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New Axioms for Mereology

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In this paper I shall present several new axioms and axiom systems for mereology with an account of their origin. I shall also outline a proof that the two most interesting of these are adequate sole axioms for mereology.

Professor Lejewski discovered the first mereological axioms which employ the terms '*Kl*' and '*ov*' (see [2], p. 43, theses E2 and E4). His axioms have thirteen ontological units each, and so are longer than most of the known single axioms. I have discovered two new axioms for these terms:

2.1 $[AB] \mathrel{::\cdot} A \in ov(B) .\equiv:: [\exists Ca] :: C \in a :: [DE] \mathrel{\because} [A] \therefore A \in E .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) :: E \in E . C \in ov(D) ::\supset. A \in ov(D) . B \in ov(D)$

3.1 $[Aa] :: A \in Kl(a) .\equiv\because [\exists B] . B \in a \because [Bb] :: [A] : A \in b .\equiv. [\exists Ccd] . A \in Kl(c) . B \in Kl(d) . C \in c . C \in d \therefore Kl(a) \in Kl(a) \therefore\supset: A \in b .\equiv. [\exists C] . C \in a . C \in b.$

Axiom 2.1 is only ten ontological units long, and so is shorter than all other single axioms now known except one for the term '*ex*', which is also ten units in length (see [3], p. 66, thesis E6). Axiom 3.1 is eleven units long. Unlike the just mentioned axiom for '*ex*' and the earlier axioms for '*Kl*' and '*ov*', Theses 2.1 and 3.1 have no functor variables. To show that 2.1 and the definitions associated with it are indeed theses of mereology, I shall derive theses corresponding to them from the following axiom system for the term '*el*'.[1]

1.1 $[AB] : A \in el(B) .\supset. B \in B$
1.2 $[ABC] : A \in el(B) . B \in el(C) .\supset. A \in el(C)$
1.3 $[A] : A \in A .\supset. A \in el(A)$
1.4 $[Aa] :: A \in Kl(a) .\equiv\therefore A \in A \therefore [B] : B \in a .\supset. B \in el(A) \therefore [B] : B \in el(A) .\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C)$
1.5 $[Aa] : A \in a .\supset. Kl(a) \in Kl(a)$
1.6 $[AB] \therefore A \in ov(B) .\equiv: A \in A : [\exists C] . C \in el(A) . C \in el(B).$

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In the deductions I shall not refer explicitly to theses of ontology except the following:

0.1 $[a] . \sim(\wedge \in a)$
0.2 $[BC] :: [A] : A \in B .\equiv. A \in C \therefore B \in B \therefore\supset. C \in C.$

From Axioms 1.1–1.6 we may deduce the following theses:

1.7 $[AB] : A \in A . B \in A .\supset. B \in el(A)$ [1.3]
1.8 $[AB] : B \in el(A) .\supset. [\exists CD] . C \in A . D \in el(B) . D \in el(C)$ [1.1, 1.3]
1.9 $[A] : A \in A .\supset. A \in Kl(A)$ [1.7, 1.8, 1.4]
1.10 $[AB] : A \in A . B \in Kl(A) .\supset. A \in B$ [1.9, 1.5]
1.11 $[ABa] : A \in Kl(a) . B \in a .\supset. B \in el(A)$ [1.4]
1.12 $[ABE] : B \in E .\supset. B \in el(Kl(A \cup E))$ [1.5, 1.11]
1.13 $[ABa] : A \in Kl(a) . B \in el(A) .\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C)$ [1.4]
1.14 $[AB] : A \in ov(B) .\equiv. [\exists C] . C \in el(A) . C \in el(B)$ [1.6, 1.1]
1.15 $[A] : A \in A .\supset. A \in ov(A)$ [1.3, 1.14]
1.16 $[ABC] : A \in el(B) . A \in ov(C) .\supset. B \in ov(C)$ [1.14, 1.2]
1.17 $[ABCa] : A \in Kl(a) . C \in a . C \in ov(B) .\supset. A \in ov(B)$ [1.11, 1.16]
1.18 $[ABa] : A \in Kl(a) . A \in ov(B) .\supset. [\exists C] . C \in a . C \in ov(B)$ [1.14, 1.13, 1.2]
1.19 $[ABCDE] :: A \in A \therefore [C] : A \in ov(C) .\supset. E \in ov(C) \therefore C \in (A \cup E) . D \in el(B) . D \in el(C) \therefore\supset. [\exists CD] . C \in E . D \in el(B) . D \in el(C)$ [1.14]
1.20 $[ABE] :: A \in A \therefore [C] : A \in ov(C) .\supset. E \in ov(C) \therefore B \in el(Kl(A \cup E)) \therefore\supset. [\exists CD] . C \in E . D \in el(B) . D \in el(C)$ [1.1, 1.13, 1.19]
1.21 $[AE] :: A \in A \therefore [C] : A \in ov(C) .\supset. E \in ov(C) \therefore\supset. Kl(A \cup E) \in Kl(E)$ [1.5, 1.12, 1.20, 1.4]
1.22 $[AE] :: A \in A \therefore [C] : A \in ov(C) .\supset. E \in ov(C) \therefore\supset. E \in Kl(A \cup E)$ [1.15, 1.21, 1.10]
1.23 $[AE] :: A \in A \therefore [C] : A \in ov(C) .\supset. E \in ov(C) \therefore\supset. A \in el(E)$ [1.22, 1.11]
1.24 $[AB] :: A \in el(B) .\equiv\therefore A \in A \therefore [C] : A \in ov(C) .\supset. B \in ov(C)$ [1.16, 1.23]
1.25 $[AB] :: A \in ov(B) .\equiv\therefore [\exists C] \therefore C \in C \therefore [D] : C \in ov(D) .\supset. A \in ov(D) . B \in ov(D)$ [1.14, 1.24]
1.26 $[ABa] :: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) \therefore B \in a \therefore\supset. B \in el(A)$ [1.23]
1.27 $[ABa] :: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) \therefore B \in el(A) \therefore\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C)$ [1.3, 1.14]
1.28 $[Aa] :: A \in A \therefore [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) \therefore\supset. A \in Kl(a)$ [1.26, 1.27, 1.4]
1.29 $[Aa] :: A \in Kl(a) .\equiv\therefore A \in A \therefore [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B)$ [1.18, 1.17, 1.28]

1.30 $[Ca] :: C \in a .\supset \therefore [A] \therefore A \in Kl(a) .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B)$ [1.15, 1.29]

1.31 $[ABCa] ∷ C \in a ∷ [DE] ∵ [A] \therefore A \in E .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) :: E \in E . C \in ov(D) ::\supset. A \in ov(D) . B \in ov(D) ∷\supset. A \in ov(B)$ [1.30, 1.5, 1.25]

1.32 $[AB] ∷ A \in ov(B) .\equiv∷ [\exists Ca] ∷ C \in a ∷ [DE] ∵ [A] \therefore A \in E .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) :: E \in E . C \in ov(D) ::\supset. A \in ov(D) . B \in ov(D)$ [1.25, 1.31].

The following theses are true in mereology, since they correspond to Theses 1.32, 1.29, and 1.24 respectively:

2.1 $[AB] ∷ A \in ov(B) .\equiv∷ [\exists Ca] ∷ C \in a ∷ [DE] ∵ [A] \therefore A \in E .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) :: E \in E . C \in ov(D) ::\supset. A \in ov(D) . B \in ov(D)$

2.2 $[Aa] :: A \in Kl(a) .\equiv\therefore A \in A \therefore [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B)$

2.3 $[AB] :: A \in el(B) .\equiv\therefore A \in A \therefore [C] : A \in ov(C) .\supset. B \in ov(C)$.

If we take Thesis 2.1 as an axiom and Theses 2.2 and 2.3 as definitions, we obtain a system of mereology equivalent to that based on Axioms 1.1–1.6. We can show this by deducing from 2.1–2.3 the following theses:

2.4 $[A] : A \in A .\supset. A \in el(A)$ [2.3]

2.5 $[ABC] : A \in el(B) . B \in el(C) .\supset. A \in el(C)$ [2.3]

2.6 $[ABCa] :: C \in a \therefore [D] : C \in ov(D) .\supset. A \in ov(D) . B \in ov(D) \therefore\supset. A \in ov(B)$ [2.1]

2.7 $[A] : A \in A .\supset. A \in ov(A)$ [2.6]

2.8 $[AB] : A \in el(B) .\supset. B \in ov(A)$ [2.7, 2.3]

2.9 $[AB] : A \in el(B) .\supset. B \in B$ [2.8]

2.10 $[Ca] :: C \in a .\supset\therefore [A] \therefore A \in Kl(a) .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B)$ [2.7, 2.2]

2.11 $[Ca] :: C \in a .\supset\therefore [\exists E] \therefore [A] : A \in E .\equiv. A \in Kl(a) \therefore E \in E$ [0.1, 2.1, 2.10]

2.12 $[Aa] : A \in a .\supset. Kl(a) \in Kl(a)$ [2.11, 0.2]

2.13 $[AB] :: A \in ov(B) .\supset\therefore [\exists Ca] \therefore C \in a \therefore [D] : C \in ov(D) .\supset. A \in ov(D) . B \in ov(D)$ [2.1, 2.10, 2.12]

2.14 $[AB] :: A \in ov(B) .\equiv\therefore [\exists C] \therefore C \in C \therefore [D] : C \in ov(D) .\supset. A \in ov(D) . B \in ov(D)$ [2.13, 2.6]

2.15 $[AB] : A \in ov(B) .\equiv. [\exists C] . C \in el(A) . C \in el(B)$ [2.14, 2.3]

2.16 $[AB] \therefore A \in ov(B) .\equiv: A \in A : [\exists C] . C \in el(A) . C \in el(B)$ [2.15]

2.17 $[Aa] : A \in Kl(a) .\supset. [\exists B] . B \in a$ [2.7, 2.2]

2.18 $[ABa] : A \in Kl(a) . B \in a .\supset. B \in el(A)$ [2.2, 2.3]

2.19 $[AB] : A \in ov(B) .\equiv. [\exists C] . C \in el(A) . C \in ov(B)$ [2.4, 2.3]

2.20 $[A] : A \in A .\supset. A \in Kl(el(A))$ [2.19, 2.2]

2.21 $[AB] : A \in el(B) .\supset. [\exists a] . B \in Kl(a) . A \in a$ [2.9, 2.20]

2.22 $[AB] : A \in el(B) .\equiv. [\exists a] . B \in Kl(a) . A \in a$ [2.21, 2.18]

2.23 $[AB] : A \in ov(B) .\equiv. [\exists Ccd] . A \in Kl(c) . B \in Kl(d) . C \in c . C \in d$ [2.15, 2.22]

2.24 $[ABab] :: A \in Kl(a) \therefore [A] : A \in b .\equiv. [\exists Ccd] . A \in Kl(c) . B \in Kl(d) . C \in c . C \in d \therefore\supset: A \in b .\equiv. [\exists C] . C \in a . C \in b$ [2.23, 2.2]

2.25 $[ABa] :: B \in a \because [Bb] :: [A] : A \in b .\equiv. [\exists Ccd] . A \in Kl(c) . B \in Kl(d) . C \in c . C \in d \therefore Kl(a) \in Kl(a) \therefore\supset: A \in b .\equiv. [\exists C] . C \in a . C \in b \because\supset. A \in Kl(a)$ [2.23, 2.12, 2.10]

2.26 $[Aa] :: A \in Kl(a) .\equiv\because [\exists B] . B \in a \because [Bb] :: [A] : A \in b .\equiv. [\exists Ccd] . A \in Kl(c) . B \in Kl(d) . C \in c . C \in d \therefore Kl(a) \in Kl(a) \therefore\supset: A \in b .\equiv. [\exists C] . C \in a . C \in b$ [2.17, 2.24, 2.25]

2.27 $[ABa] : A \in Kl(a) . B \in el(A) .\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C)$ [2.8, 2.2, 2.15]

2.28 $[ABCa] :: [B] : B \in a .\supset. B \in el(A) \therefore C \in a . C \in ov(B) \therefore\supset. A \in ov(B)$ [2.3]

2.29 $[ABa] :: [B] : B \in el(A) .\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C) \therefore A \in ov(B) \therefore\supset. [\exists C] . C \in a . C \in ov(B)$ [2.15, 2.5]

2.30 $[Aa] :: A \in A \therefore [B] : B \in a .\supset. B \in el(A) \therefore [B] : B \in el(A) .\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C) \therefore\supset. A \in Kl(a)$ [2.29, 2.28, 2.2]

2.31 $[Aa] :: A \in Kl(a) .\equiv\therefore A \in A \therefore [B] : B \in a .\supset. B \in el(A) \therefore [B] : B \in el(A) .\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C)$ [2.18, 2.27, 2.30].

Theses 2.9, 2.5, 2.4, 2.31, 2.12, and 2.16 correspond respectively to Theses 1.1, 1.2, 1.3, 1.4, 1.5, and 1.6. Therefore the two axiom systems are equivalent, and Thesis 2.1, with the help of the Definitions 2.2 and 2.3, is adequate as a sole axiom for mereology.

The following theses are true in mereology, since they correspond to Theses 2.26, 2.23, and 2.3 respectively:

3.1 $[Aa] :: A \in Kl(a) .\equiv\because [\exists B] . B \in a \because [Bb] :: [A] : A \in b .\equiv. [\exists Ccd] . A \in Kl(c) . B \in Kl(d) . C \in c . C \in d \therefore Kl(a) \in Kl(a) \therefore\supset: A \in b .\equiv. [\exists C] . C \in a . C \in b$

3.2 $[AB] : A \in ov(B) .\equiv. [\exists Ccd] . A \in Kl(c) . B \in Kl(d) . C \in c . C \in d$

3.3 $[AB] :: A \in el(B) .\equiv\therefore A \in A \therefore [C] : A \in ov(C) .\supset. B \in ov(C)$.

If we take Thesis 3.1 as an axiom and Theses 3.2 and 3.3 as definitions, we obtain a system of mereology equivalent to that based on Theses 2.1–2.3. We can show this by deducing from 3.1–3.3 the following theses:

3.4 $[Ba] : B \in a .\supset. Kl(a) \in Kl(a)$ [3.1, 0.1]

3.5 $[Aa] \therefore A \in Kl(a) .\supset: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B)$ [3.1, 3.2, 3.4]

3.6 $[A] : A \in A .\supset. A \in Kl(A)$ [3.1]

3.7 $[A] : A \in A .\supset. A \in ov(A)$ [3.6, 3.2]

3.8 $[Aa] :: A \in A \therefore [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) \therefore\supset. [\exists C] . C \in a$ [3.7]

3.9 $[Ca] :: C \in a .\supset\therefore [A] \therefore A \in Kl(a) .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B)$ [3.5, 3.2, 3.1]

3.10 $[Aa] :: A \in A \therefore [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) \therefore\supset. A \in Kl(a)$ [3.8, 3.9]

3.11 $[Aa] :: A \in Kl(a) .\equiv\therefore A \in A \therefore [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B)$ [3.5, 3.10]

3.12 $[ABa] : A \in Kl(a) . B \in a .\supset. B \in el(A)$ [3.5, 3.3]

3.13 $[A] : A \in A .\supset. A \in el(A)$ [3.3]

3.14 $[AB] . A \in ov(B) .\equiv. [\exists C] . C \in el(A) . C \in ov(B)$ [3.13, 3.3]

3.15 $[A] : A \in A .\supset A \in Kl(el(A))$ [3.14, 3.10]

3.16 $[AB] : A \in el(B) .\supset. B \in Kl(el(B))$ [3.7, 3.3, 3.15]

3.17 $[AB] : A \in el(B) .\equiv. [\exists a] . B \in Kl(a) . A \in a$ [3.16, 3.12]

3.18 $[AB] : A \in ov(B) .\equiv. [\exists C] . C \in el(A) . C \in el(B)$ [3.2, 3.17]

3.19 $[AB] :: A \in ov(B) .\equiv\therefore [\exists C] \therefore C \in C \therefore [D] : C \in ov(D) .\supset. A \in ov(D) . B \in ov(D)$ [3.18, 3.3]

3.20 $[ABCa] ::\therefore C \in a :: [DE] :\because [A] \therefore A \in E .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) :: E \in E . C \in ov(D) ::\supset. A \in ov(D)\ B \in ov(D) ::\supset. A \in ov(B)$ [3.9, 3.4, 3.19]

3.21 $[AB] ::\therefore A \in ov(B) .\equiv:: [\exists Ca] :: C \in a :: [DE] :\because [A] \therefore A \in E .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) :: E \in E . C \in ov(D) ::\supset. A \in ov(D) . B \in ov(D)$ [3.19, 3.20].

Theses 3.21, 3.11, and 3.3 correspond respectively to Theses 2.1, 2.2, and 2.3. Therefore the system of Theses 3.1–3.3 is equivalent to the system 2.1–2.3, and Thesis 3.1, with the help of Definitions 3.2 and 3.3, is adequate as a sole axiom for mereology.

Late in 1979, I discovered a twelve unit axiom for '*Kl*':

4.1 $[Aa] :: A \in Kl(a) .\equiv:\because A \in A :\because [f] :: [AB] : A \in f(B) .\equiv. [\exists a] . B \in Kl(a) . A \in a \therefore A \in Kl(A) \therefore\supset\therefore [b] \therefore A \in f(Kl(b)) .\equiv: [BE] : E \in a . B \in f(E) .\supset. [\exists CD] . C \in a . D \in f(B) . D \in f(C).$

When Professor Lejewski learned of this axiom, he showed me a new axiom for '*el*' which contains only eleven units. I had already suggested that since my axiom is not externally independent,[2] and since I was able to find several other axioms the same length as Thesis 4.1, all of which had similar methodological flaws, a shorter thesis might be found. Prompted by Lejewski's discovery, though without making actual use of it in my deductions, I discovered the eleven unit axiom

4.2 $[Aa] ::\therefore A \in Kl(a) .\equiv:: [f] :: [AB] : A \in f(B) .\equiv. [\exists a] . B \in Kl(a) . A \in a :\supset:\because [b] :\because A \in f(Kl(b)) .\equiv:: [E] :: E \in a .\supset\therefore E \in Kl(E) \therefore [B] : B \in f(E) .\supset. [\exists CD] . C \in b . D \in f(B) . D \in f(C).$

For some time I have been searching for axiom systems consisting of two theses, one an equivalence and the other allowing us to prove the existence and uniqueness of the $Kl(\underline{a})$ providing some object is an $\underline{a}$. Leśniewski himself discovered the first system of this kind.[3] An example of such a system closely related to Theses 1.1–1.5 is the following[4]:

5.1 $[AB] :: A \in el(B) .\equiv .\therefore B \in B \therefore [C] : B \in el(C) .\supset. A \in el(C)$

5.2 $[Da] :\because [A] :: A \in D .\equiv .\therefore [B] : B \in a .\supset. B \in el(A) \therefore [B] : B \in el(A) .\supset. [\exists CD] . C \in a . D \in el(B) . D \in el(C) ::\supset. D \in D.$

The following is one of the shortest systems of this kind I was able to find:

6.1 $[AB] :: A \in ov(B) .\equiv .\therefore [\exists C] \therefore C \in C \therefore [D] : C \in ov(D) .\supset. A \in ov(D) . B \in ov(D)$

6.2 $[Da] :: [A] \therefore A \in D .\equiv: [B] : A \in ov(B) .\equiv. [\exists C] . C \in a . C \in ov(B) \therefore\supset. D \in D.$

When I analysed Professor Lejewski's new axiom for '*el*', I found he had discovered a new technique for constructing single axioms which is particularly applicable to systems of the above type. Almost immediately I found Axiom 2.1 by applying his method to my system 6.1 and 6.2. After some reflection, I discovered Axiom 3.1, which is also closely related to Theses 6.1 and 6.2. Although Axioms 3.1 and 4.2 both have eleven ontological units, 3.1 is, one might say, a better axiom, since it lacks functor variables.

I have sought an axiom for '*ex*' which lacks functor variables, since these appear in the two axioms for this term discovered in the fifties.[5] But the shortest thesis I can find using '*ex*', lacking functor variables, and able to serve as an axiom for mereology, is the thesis

7.1 $[AB] \,{:}{:}{:}\, A \in ex(B) .\equiv \,{:}{\because}{:}\, [C] \,{:}{\because}{:}\, [\exists a] \,{:}{\because}{:}\, A \in a .\vee. B \in a \,{:}{\because}{:}\, [E] \,{:}{\because}{:}\, [A] :\because A \in E .\equiv:: A \in A :: [B] \therefore A \in ex(B) .\equiv: [C] : C \in a .\supset. C \in ex(B) :: E \in E ::\supset. [\exists D] . E \in ex(D) . \sim(C \in ex(D)).$

Thesis 7.1 is eleven units long, one unit longer than the above-mentioned ten-unit axiom for '*ex*', which makes use of functor variables.

NOTES

1. This axiom system is not independent; 1.3 is a consequence of the other axioms. See [1], where the axiom system in question is obviously equivalent to that consisting of theses 1.1–1.5.

2. External independence, internal independence, and other terms describing axioms of mereology are defined and discussed in [5].

3. See the system for '*ex*' in [4], p. 142, and in [3], p. 65. The functor variable is easily eliminated from this system.

4. This system is not internally independent. The proof in [1] allows us to replace the sign of equivalence in Thesis 5.1 with an implication sign.

5. See Thesis E3 in [2], p. 43, and Thesis E6 in [3], p. 66.

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