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Constructing Sequent Rules for Generalized Propositional Logics

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1 Introduction Sequents were first introduced by Gentzen in [1]. His sequent system was developed as a convenient framework for proving his "normal form theorem". Different but related techniques have been employed also by Herbrand [2], Beth [3], Hintikka [4], Schütte [5], and Kanger [6]. This paper is based on that version of the propositional part of the sequent system presented in Chapter VI of Kleene's [7].

The concept of a propositional logic, *PL*, will be defined and a method, to be referred to as the Kleene Search Procedure, will be used to determine the validity of formulas of *PL*. This method utilizes a set of sequent rules which are derived *in a purely mechanical fashion* from the truth tables which are the intended interpretations of the connectives of *PL*. The results are then used to show how a formal system, *GPL*, which is a sequent calculus can be constructed with very simple axioms and these sequent rules to yield the valid formulas of *PL*.

The term 'propositional logic' is used here in the narrow sense of classical two-valued propositional logic with some designated collection of connectives, not necessarily the usual ones and not necessarily two-place. Thus, for example, intuitionistic propositional logics and many-valued propositional logics are not included.

2 *Propositional logics*

Definition 1 A propositional logic shall consist of:

(i) A language *PL* containing: (a) propositional atoms $p, q, r, p_1, q_1, \ldots$, and (b) a collection of symbols which shall be called "connectives". With each connective is associated a positive integer k and we say that the connective is k-ary. Intuitively we think of the connective as

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binding k sentences simultaneously to form a new sentence. (c) Certain auxiliary symbols: (,), [,], →.

(ii) An intended interpretation for each connective, that is, a function $\mathsf{T}^k \rightarrow \mathsf{T}$, where $\mathsf{T} = \{T, F\}$ for each k-ary connective.

The usual definitions of complete assignment, valid formula, satisfiable formula, well-formed formula, etc. will be assumed.

Definition 2 We shall call the formula expression $\Delta \rightarrow \Omega$, where Δ and Ω are two finite strings of zero or more formulas, a *sequent.* We call Δ its *antecedent* and Ω its *succedent.*

3 Constructing logical sequent rules We are going to describe a procedure for determining for a given propositional logic *PL* whether or not a formula of *PL* is valid.

With each connective C of *PL* we shall associate two rules: an antecedent rule $C\rightarrow$ and a succedent rule $\rightarrow C$, which we shall call the sequent rules for that connective. These rules are indicated by certain expressions which are described as follows:

Let $C^k(A_1, \ldots, A_k)$ represent a formula built up by applying the k-ary connective C^k to the formulas $A_1, \ldots, A_k$. The expression for the antecedent rule $C^k\rightarrow$ is described thus: the expression will consist of a horizontal line followed by $C^k\rightarrow$ and which has $C^k(A_1, \ldots, A_k), \Gamma \rightarrow \theta$ below the line. Here Γ and θ represent arbitrary finite sets of formulas. Above the line will be a finite number of sequents obtained as follows: let τ be the truth function associated with C^k and assume that τ is defined by a truth table containing 2^k lines with the variables $p_1, \ldots, p_k$ at the head of the first k columns and $C^k(p_1, \ldots, p_k)$ at the head of the $(k + 1)^{\text{th}}$ or defining column of the table. To each line of the table which contains a T in column $k + 1$ we associate a sequent $\Gamma_i \rightarrow \theta_i$ above the line. The antecedent of $\Gamma_i \rightarrow \theta_i$ contains Γ and if the j^{th} column of the table contains a T, then the antecedent contains A_j. The succedent θ_i contains θ and if the j^{th} column of the table contains F, the succedent contains A_j, $1 \leqslant j \leqslant k$.

The expression for $\rightarrow C^k$ is to be written in the same way except that the sequent below the line will be $\Gamma \rightarrow \theta,\ C^k(A_1, \ldots, A_k)$, the sequents above the line are to be written as above except that each sequent will correspond to an F in column $k + 1$ in this case.

Having thus written expressions for two sequent rules corresponding to C^k, each is to be modified as follows: Each pair of sequents above the line will be compared. If any two sequents are exactly alike except that an A_j which occurs in the antecedent of one occurs in the succedent of the other, then the A_j is deleted from both sequents, the sequents then become identical and one of them is omitted. This must be done by comparing all of the original pairs before deletion and dropping as a given sequent may compare favorably in this way to two or more different sequents with regard to different A_j's. After the first round of comparing all pairs, the resulting set of sequents should then be compared in pairs in a second round. This process is to go on until the sequents that remain at the end of a round are the same as those that were present at the beginning of that round, that is, until no more deletions can be made. The resulting set of sequents will then be the set of sequents to be used above the line.

When the procedure described above is applied to the classical propositional logic, *CPL*, with the connectives $\neg$, &, $\vee$, $\supset$, $\equiv$ (for negation, conjunction, disjunction, material implication, and material equivalence, respectively) the sequent rules for those connectives which appear on p. 289 of [7] are obtained.

Examples:

(a) First we obtain the expressions for the sequent rules for material implication assuming the usual truth table definition.

(i) $\supset\rightarrow$: From the table this rule is written as:

$$\frac{A_1, A_2, \Gamma \rightarrow \theta; A_2, \Gamma \rightarrow A_1, \theta; \Gamma \rightarrow A_1, A_2, \theta}{A_1 \supset A_2, \Gamma \rightarrow \theta} \supset\rightarrow$$

Table 1 is used in simplifying the expression. To simplify the table the Γ and θ will not be written. In the table we place an X down the main diagonal since a sequent need not be compared to itself. Also, it is necessary only to work above the main diagonal. When two sequents do not compare as indicated above, an X is placed in the row and column of the two sequents. If they do compare, then in that row and column we put the sequent that replaces the two sequents.

Table 1.

	$A_1, A_2\rightarrow$	$A_2 \rightarrow A_1$	$\rightarrow A_1, A_2$
$A_1, A_2\rightarrow$	X	$A_2\rightarrow$	X
$A_2 \rightarrow A_1$		X	$\rightarrow A_1$
$\rightarrow A_1, A_2$			X

From Table 1 we see that after comparing all three original sequents we are left with $A_2\rightarrow$ and $\rightarrow A_1$ which clearly do not simplify any further. Thus, our $\supset\rightarrow$ rule is:

$$\frac{A_2, \Gamma \rightarrow \theta; \Gamma \rightarrow \theta, A_1}{A_1 \supset A_2, \Gamma \rightarrow \theta} \supset\rightarrow$$

(ii) $\rightarrow\supset$: Since the truth table for $\supset$ contains only one *F* in the defining column the sequent rule $\rightarrow\supset$ is:

$$\frac{A_1, \Gamma \rightarrow \theta, A_2}{\Gamma \rightarrow \theta, A_1 \supset A_2} \rightarrow\supset$$

Note that when more than one round is needed, only the new sequents that arose in the previous round need to be compared.

(b) In this example *C* is a three-place connective defined by Table 2. We use *p, q,* and *r* in place of P_1, P_2, and P_3, respectively.

Table 2.

p	q	r	$C(p, q, r)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

(i) $C\rightarrow$: The $C\rightarrow$ expression is written from Table 2 as follows:

$$\frac{p, q, r, \Gamma\rightarrow\theta; p, q, \Gamma\rightarrow\theta, r; p, r, \Gamma\rightarrow\theta, q; q, r, \Gamma\rightarrow\theta, p; q, \Gamma\rightarrow\theta, p, r; r, \Gamma\rightarrow\theta, p, q; \Gamma\rightarrow\theta, p, q, r}{C(p\cdot q\cdot r), \Gamma\rightarrow\theta}\ C\rightarrow$$

Deleting Γ and θ our table for round one is Table 3.

Table 3.

	$p, q, r\rightarrow$	$p, q\rightarrow r$	$p, r\rightarrow q$	$q, r\rightarrow p$	$q\rightarrow p, r$	$r\rightarrow p, q$	$\rightarrow p, q, r$
$p, q, r\rightarrow$	x	$p, q\rightarrow$	$p, r\rightarrow$	$q, r\rightarrow$	x	x	x
$p, q\rightarrow r$		x	x	x	$q\rightarrow r$	x	x
$p, r\rightarrow q$			x	x	x	$r\rightarrow q$	x
$q, r\rightarrow p$				x	$q\rightarrow p$	$r\rightarrow p$	x
$q\rightarrow p, r$					x	x	$\rightarrow p, r$
$r\rightarrow p, q$						x	$\rightarrow p, q$
$\rightarrow p, q, r$							x

Thus, after round one our seven original sequents are replaced by nine new sequents. The table for round two is Table 4.

Table 4.

	$p, q\rightarrow$	$p, r\rightarrow$	$q, r\rightarrow$	$q\rightarrow r$	$r\rightarrow q$	$q\rightarrow p$	$r\rightarrow p$	$\rightarrow p, r$	$\rightarrow p, q$
$p, q\rightarrow$	x	x	x	x	x	$q\rightarrow$	x	x	x
$p, r\rightarrow$		x	x	x	x	x	$r\rightarrow$	x	x
$q, r\rightarrow$			x	$q\rightarrow$	$r\rightarrow$	x	x	x	x
$q\rightarrow r$				x	x	x	x	x	x
$r\rightarrow q$					x	x	x	x	x
$q\rightarrow p$						x	x	x	$\rightarrow p$
$r\rightarrow p$							x	$\rightarrow p$	x
$\rightarrow p, r$								x	x
$\rightarrow p, q$									x

The nine sequents after round one have now been reduced to the three sequents: $q\rightarrow$, $r\rightarrow$, and $\rightarrow p$.

These three sequents cannot be further simplified and our $C\rightarrow$ expression can now be written in its simplified form as:

$$\frac{q, \Gamma \rightarrow \theta; r, \Gamma \rightarrow \theta; \Gamma \rightarrow \theta, p}{C(p, q, r), \Gamma \rightarrow \theta} \; C\!\rightarrow.$$

(ii) $\rightarrow C$: As there is only one F in the defining table for $C(p, q, r)$, the $\rightarrow C$ expression will be:

$$\frac{p, \Gamma \rightarrow \theta, q, r}{\Gamma \rightarrow \theta, C(p, q, r)} \rightarrow C.$$

Note that when a connective, C, is k-ary, no more than k rounds are required to simplify either the $C\rightarrow$ or $\rightarrow C$ expression.

4 The Kleene Search Procedure We extend some of our definitions by specifying that $\Gamma \rightarrow \Omega$ shall take the value F under a complete assignment if each formula of Γ takes the value T under the assignment and each formula of Ω takes the value F under the assignment; otherwise $\Gamma \rightarrow \Omega$ takes the value T. We say that the sequent is *falsifiable* if for some assignment it takes the value F, otherwise it is said to be *valid.*

Lemma 3 *Each of the two sequent rules for a given connective is such that the sequent written below the line is falsifiable if and only if at least one of the sequents written above the line is falsifiable. Equivalently, the sequent written below the line is valid if and only if each of the sequents written above the line is valid.*

Proof: Suppose that $C^k(A_1, \ldots, A_k)$, $\Gamma \rightarrow \theta$ is falsifiable, then there is a complete assignment to the propositional atoms of Γ, θ, and $C(A_1, \ldots, A_k)$ such that $C^k(A_1, \ldots, A_k)$ and all formulas of Γ take the value T and all the formulas of θ take the value F. This complete assignment makes each of the formulas $A_1, \ldots, A_k$ have a particular truth value in such a way that $C^k(A_1, \ldots, A_k)$ has the truth value T. But this implies that one of the sequents above the line is falsifiable since they are chosen so that whenever they take the value T, so does $C^k(A_1, \ldots, A_k)$. The proof in the other direction is similar and the other case is analogous.

We can now adopt what is essentially the plan described by Kleene [7] for searching for counterexamples to the formulas of PL. Given a formula of PL, the procedure describes the construction of a "sequent tree" which is used for finding counterexamples when they exist and which indicates that there is no counterexample when that is the case. We call this procedure the *Kleene Search Procedure* and we adopt the terminology of that procedure.

Lemma 4 *In a sequent tree constructed upward using the Kleene Search Procedure every path will become terminated by round K, where K is the maximum number of connectives appearing in the formulas of the endsequent. Corresponding to any unclosed terminated path, there is a counterexample to the endsequent.*

Proof: In each round of the procedure all the composite formulas appearing at the branch points at the beginning of that round are used as principal formula along each path emanating from that point exactly once. When this happens, the principal operator of that formula is dropped, thus reducing the number of occurrences of propositional connectives in the formula by one. Since no step can be carried out when all of the formulas in the sequent are atomic, there can be at most K rounds in the construction of the sequent tree. Thus, at the end of round K all paths are either terminated and closed or terminated and unclosed. If a path is terminated and unclosed, then the last sequent in the path consists of an antecedent and succedent having only atoms and no atom occurring in both the antecedent and succedent. If we assign T to each atom in the antecedent and F to each atom in the succedent and T to the remaining atoms, then by the preceding lemma as we trace the path downward to the endsequent each formula receives its desired truth value at each step downward and since the tree is finite the endsequent receives the value F under the given assignment and thus is falsified.

We now have the following result.

Theorem 5 *For any propositional logic, PL, there is a mechanical procedure for testing the validity or invalidity of any sequent (and hence any formula of PL).*

Proof: Let $E_1, \ldots, E_k \rightarrow F_1, \ldots, F_m$ be a sequent of PL. Applying the Kleene Search Procedure through K (or fewer) rounds yields a sequent tree all of whose paths are terminated and closed or at least one of whose paths is terminated and unclosed. In the first case the sequent is valid, in the second case it is invalid.

*5 **The formal system GPL*** We would like to employ these results to construct a formal system, GPL, for the valid formulas of PL. We will construct a Gentzen-type system (in the sense of Kleene) for PL. To obtain the system six new rules of inference are added to the set of rules previously associated with our propositional connectives. The new rules are: thinning (in the antecedent or succedent), contraction (in the antecedent or succedent), and interchange (in the antecedent or succedent).

Thinning: $$\frac{\Gamma \rightarrow \theta}{C, \Gamma \rightarrow \theta}\ T\!\rightarrow \qquad \frac{\Gamma \rightarrow \theta}{\Gamma \rightarrow \theta, C}\ \rightarrow\! T$$

Contraction: $$\frac{C, C, \Gamma \rightarrow \theta}{C, \Gamma \rightarrow \theta}\ C\!\rightarrow \qquad \frac{\Gamma \rightarrow \theta, C, C}{\Gamma \rightarrow \theta, C}\ \rightarrow\! C$$

Interchange: $$\frac{\Gamma_1, A, B, \Gamma_2 \rightarrow \theta}{\Gamma_1, B, A, \Gamma_2 \rightarrow \theta}\ I\!\rightarrow \qquad \frac{\Gamma \rightarrow \theta_1, A, B, \theta_2}{\Gamma \rightarrow \theta_1, B, A, \theta_2}\ \rightarrow\! I.$$

Here A, B, C are any formulas and Γ and θ any lists of formulas. These rules we call structural, the former rules logical.

Our formal system GPL is a sequent calculus whose axioms are $P \rightarrow P$ where P can be any atom of PL and whose rules are the sequent rules, both structural and logical, introduced above. However, now we read downward instead of upward. A proof in GPL is written in tree form in the obvious way, each step down the tree being justified by the corresponding sequent rule.

If A is a valid formula we construct the sequent tree for $\rightarrow A$. All the paths of the tree will be terminated and closed. The structural rules can be applied to these terminated paths which proceed upward to a final sequent of the form $P \rightarrow P$, where P is an atom. If we read the tree downward, we have a proof in *GPL* of A.

The procedure for constructing the sequent trees leads us to the following result which is exactly Kleene's Lemma 13.

Lemma 6 (Ancester and Subformula Property) *In a proof in GPL of any given sequent:*

(a) *Each formula occurrence is identifiable as an ancestor of a specific formula occurrence in the endsequent; and the former formula is a subformula of the latter.*
(b) *Each formula part (or whole) is identifiable as an ancestral image of a specific part in the endsequent; the former formula part is identical with the latter.*
(c) *Each occurrence of an operator is identifiable as an ancestor of a specific occurrence of the same operator in the endsequent.*

To paraphrase Kleene and Gentzen: a proof in *GPL* is in a certain, but by no means unique, normal form. In it the only concepts introduced are those which occur in its final result, and hence which must be applied in proving that result. Its result is built up progressively out of the components of the result (subformulas); nothing is first built up and then torn down. It makes no detours. We shall speak of such a proof as *direct.*

Theorem 7 *For any propositional logic PL there exists a Gentzen-type sequent calculus GPL with axioms $P \rightarrow P$, P an atom, and whose rules are logical sequent rules for the introduction of connectives and structural rules and whose proofs are direct and have the subformula property.*

6 Remarks This paper relies heavily upon [7]. What is of interest here, however, is the mechanical procedure for generating and simplifying our sequent rules. It has been a tradition in mathematics that in an axiom system the axioms should possess a certain quality of basic simplicity while remaining adequate for the purposes of the system. The axioms of *GPL* certainly would appear to satisfy this condition. However, in many systems the rules of inference which are used to derive theorems from the axioms and which operate at the metasystem level have been devised with great ingenuity and might appear to some to possess a mysterious effectiveness in their functioning. In contrast, the rules presented here are arrived at in a perfectly straightforward manner while yielding a collection of systems all of which possess very desirable metamathematical characteristics.

It should be remarked that the results presented here have been extended by the author to certain classes of systems containing generalized quantifiers and to corresponding systems of many-valued logics.

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