1	1	0	u
0	1	1	1
u	1	u	u

$\leftrightarrow$	1	0	u
1	1	0	u
0	0	1	u
u	u	u	u

Kleene's logic was originally conceived to accommodate undecidable mathematical statements. The third truth-value, intuitively, represents 'undecidable' (u) and, as such, its assignment to a wff is not intended to indicate that the wff is neither true nor false. Rather, its purpose is to signal a state of partial ignorance. Indeed, enshrined in Kleene's logic is the principle that where one can determine the truth-value (true or false) of a compound wff from its components, the wff should be assigned that truth-value, regardless of whether or not certain of its components are undecidable. So, for example, $A \;\&\; B$ will be assigned the value 1 if both A and B are assigned the value 1 and it will be assigned the value 0 if one of A or B is assigned the value 0, and this will be so even if u is assigned to the other.

Klenne's logic seems, therefore, to be quite appropriate to the task in hand, especially in view of our intended models of nominalization based upon Scott domains. It is worth pointing out, however, that we could employ the connectives of Bochvar without any major technical changes.

This brings us to that part of our program which bears upon the comprehension principle itself.[2] Our stated desire is to formulate a theory which authorizes the nominalization of all the wff of L. Clearly not all partial frames will facilitate this and those that do will occupy a central place in our theory. For this reason we single them out for special attention.

Definition 2.2 A partial frame M is *closed* iff for each wff A, with free-variables $M_0, \ldots, M_{n-1}$, the function, $\lambda e_0, \ldots, e_{n-1}. [A]^{M}_{g(e_i|M_i)}$ $(0 \leq i \leq n-1)$, is an element of R_n.

The intention here is that the e_i are to range over elements of E appropriate to the M_i; so, in particular, if M_i is an n-place relation variable then $e_i \in R_n$ (considered as a subset of E, where this identification is, of course, sanctioned by the injective nature of F).

Such frames permit the nominalization of all the wff of L but can we characterize them via an explicit principle of comprehension. The obvious move is to postulate the following principle:

> For each wff A, with free variables $M_0, \ldots, M_{n-1}$ $(\exists X)(\forall M_0, \ldots, M_{n-1})(X(M_0, \ldots, M_{n-1}) \leftrightarrow A)$ where 'X' does not occur among the free variables of A.

But such a principle cannot possibly be true since, under Klenne's interpretation of the biconditional, it is impossible for $A \leftrightarrow \sim A$ for any wff. We must be guided, in our search for an appropriate biconditional, by the definition of closure. Indeed, a little reflection on the definition suggests that we require a biconditional $\equiv$ which possesses the property: $b = c$ implies $b \equiv c$. For example, the biconditionals of Lukasiewicz and Reichenbach satisfy this constraint.

$=_L$	1	0	u
1	1	0	u
0	0	1	u
u	u	u	1

Łukasiewicz

$=_R$	1	0	u
1	1	0	0
0	0	1	0
u	0	0	1

Reichenbach

A natural step here is to reformulate the above principle of comprehension with $=_L$ or $=_R$ in the position of the main connective. Similar moves have been adopted, in a somewhat different context, by Brady [4] and Feferman [10]. To achieve this formulation, we must extend L by the addition of a new biconditional $\equiv$: if A and B are in $L_{\equiv}$ then so is $A \equiv B$.

CS* If A is any wff of LNP with free variables $M_0, \ldots, M_{n-1}$ (excluding X) then $(\exists X)(\forall M_0, \ldots, M_{n-1})(X(M_0, \ldots, M_{n-1}) \equiv A)$.[3]

The additional semantic clause for '$\equiv$' obviously depends upon the interpretation of $\equiv$.

(6_L) $[A \equiv B]_g^M = [A]_g^M =_L [B]_g^M$
(6_R) $[A \equiv B]_g^M = [A]_g^M =_R [B]_g^M$.

The following result is immediate.

Lemma 2.3 *For each of the interpretations, $=_L$ and $=_R$, of '$\equiv$' we have $[A \equiv B]_g^M = 1$ iff $[A]_g^M = [B]_g^M$.*

Theorem 2.4 *M is closed iff each instance of CS^* holds under either interpretation of $\equiv$.*

Proof: CS^* holds iff for each wff A, (of L)

$$[\exists X \forall M_0, \ldots, M_{n-1}(X(M_0, \ldots, M_{n-1}) \equiv A)]g = 1$$

for each assignment function g. This is equivalent to the claim that there is some $r \in R$ such that for each $e_0, \ldots, e_{n-1}$ of E (of the appropriate type)

$$[X(M_0, \ldots, M_{n-1}) \equiv A]g(r|X)(e_i|M_i) = 1 \ .$$

But since $[A \equiv B]g = 1$ iff $[A]g = [B]g$, this statement is equivalent to the closure of M.

The theory we shall study is thus characterized by closed partial frames: let P^* be the theory (set of $L_\equiv$ wffs) valid in all closed partial frames. We now explore some consequences of this theory.

In standard second-order logic the addition of equality to the language results in a conservative extension, since identity is provably equivalent to indiscernibility given by

$$(X \simeq Y) \leftrightarrow (\forall Z)(Z(X) \rightarrow Z(Y))$$

where Z is the first variable, in alphabetical ordering, not equal to X or Y. One principle of some importance is the following principle of indiscernibility (IND^*):

Neither

IND* $\quad (\forall X \forall Y)(X \simeq Y \rightarrow (\forall x)(X(x) \leftrightarrow Y(x))$

nor its converse

EXT* $\quad (\forall X)(\forall Y)((\forall x)((X(x) \leftrightarrow Y(x)) \rightarrow X \simeq Y)$

is valid in all closed frames.

This failure of EXT^* should not, of course, come as any surprise, for in a regime where partial predicates are employed the failure of extensionality is to be expected.

One relation put to much use in Turner [17] is that of predication. It is therefore of some interest to see whether or not predication is a relation whose existence is guaranteed by our schema of comprehension.

PRED* $\quad (\exists X)(\forall Y)(\forall x)(X(Y, x) \equiv Y(x))$.

It takes but a little insight to see that $PRED^*$ is indeed valid. We see this as a positive part to our theory for without it the treatment of nominalization, proffered in the above paper, would be a great deal more unnatural and cumbersome.

It is now incumbent upon us to compare our theory to that of Cocchiarella. Our theory differs from that of Cocchiarella both in terms of the schema of comprehension validated and the underlying logic supported. To spell out the difference in more detail we need first to say a few words about Cocchiarella's approach. The semantics of L is furnished by Cocchiarella's notion of a 'Fregean Frame'.

Definition 2.6 A *Fregean Frame* M consists of a triple $\langle E, R, F\rangle$ where E is a nonempty set, $R = \bigcup_{n\geq 0} R_n$ (where for each $n \geq 0$, R_n is a nonempty set of functions from E^n to $\{1, 0\}$) and F is an injective function from R to E.

This notion of a frame seems to reflect a Fregean tradition in the analysis of nominalized predicates. For Frege, nominalized predicates function as singular terms, but as singular terms, they do not denote the same thing as they do in their primary linguistic role. Instead, nominalized predicates denote objects which are correlated with the actual denotations of the predicates: the function F associates with each predicate denotation its corresponding correlate. The actual semantics for L is provided much as with partial frames but with the crucial difference that the classical connectives supercede those of Kleene's three-valued logic. Extensionality (EXT^*) is a direct consequence of working with total functions.

To sanction the nominalization of complex-relations (i.e., those induced by the wff of L) Cocchiarella considers various principles of comprehension which are extensions of the minimal schema of second-order logic (CP).

For each wff A, of standard second-order logic,

CP $(\exists X)(\forall x_0, \ldots, x_{n-1})(X(x_0, \ldots, x_{n-1}) \leftrightarrow A)$ where $x_0, \ldots, x_{n-1}$ occur free in A but X does not.

Cocchiarella extends CP to L by permitting A to be any wff of L. This generates Cocchiarella's principle (CP^*) and results in the (consistent) theory T^{**}. This theory, however, has some rather curious features, among which is the *refutability* of the principle of indiscernibility. Cocchiarella seems to believe that the failure of IND^*, in T^{**}, is more damaging for the Platonic view of nominalized predicates than for the Fregean but I find his arguments here less than persuasive.

In any case, Cocchiarella seems to favour a theory based upon a stratified comprehension principle, a principle which actually guarantees the truth of IND^*. The system based upon this stratified principle is, in reality, a second-order analogue of Russell's simple theory of types. The principle utilizes the notion of a (homogeneously) stratified wff:

Definition 2.7 A wff A of LNP is *homogeneously stratified* if there is an assignment f of natural numbers, to the variables of A, such that for each atomic wff $K(t_0, \ldots, t_{n-1})$ of A,

(1) $f(t_i) = f(t_j)$ for $0 \leq i, j \leq n-1$
(2) $f(K) = 1 + f(t_0)$.

We are now well-placed to state Cocchiarella's principle *HSCP**.

HSCP* $\quad (\exists X)(\forall M_0, \ldots, M_{n-1})(X(M_0, \ldots, M_{n-1}) \leftrightarrow A)$

where

(i) the free variables of A include $M_0, \ldots, M_{n-1}$
(ii) 'X' does not occur free in A
(iii) the entire biconditional is homogeneously stratified.

This principle of comprehension thus permits the nominalization of those complex relations induced by homogenously stratified well-formed formulas. Moreover, the principle of indiscernibility, *IND**, is a consequence of *HSCP**.

Unfortunately, *PRED** is not. This latter fact is particularly disturbing given the potential application of such a theory to an analysis of English nominalization. Chierchia [6] has to adopt some rather unorthodox (not to say suspicious) maneuvers to circumvent this problem.

We summarize the main differences between our theory and that of Cocchiarella in the following table:

	BIVALENCE	*PRED**	*IND**	*EXT**	*COMPREHENSION SCHEMA*
P^*	no	yes	no	no	Unrestricted
HST^*	yes	no	yes	yes	Homogeneously Stratified

3 Scott frames It is certainly not obvious that the theory P^* is consistent. Do closed frames exist? In this section we provide an existence proof for closed frames. We shall employ a technique due to Scott for constructing models of the untyped lambda calculus. The idea of using Scott's construction in the analysis of nominalization first occurs in Turner [17] where some motivation is proffered for the employment of the construction in the analysis of nominalized predicates. We shall not repeat the discussion here but only make use of the construction to establish the existence of closed frames.

Definition 3.1 A *semantic domain* is a partially ordered set, with a least element u, which admits the least upper bounds of ω-sequences.

We need to spell this out in somewhat more detail. Let D be a semantic domain and let $\sqsubseteq$ be the ordering of the domain. The element u is the *least element* of the domain D if $u \sqsubseteq d$ for each d in D. An ω-sequence is of the form $d_0 \sqsubseteq d_1 \ldots \sqsubseteq d_n \sqsubseteq \ldots, d_i \in D$ for $i \geq 0$. An element d is an *upper bound* of the sequence if $d_i \sqsubseteq d$ for each $i \geq 0$; it is a least upper bound if $d \sqsubseteq d'$ for any other upper bound d' of the sequence. We write the least upper bound of the sequence as $\bigsqcup_i d_i$.

Our first task is to discuss several ways of building new domains from existing domains. The first construction is the analogue of disjoint union for sets. Let (D_i) $i \in I$ be a family (possibly infinite) of semantic domains. Then define

$$\bigoplus_{i \in I} D_i = \{\langle d, i \rangle : d \in D_i\} \cup \{u\}$$

where the ordering is given by

$$d \sqsubseteq d' \Leftrightarrow (d = u \text{ or } (\exists i \in I, \exists x, y \in D_i \text{ such that } x \sqsubseteq y \text{ and } d = \langle x, i \rangle, d' = \langle y, i \rangle)).$$

The order is inherited from the D_i with the additional demand that a new least element is added. This structure forms a semantic domain. When there are just two domains D_1 and D_2 we shall write the disjoint union as $D_1 + D_2$.

Our second construction is the Cartesian product construction. Let D and D' be semantic domains. Then define

$$D \times D' = \{\langle d, d' \rangle : d \in D \text{ and } d' \in D'\}$$

where $\langle d_0, d_0' \rangle \sqsubseteq \langle d_1, d_1' \rangle \Leftrightarrow d_i \sqsubseteq d_i'$, $i = 0, 1$.

The least element is $\langle u, u' \rangle$.

We leave to the reader to check that $D \times D'$ forms a semantic domain.

Our third construction involves the function spaces themselves. Consider the set R_n. For obvious reasons, R_n cannot be all the functions from E^n to B. We shall, in fact, restrict ourselves to the class of "continuous" functions.

Definition 3.2 A function $f: D \to D'$ is *continuous* iff for each ω-sequence $\langle d_n \rangle_{n \in \omega}$ in D, $f\left(\bigsqcup_n d_n\right) = \bigsqcup_n (f(d_n)$.

If we are to restrict the class of functions from $E^n \to B$ to the continuous ones in our domains we must indicate how this class is, itself, to be viewed as a semantic domain. Since we know how to view E^n as a semantic domain it is sufficient to indicate how the set of continuous functions from D to D' (where D and D' are arbitrary semantic domains) is a semantic domain. The first step is to define an ordering on the set $[D \to D']$ of continuous functions from D to D'. We do this in the obvious way. For $f, g \in [D \to D']$ define

$$f \sqsubseteq g \leftrightarrow (\forall d \in D)(f(d) \sqsubseteq' g(d)) \ .$$

This is clearly a partial ordering with bottom element the function $\lambda d.u_{D'}$. But how do we compute least upper bounds of ω-sequences in $[D \to D']$? Let $\langle f_n \rangle_{n \in \omega}$ be such an ω-sequence. Define the least upper bound of $\langle f_n \rangle_{n \in \omega}$, by $\bigsqcup_{n \in \omega} f_n = \lambda d. \cup \{f_n(d) : n \in \omega\}$.

Once again we leave the details to the reader who must check that this function is well-defined, continuous, and the least upper bound of the sequence $\langle f_n \rangle_{n \in \omega}$. We also leave to the reader the detailed checking (or looking up) of the following standard result:

Theorem 3.3 *The set of continuous functions from one semantic domain to a second form a semantic domain under the above ordering.*

The set, *Asq*, of assignment functions forms a domain in this way. We now describe a technique for constructing (closed) partial frames. We shall in fact construct a domain which satisfies the equation[4]

$$E = B + [E \to E] \ .$$

The technique can be applied to provide us with a solution to the equation

$$E = B + \bigoplus_{n \geq 0} [E^n \to B]$$

but the details are very messy. Such a solution would provide us with a partial frame since clearly the union of the domains $[E^n \to B]$ is a subset of $\bigoplus_{n\geq 0} [E^n \to B]$. The equality represents a one-to-one and onto continuous function between E and $B + \bigoplus_{n\geq 0} [E^n \to B]$.

The first step in the construction of the domain E is to define a sequence of domains which is achieved by induction as follows:

$$\begin{aligned} E_0 &= B \\ E_{n+1} &= B + [E_n \to E_n],\ n > 0 \ . \end{aligned}$$

Our objective is to embed each E_n in E_{n+1}. To this end we introduce a sequence of continuous functions.

$$\begin{aligned} p_n &: E_n \to E_{n+1} \\ q_n &: E_{n+1} \to E_n \end{aligned}$$

by induction on $n \geq 0$. For the base step we define p_0 and q_0 as follows:

$$p_0(a) = a \text{ for } a \text{ in } B$$

and

$$q_0(x') = \begin{cases} x' & \text{for } x' \in B \\ u & \text{if } x' \in [E \to E] \ . \end{cases}$$

Now assume that p_n and q_n have been defined already. Then define

$$p_{n+l}(x) = \begin{cases} x & \text{for } x \in B \\ p_n \mathrm{o} x \mathrm{o} q_n & \text{for } x \text{ in } [E_n \to E_n] \end{cases}$$

$$q_{n+1}(x') = \begin{cases} x' & x' \text{ in } B \\ q_n \mathrm{o} x' \mathrm{o} p_n & \text{for } x' \text{ in } [E_{n+1} \to E_{n+1}] \ . \end{cases}$$

These functions (as an easy inductive argument shows) satisfy $q_n(p_n(f)) = f$ and $p_n(q_n(g)) \sqsubseteq g$. So, in particular, the function p_n is one-to-one. This permits us to view E_n as a subdomain of E_{n+1} (under the projection p_n). In fact, we can extend the mappings p_n, q_n to mappings (continuous) $p_{nm} : E_n \to E_m$ as follows:

$$p_{nm}(f) = \begin{cases} p_{m-1}\circ \ldots \circ p_n & n < m \\ f & n = m \\ q_m\circ \ldots \circ q_{n-1} & m < n \ . \end{cases}$$

Once again it is easy to check that $p_{mn}(p_{nm}(f)) = f$ and $p_{nm}(p_{mn}g)) \sqsubseteq g$ for $0 \le n \le m$.

All this facilitates the construction of a domain satisfying our equation.

Definition 3.4 The domain E_∞ consists of the set $\{\langle f_n\rangle_{n\in\omega}: f_n \in E_n$ and $q_n(f_{n+1}) = f_n\}$ where for $\langle f_n\rangle$, $\langle g_n\rangle$ in E $\langle f_n\rangle_{n\in\omega} \sqsubseteq \langle g_n\rangle_{n\in\omega} \Leftrightarrow (\forall n)(f_n \sqsubseteq g_n)$.

Under this ordering E_∞ forms a semantic domain. Moreover, we can embed each of the E_n $(n \ge 0)$ in E_∞ by stipulating $p_{n\infty}$: $E_n \to E_\infty$ and $p_{\infty n} : E_\infty \to E_n$. These are defined as follows:

$$p_{n\infty}(f) = \langle p_{nk}(f)\rangle_{k\in\omega}$$
$$p_{\infty n}(f) = f_n \ .$$

Once more we leave the reader to check that $p_{\infty n}(p_{n\infty}(f)) = f$ and $p_{n\infty}(p_{\infty n}(f')) \sqsubseteq f'$. We can, therefore, regard each E_n as a subdomain of E_∞ under the embedding $p_{n\infty} : E_n \to E_\infty$.

The following lemmas are standard; the proofs can be found in Barendregt [3].

Lemma 3.5 *For each $f \in E_\infty$*

(i) *If $f \in E_n$ then $f = f_n$*
(ii) *If $f \in E_n$ then $p_n(f) = f$*
(iii) *If $f \in E_{n+1}$ then $q_n(f) \sqsubseteq f$.*

Lemma 3.6 *In E_∞*

(i) $(f_n)_m = f_{min(n,m)}$
(ii) *If $n \le m$ then $f_n \sqsubseteq f_m \sqsubseteq f$*
(iii) $f = \bigsqcup_{n\in\omega} f_n$.

We are now in a position to define application in E_∞. Let $f \in E_\infty$ and $e \in E_\infty$, then define

$$f(e) = \bigsqcup_{n\in\omega} f_{n+1}(e_n)$$

Lemma 3.7 *Application in E_∞ is continuous and satisfies*

$$f_{n+1}(e) = f_{n+1}(e_n) = (f(e)_n)_n \ .$$

For the proof we once again refer the reader to Barendregt [3]. This brings us to the results of Scott [14].

Theorem 3.8 *For each $f \in [E_\infty \to E_\infty]$ there exists an x_f in E_∞ such that for each e in E, $f(e) = x_f(e)$.*

The x_f in question is given by $x_f = \bigsqcup_{n\in\omega} \lambda y \in E_n.(f(y)_n)$. This follows by a relatively straightforward computation for $x_f(e)$.

Theorem 3.9 *The semantic domain E_∞ is isomorphic to the domain* $[B + [E_\infty \to E_\infty]]$.

The isomorphism [one-to-one, onto and continuous) ψ: $[B + [E_\infty \to E_\infty]] \to E_\infty$ is given by

$$\psi(f) = \begin{cases} f & \text{if } f \in B \\ x_f & \text{otherwise} \end{cases} .$$

The main definition of this section is the following.

Definition 3.10 A *Scott Frame* for L is any partial frame in which E is a nonempty domain and the set R_n (for each $n \geq 0$) is the domain of continuous functions from E^n to B. In addition, the function F is a continuous, injective function from $\bigoplus_{n \geq 0} R_n$ into E.

Our aim is to establish the closure of such frames. The next two lemmas constitute the first move in this direction.

Lemma 3.11 *Each of the operators* $\wedge$, $\neg$ *is monotonic and continuous viewed as functions*

$$\wedge\colon\ B \times B \to B$$
$$\neg\colon\ B \to B\ .$$

Proof: Since B is finite it is sufficient to show monotonicity and this much is clear.

Lemma 3.12 *The function*

$$\bigwedge\colon\ [D \to B] \to B$$

given by

$$\bigwedge(f) = \begin{array}{l} 1 \ \textit{if}\ f(d) = 1 \ \textit{for each}\ d \in D \\ 0 \ \textit{if}\ f(d) = 0 \ \textit{for some}\ d \in D \\ u \ \textit{otherwise} \end{array}$$

is monotonic and continuous.

Proof: Let $\langle f_n \rangle_{n \geq 0}$ be an ω-sequence in $[D \to B]$. We have to show

$$\bigwedge\left(\bigsqcup_n f_n\right) = \bigsqcup_n \bigwedge (f_n)\ .$$

We have to prove $\bigwedge\left(\bigsqcup_n f_n\right) = 1$ iff $\bigsqcup_n \bigwedge(f_n) = 1$ and $\bigwedge\left(\bigsqcup_n f_n\right) = 0$ iff $\bigsqcup_n \bigwedge(f_n) = 0$.

Now $\bigwedge\left(\bigsqcup_n f_n\right) = 1$ iff for each $d \in D$, $\left(\bigsqcup_n f_n\right)(d) = 1$. By definition this is equivalent to $\bigsqcup_n (f_n(d)) = 1$. By the structure of B there must be some n such that for each $m \geq n$, $f_m(d) = 1$. So for each $d \in D$, there exists an n such that for each $m \geq n$, $f_m(d) = 1$. In particular, for $u_D \in D$ (the undefined element of D) there is such an n. But if $f_m(u_D) = 1$ then by the continuity (and hence monotonicity) of f_m, $f_m(d) = 1$ for each $d \in D$. As a consequence we can find an n such that for each $d \in D$ and each $m \geq n$, $f_m(d) = 1$. But this is exactly the condition to guarantee that $\bigwedge(f_m) = 1$ for each $m \geq n$. Hence $\bigsqcup_n \bigwedge(f_n) = 1$. On the other hand, if $\bigsqcup_n \bigwedge(f_n) = 1$ then, for some n, $\bigwedge(f_m) = 1$, for each $m \geq n$. Consequently, for each $m \geq n$, $f_m(d) = 1$ for each $d \in D$. Hence, $\bigsqcup_n f_n(d) = 1$ for each $d \in D$. Therefore, $\bigwedge\left(\bigsqcup_n f_n\right) = 1$.

Next assume $\bigwedge\left(\bigsqcup_n f_n\right) = 0$. Then for some $d \in D$, $\left(\bigsqcup_n f_n\right)(d) = 0$. But then $f_m(d) = 0$ for $m \geq$ some n. Subsequently, $\bigwedge \mathrm{f_m} = 0$ for each $m \geq n$, and so $\bigsqcup_n \bigwedge f_n = 0$. Conversely, if $\bigsqcup_n \bigwedge f_n = 0$ then there exists an n such that $\bigwedge f_m = 0$ for each $m \geq n$. Therefore, for each $m \geq n$ there exists a d such that $f_m(d) = 0$. In particular, there exists a d' such that $f_n(d') = 0$. But $f_n \sqsubseteq f_m$, for $m \geq n$, and so $f_m(d') = 0$ for $m \geq n$. Hence there exists a d such that for each $m \geq n$, $f_m(d) = 0$. It follows that $\bigwedge\left(\bigsqcup_n f_n\right) = 0$.

Theorem 3.13 *Each wff A of L, considered as a function*

$$[A] : Asg \to B$$

is monotonic and continuous.

Proof: By induction on A. The atomic case follows directly from the definition of [ ]. Let $\langle g_n \rangle$ be any ω-sequence in Asg. Then $[\sim\mathrm{A}] \bigsqcup_n g_n = \neg\left([A] \bigsqcup_n g_n\right) = \neg\left(\bigsqcup_n [A]g_n\right)$ by induction hypothesis. But by the continuity of $\neg$ this equals $\bigsqcup_n (\neg [A]g_n)$.

For the conjunction case consider $[A \;\&\; B] \bigsqcup_n g_n$. By definition this equals $[A] \bigsqcup_n g_n \wedge [B] \bigsqcup_n g_n$. By induction this gives $\left(\bigsqcup_n [A]g_n\right) \wedge \left(\bigsqcup_n [B]g_n\right)$. By the continuity of $\wedge$ we obtain $\bigsqcup_n ([A]g_n \wedge [B]g_n)$ which equals $\bigsqcup_n ([A \;\&\; B]g_n)$.

This leaves us to deal with the quantification clauses. We deal with clause (4) since clause (5) is identical.

Consider $[\forall x\, A] \bigsqcup_n g_n$. By definition we obtain

$$\bigwedge \lambda e \,.\, [A]\left(\bigsqcup_n g_n\right)(e|x) \;.$$

This equals $\bigwedge\left(\lambda e \,.\, [A] \bigsqcup_n g_n(e|x)\right)$ which, by induction, equals $\bigwedge\left(\lambda e \,.\, \bigsqcup_n [A]g_n(e|x)\right)$. By continuity of $\bigwedge$ we obtain

$$\bigsqcup_n \left(\bigwedge (\lambda e \,.\, [A]g_n(e|x))\right) \;.$$

This is precisely $\bigsqcup_n ([\forall x\, A]g_n)$, as required.

Theorem 3.14 *For each wff A of L, with free variables $M_0, \ldots, M_{n-1}$, the function*

$$r_A = \lambda e_0, \ldots, e_{n-1} \,.\, [A]_g^M(e_i|M_i),\ 0 \leq i \leq n-1$$

is monotonic and continuous.

Proof: It is sufficient to consider the case of one variable. Let $\langle e_n \rangle_{n\in\omega}$ be an ω-sequence in E (of appropriate type for M). Then $r_A\left(\bigsqcup_n e_n\right) = [A]g\left(\bigsqcup_n e_n|M\right) = [A]\bigsqcup_n g_n$ where $g_n = g(e_n|M)$, by continuity of the extension to assignments function. But $[A]\bigsqcup_n g_n$ equals, by the previous theorem, $\bigsqcup_n [A]g_n$, which is exactly $\bigsqcup_n [A]g(e_n|M)$; and this equals $\bigsqcup_n r_A(e_n)$.

This completes our proof of the existence of closed frames.

We are not claiming here that Scott Frames offer an intuitively acceptable analysis of the nominalization of complex-predicate phrases. The interpretation of phrases involving the quantifiers is somewhat curious. Our intention here is to employ Scott Frames only to establish the existence of nontrivial closed Partial Frames.

NOTES

1. We shall in the sequel sometimes drop the reference to M in $[\ ]_g^M$.
2. I shall frequently use the phrase 'principle of comprehension' in place of the more common term 'comprehension schema'. In my usage the two terms are meant to be synonymous.
3. The '*' indicates that it is a principle of the language L rather than the language of standard second-order logic.
4. We could solve the equation $E = A + [E \rightarrow E]$ where A is any domain of 'individuals' containing a copy of B.

REFERENCES

[1] Aczel, P., "Frege structures and the notions of proposition, truth and set," pp. 31-59 in *The Kleene Symposium*, eds., J. Barwise, H. J. Keisler, and K. Kunen, North-Holland, Amsterdam, 1980.

[2] Aczel, P. and S. Feferman, "Consistency of the unrestricted abstraction principle using an intensional equivalence operator," pp. 67-98 in *H. B. Curry: Essays on Combinatory Logic, Lambda Calculus & Formalism*, ed., J. P. Seldin and J. R. Hindley, Academic Press, London, 1980.

[3] Barendregt, H., *The Type-Free Lambda Calculus*, *Handbook of Mathematical Logic*, North Holland, Amsterdam, 1977.

[4] Brady, R. T., "The consistency of the axioms of abstraction and extensionality in three-valued logic," *Notre Dame Journal of Formal Logic*, vol. 12 (1971), pp. 447-453.

[5] Bunder, M. W., "The naturalness of illative combinatory logic as a basis for mathematics," pp. 55-64 in *H. B. Curry: Essays on Combinatory Logic, Lambda Calculus & Formalism*, ed., J. P. Seldin and J. R. Hindley, Academic Press, London, 1980.

[6] Chierchia, G., "Nominalization and Montague grammar," *Linguistics and Philosophy*, vol. 5 (1982), p. 3.

[7] Cocchiarella, N. B., "Fregean semantics for a realist ontology," *Notre Dame Journal of Formal Logic*, vol. 15 (1974), pp. 552-568.

[8] Cocchiarella, N. B., "On the logic of nominalized predicates, and its philosophical interpretations," *Erkenntnis*, vol. 13 (1978), pp. 339-369.

[9] Cocchiarella, N. B., "The theory of homogeneous simple types as a second order logic," *Notre Dame Journal of Formal Logic*, vol. 20 (1979), pp. 505-524.

[10] Feferman, J., "Toward useful type-free theories, I," unpublished manuscript, Stanford, 1981.

[11] Gilmore, P.C., "The consistency of partial set theory without extensionality, Axiomatic set theory," pp. 147-153 in *Proceedings of Symposia in Pure Mathematics*, XIII, Part II, American Mathematical Society, 1974.

[12] Gilmore, P.C., "Combining unrestricted abstraction with universal quantification," pp. 99-123 in *H. B. Curry: Essays on Combinatory Logic, Lambda Calculus & Formalism*, ed., J. P. Seldin and J. R. Hindley, Academic Press, London 1980.

[13] Kleene, S. C., *Introduction to Metamathematics*, North-Holland, Amsterdam, 1952.

[14] Scott, D., "Continuous lattices" pp. 97-136 in *Toposes, Algebraic Geometry and Logic*, ed., F. W. Lawvere, *Lecture Notes in Mathematics*, vol. 274, Springer Verlag, New York, 1972.

[15] Scott, D., "Lambda calculus and recursion theory," in *Proceedings of Third Scandinavian Logic Symposium*, ed., S. Kanger, North Holland, Amsterdam, 1975.

[16] Smyth, M. B., "The largest Cartesian closed category of domains," unpublished manuscript, University of Edinburgh, 1982.

[17] Turner, R., "Montague semantics, nominalization & Scott's domains," *Linguistics & Philosophy*, vol. 6 (1983), pp. 259-288.

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