Some Results on Quantifiers

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Introduction   We shall study (generalized) quantifiers in the framework introduced by Barwise and Cooper in [1], the logical investigation of which has been continued in van Benthem [3] and Keenan and Stavi [2]. The present paper, although self-contained, is in the spirit of [3]. Its main characteristic is a systematic use of a method of proof introduced by van Benthem, which is based on a representation of quantifiers as relations on the natural numbers. With this method we give simplified proofs of a number of van Benthem's results and prove some new ones. In particular, the number-theoretic representation proves convenient for studying monotonicity behavior of quantifiers, and our main result is a characterization of the first-order definable quantifiers in terms of monotonicity.

1 Preliminaries   The basic idea in [1] is that quantifiers correspond to *noun phrases* in natural language. A *quantifier* has the syntactic form

$$D\eta,$$

where D is a *determiner* and η a *set term* (an expression denoting a set of individuals), and denotes, in a model $M = \langle E, \|\cdot\| \rangle$, a set of sets of individuals, namely, those subsets X of E for which $D\eta(X)$ holds (or, more accurately, $D\eta(\zeta)$ holds when $\|\zeta\| = X$). Familiar determiners are, e.g., *all, some, most, exactly one,* and we have (in M)

$$\|all \ \eta \| = \{X \subseteq E: \|\eta\| \subseteq X\}$$
$$\|some \ \eta \| = \{X \subseteq E: \|\eta\| \cap X \neq \phi\}$$
$$\|most \ \eta \| = \{X \subseteq E: |\|\eta\| \cap X| > |\|\eta\| - X|\}$$
$$\|exactly \ one \ \eta \| = \{X \subseteq E: |\|\eta\| \cap X| = 1\}.$$

Since $\|D\eta\| = \|D\|(\|\eta\|)$, this means that the determiners themselves denote functions from $P(E)$ (the power set of E) to $P(P(E))$, so that, e.g., $\|some\|$ is the function defined on $P(E)$ by

$$\|some\|(Y) = \{X \subseteq E: Y \cap X \neq \phi\}.$$

Using this idea Barwise and Cooper give a new formulation of logics (or languages) with generalized quantifiers (for the relation between their formulation and the traditional one cf. [4]). The further details of the syntax and semantics of these logics will not concern us here, however.

In general, the interpretation $\|D\|$ of a determiner symbol 'D' may vary from model to model. In the present paper, however, we shall treat determiner symbols as *constants* in the sense that their interpretation is fixed in each universe E. We may then write

$$D^E$$

instead of $\|D\|^M$ when $M = \langle E, \|\cdot\| \rangle$, and identify the determiner with its interpretation D^E for all (nonempty) E. So a determiner D is, in what follows, a functor assigning to each $E \neq \phi$ a function from $P(E)$ to $P(P(E))$.

Not all such functors deserve to be called 'determiners', however. Below three conditions are stated, and a *determiner* is then defined to be a functor of the above type satisfying all three conditions. The linguistic and philosophical justification of these conditions is discussed in [1], [2], [3], and [4], and will not be repeated here.

Let D be an arbitrary functor as above. The first condition is a strengthening of the requirement that the interpretation of a determiner symbol is determined by the universe:

(CONST)   If $A, B \subseteq E_1 \subseteq E_2$, then $B \in D^{E_1}(A) \Leftrightarrow B \in D^{E_2}(A)$.

This condition enables us to view determiners from a new and, as it has turned out (cf. [3]), fruitful perspective, namely, as binary *relations between sets*: we let

$$DAB$$

mean that, for *some* universe E with $A, B \subseteq E$, $B \in D^E(A)$. By CONST, D is well-defined as a relation. This view of (and notation for) determiners will be used frequently in what follows.

The second condition is the crucial *conservativity* condition, introduced by Barwise and Cooper in [1]:

(CONSERV)   For all A, B, $DAB \Leftrightarrow DA(A \cap B)$.

(The term 'conservativity' is from [3], where it is used for a condition combining CONST and CONSERV above; some reasons for keeping them apart are given in [4]. Barwise and Cooper express CONSERV by saying that the quantifier $D^E(A)$ *lives on A*.)

Our final condition is familiar from traditional logic with generalized quantifiers:

(ISOM)   If E is a universe, f a permutation of E, and $A, B \subseteq E$, then $B \in D^E(A) \Leftrightarrow f[B] \in D^E(f[A])$.

A seemingly more general condition would be obtained by using two universes instead of one, and letting f be a bijection between them. However, it is not hard to see (cf. [4]) that, under CONST, this condition is in fact equivalent to ISOM.

This concludes the definition of determiners. We may note that the class of determiners is closed under Boolean operations: conjunction, disjunction, negation.

It is often convenient to exclude certain trivial determiners:

(NONTRIV)   There are universes E_1, E_2 and $A_1 \subseteq E_1$, $A_2 \subseteq E_2$ such that $D^{E_1}(A_1) \neq \phi$ and $D^{E_2}(A_2) \neq P(E_2)$.

Unless otherwise stated, we always assume in what follows that NONTRIV holds for the determiners involved.

In some contexts, certain strengthenings of NONTRIV seem appropriate, e.g., the following:

(VAR₁)   For each universe E there are $A_1, A_2 \subseteq E$ such that $D^E(A_1) \neq \phi$ and $D^E(A_2) \neq P(E)$.

(VAR₂)   For each universe E there are $E' \supseteq E$ and $A \subseteq E'$ such that $\phi \neq D^{E'}(A) \neq P(E')$.

VAR₁ is from van Benthem [3], and VAR₂ has been proposed by Zwarts [5]. van Benthem also considers a further strengthening:

(VAR*)   For all universes E and all *nonempty* $A \subseteq E$, $\phi \neq D^E(A) \neq P(E)$.

These VAR constraints require a determiner to be more or less "active" on every universe. Note that VAR₁ and VAR₂ are independent, but that both imply NONTRIV and are implied by VAR*. Use of any of the VAR constraints will always be explicitly stated in what follows.

2 Determiners as relations on ω   It follows from CONST, CONSERV, and ISOM that only the *cardinals* $|A - B|$ and $|A \cap B|$ matter for whether DAB holds or not. In general these cardinals may be infinite. However, we now lay down the following constraint:

(FIN)   Only finite universes are considered.

This is a drastic restriction, no doubt. It is partly motivated by the fact that a great deal of the interest of the present theory of determiners comes from applications to natural language, where this restriction is reasonable. But FIN is also a characteristic of certain logical results on quantifiers which appear to be interesting in their own right, and which fail without it. In particular, the representation of determiners as relations on ω developed in this section presupposes FIN.

With FIN, then, the cardinals $|A - B|$ and $|A \cap B|$ above are finite, and it follows that to every determiner D there corresponds a unique relation between natural numbers defined by
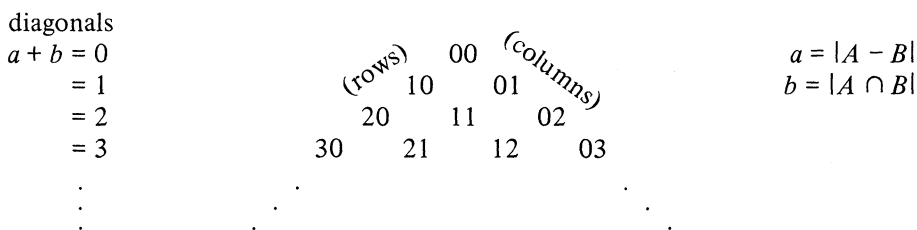
$R(a, b) \Leftrightarrow$ there are A, B with $|A - B| = a$ and $|A \cap B| = b$
such that DAB.

Note that NONTRIV implies that $\phi \neq R \neq \omega \times \omega$. Conversely, given a relation $R \subseteq \omega \times \omega$, the corresponding determiner is given by

$$DAB \Leftrightarrow R(|A - B|, |A \cap B|).$$

To facilitate reading we shall use the *same* symbol ('D', 'D'', etc.) for the determiner and the corresponding relation on ω. A determiner D is thus sometimes viewed as a relation between sets (the *set-theoretic* framework), and sometimes as a relation between natural numbers (the *number-theoretic* framework).

In the number-theoretic framework, then, a determiner can be represented as a subset of the *tree*

diagonals

$a + b = 0$			(rows)	00	(columns)		$a =	A - B	$
$= 1$				10	01		$b =	A \cap B	$
$= 2$			20		11	02
$= 3$		30		21	12	03

For example, the determiner *all* is the first (rightmost) row in the tree, *exactly one* is the second column, and *some* consists of all columns except the first. Note that conjunction of determiners (as relations between sets) corresponds to intersection in the tree, disjunction to union, and negation to complement.

Properties of determiners can also be given two formulations: a set-theoretic and a number-theoretic one. For example, one easily checks that the VAR constraints, which we formulated set-theoretically in Section 1, have the following number-theoretic versions:

(VAR$_1$) Of the pairs $(0,0)$, $(1,0)$, and $(0,1)$, at least one is in D and at least one is not in D.

(VAR$_2$) There is at least one diagonal (in the tree) with one point in D and another point not in D.

(VAR*) On every diagonal except 0 there is at least one point in D and another point not in D.

Here we have used the "tree representation" to express number-theoretic properties of D. This is often convenient but not, of course, necessary; VAR*, for example, can also be formulated thus (with quantification over ω):

(VAR*) $\forall xy(x + y \neq 0 \rightarrow \exists zu(D(x + y - z, z) \wedge \sim D(x + y - u, u)))$.

In Sections 3 and 4 we shall consider some properties of determiners which are natural in the set-theoretic framework. For these simple properties there is an easy mechanical translation procedure into the number-theoretic framework, which we now sketch. To begin with, all properties to be considered are *universal* in the sense that they can be written with only universal set quantifiers in front and no other quantifiers. Consider a property of D of

the form

(1) $\forall E \neq \phi \ \forall A, B \subseteq E \psi,$

where ψ is a truth-functional combination of expressions of the forms DXY and $X \subseteq Y$, where X and Y are Boolean combinations of A, B within the universe E. Looking at the A, B, E in ψ as sets for the moment, we let $x = |A - B|$, $y = |A \cap B|$, $z = |B - A|$, $u = |E - (A \cup B)|$. Then, for every Boolean combination X of A, B in the universe E we can find an arithmetical term t_X in x, y, z, u such that $t_X = |X|$. Now replace, in ψ, every expression DXY by $D(t_{X-Y}, t_{X \cap Y})$ and every expression $X \subseteq Y$ by $t_{X-Y} = 0$. Finally, if the result of this is ψ^*, it is easily seen that the number-theoretic formulation of (1) is

(1') $\forall xyzu \ \psi^*.$

For example, translating the properties

(2) $\forall AB(DAB \rightarrow (DBB \wedge D(A \cap B)B))$
(3) $\forall AB(DAB \wedge DBA \rightarrow A = B)$

by the above method yields

(2') $\forall xyz(D(x, y) \rightarrow (D(0, y + z) \wedge D(0, y)))$
(3') $\forall xyz(D(x, y) \wedge D(z, y) \rightarrow x = z = 0).$

A similar procedure works with more than two set variables. In particular, with three set variables $A, B,$ and C we use the correspondence indicated in Figure 1.
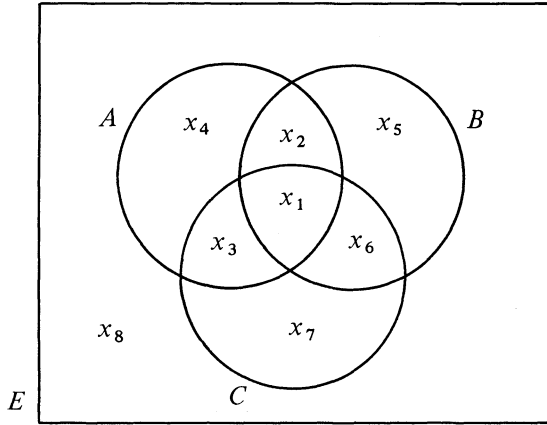

Figure 1.

Then, for example, *transitivity* of D, i.e.,

(4) $\forall ABC(DAB \wedge DBC \rightarrow DAC),$

translates as

(4') $\forall x_1 \ldots x_6(D(x_3 + x_4, x_1 + x_2) \wedge D(x_2 + x_5, x_1 + x_6)$
$\rightarrow D(x_2 + x_4, x_1 + x_3)).$

Since we only deal with universal properties we shall often omit the universal (set or number) quantifiers from expressions such as (4) and (4$'$).

3 Some elementary properties of determiners   The following is a list of properties of D (most of which are studied in [3]) in their set-theoretic formulations, together with their translations, by the procedure from Section 2, into the number-theoretic framework.

Reflexivity	DAA	$D(0, x + y)$
Quasireflexivity	$DAB \to DAA$	$D(x, y) \to D(0, x + y)$
Weak reflexivity	$DAB \to DBB$	$D(x, y) \to D(0, y + z)$
Symmetry	$DAB \to DBA$	$D(x, y) \to D(z, y)$
Asymmetry	$DAB \to \sim DBA$	$D(x, y) \to \sim D(z, y)$
Antisymmetry	$DAB \wedge DBA \to A = B$	$D(x, y) \wedge D(z, y) \to x = z = 0$
Transitivity	$DAB \wedge DBC \to DAC$	$D(x_3 + x_4, x_1 + x_2)$
		$\wedge D(x_2 + x_5, x_1 + x_6)$
		$\to D(x_2 + x_4, x_1 + x_3)$
Circularity	$DAB \wedge DBC \to DCA$	$D(x_3 + x_4, x_1 + x_2)$
		$\wedge D(x_2 + x_5, x_1 + x_6)$
		$\to D(x_6 + x_7, x_1 + x_3)$
Euclidity	$DAB \wedge DAC \to DBC$	$D(x_3 + x_4, x_1 + x_2)$
		$\wedge D(x_2 + x_4, x_1 + x_3)$
		$\to D(x_2 + x_5, x_1 + x_6)$
Antieuclidity	$DAB \wedge DCB \to DAC$	$D(x_3 + x_4, x_1 + x_2)$
		$\wedge D(x_3 + x_7, x_1 + x_6)$
		$\to D(x_2 + x_4, x_1 + x_3)$

Part of the study of determiners concerns the investigation of which determiners have properties like those listed above, or combinations of such properties. In this section we prove some results of this type. Several of these results are in van Benthem [3], but whereas he uses the set-theoretic framework, we use the number-theoretic one. This often has the effect of making the proofs shorter. Indeed, certain facts which need proof in the set-theoretic formulation become obvious in the other formulation. As an example, we may note that it is *immediate* in the number-theoretic framework that

(1)   weak reflexivity implies quasireflexivity,

a fact which is not equally obvious in the set-theoretic framework. (We leave the formulation of a direct set-theoretic proof as an exercise for the reader.)

The next lemma shows that some of the number-theoretic formulations in the above list can be simplified in the sense that the number of variables can be reduced.

Lemma 3.1

(a) *Reflexivity is the property that* $D(0, x)$ *(for all x)*
(b) *Symmetry is the property that* $D(x, y) \leftrightarrow D(0, y)$
(c) *Antisymmetry is the property that* $D(x, y) \to x = 0$.

Proof: (a) is immediate, and the others are almost immediate. We prove (b) for

illustration. Suppose first that $D(x, y) \rightarrow D(z, y)$. Then, given $D(x, y)$ we get $D(0, y)$ (put $z = 0$), and given $D(0, y)$ we get $D(x, y)$ (put $z = x$). Thus $D(x, y) \leftrightarrow D(0, y)$. Conversely, assume that the latter holds. Then, given $D(x, y)$ we get $D(0, y)$, and then $D(z, y)$ (put $x = z$). Thus $D(x, y) \rightarrow D(z, y)$.

We may note that these number-theoretic versions of the three properties in Lemma 3.1 are easy to visualize in the tree. For example, symmetry means that if a point (x, y) is in D then so is the whole column through (x, y).

Lemma 3.1 also gives us new set-theoretic equivalents of the last two properties:

Corollary 3.2   *In the set-theoretic framework,*

(b) *symmetry is equivalent to* $DAB \leftrightarrow D(A \cap B)(A \cap B)$
(c) *antisymmetry is equivalent to* $DAB \rightarrow A \subseteq B$.

Proof: Translating the expressions to the right we get the number-theoretic formulations of Lemma 3.1(b)-(c).

Corollary 3.2(b) was first proved by Barwise and Cooper in [1]; 3.2(c) has also been proved by Zwarts.

Can weak reflexivity also be expressed as a universal property using only two number variables? The answer is yes, and we leave to the reader to verify that weak reflexivity is equivalent to

$$D(x, y) \rightarrow D(0, y) \wedge D(0, y) \rightarrow D(0, y + 1).$$

Now we use Lemma 3.1 to give easy proofs of a number of results on determiners. Let all_e be the determiner (not, it seems, unusual in natural language) defined by $all_e AB \leftrightarrow A \neq \phi$ and $A \subseteq B$, or, if we wish, $all_e(x, y) \leftrightarrow x = 0$ and $y > 0$.

Theorem 3.3

(a) (van Benthem) *There are no asymmetric determiners.*
(b) (Barwise and Cooper) *There are no reflexive and symmetric determiners.*
(c) (van Benthem) *The only symmetric and quasireflexive determiners are* at least k, *for* $k \geqslant 1$. *In particular, under* VAR_1 *the only symmetric and quasireflexive determiner is* some.
(d) *Under* VAR^*, *the only antisymmetric determiners are* all *and* all_e.
(e) (van Benthem) *The only antisymmetric and reflexive determiner is* all.

Proof: (a) Suppose that D is asymmetric. If, for some x, y, $D(x, y)$, then, by the number-theoretic version of asymmetry, $\sim D(x, y)$ (put $z = x$). Thus $\sim D(x, y)$ for all x, y. But this contradicts NONTRIV.

(b) By Lemma 3.1(a) and (b), a reflexive and symmetric determiner would contain all the points in the tree, contradicting NONTRIV.

(c) Evidently the determiners *at least* k, for $k \geqslant 1$, are symmetric and quasireflexive. Now suppose D is a symmetric and quasireflexive determiner. We argue in the tree. Let k be the least y for which there is an x such that (x, y) is in D. k exists by NONTRIV. By symmetry, the whole column k (i.e., the one

through $(0,k)$) is in D. By quasireflexivity, $(0,x)$ is in D for all $x \geqslant k$ (draw a picture!). Thus, by symmetry, all columns y for $y \geqslant k$ are in D, and no other points are in D, i.e., D is *at least k*. Also, by NONTRIV, $k > 0$. Furthermore, it is clear that VAR_1 excludes all *at least k,* for $k \geqslant 2$, leaving only *some.*

(d) and (e) These are immediate from Lemma 3.1(a) and (c).

The tree $\omega \times \omega$ is symmetric with respect to the line through $(0,0)$, $(1,1)$, $(2,2), \ldots$, and determiners and results about determiners can be "mirrored" in this line. To do this, define, for each determiner D, the *co-determiner* $D^\sim$ by

$$D^\sim(x,y) \Leftrightarrow D(y,x),$$

or, equivalently,

$$D^\sim AB \Leftrightarrow DA\overline{B}$$

(where $\overline{B} = E - B$ for some suitable E). We leave to the reader to check that $D^\sim$ is indeed a determiner, i.e., satisfies the three conditions from Section 1. Next, if $\mathcal{P}$ is a property of determiners we say that D is *co-* $\mathcal{P}$ iff $D^\sim$ is $\mathcal{P}$. We then get a symmetric version of Lemma 3.1, i.e.,

co-reflexivity is equivalent to $D(x,0)$ (for all x)
co-symmetry is equivalent to $D(x,y) \leftrightarrow D(x,0)$,

etc. And the symmetric version of Theorem 3.3 tells us that there are no co-asymmetric determiners, no co-reflexive and co-symmetric determiners, that under VAR_1 the only co-symmetric and co-quasireflexive determiner is *some$^\sim$*, i.e., *not all*, that the only co-reflexive and co-antisymmetric determiner is *all$^\sim$*, i.e., *no*, etc. There are also results on combinations of properties with co-properties, e.g.:

Proposition 3.4

(a) *Under VAR_1, there are no reflexive and co-reflexive determiners.*
(b) *There are no symmetric and co-symmetric determiners.*

Proof: (a) If D is both reflexive and co-reflexive, then $D(0,0)$, $D(1,0)$, and $D(0,1)$, contradicting VAR_1.

(b) If D is symmetric and co-symmetric, then, by NONTRIV, one row and one column of the tree are in D. But then the whole tree is in D, contradicting NONTRIV.

Following [1] we may also define the *dual $\breve{D}$* of a determiner D by

$$\breve{D} = {\sim}D^\sim(= {\sim}(D^\sim) = ({\sim}D)^\sim).$$

For example, $\breve{all} = some$. Barwise and Cooper call D *self-dual* if $\breve{D} = D$. In their examples from English they find instances of self-duality only among *partially* defined determiners. This is no accident, since for our (totally defined) determiners we have

Proposition 3.5   *There are no self-dual determiners.*

Proof: If D is self-dual then, for all x, y, $D(x,y) \Leftrightarrow {\sim}D^\sim(x,y) \Leftrightarrow {\sim}D(y,x)$. But this is impossible when $x = y$.

Now let us turn to the properties of transitivity, circularity, euclidity, and antieuclidity. Here the number-theoretic versions seem rather complicated, due to the large number of (number) variables. But this number can be reduced, as we shall see.

We begin with euclidity, i.e.,

(2) $D(x_3 + x_4, x_1 + x_2) \wedge D(x_2 + x_4, x_1 + x_3) \to D(x_2 + x_5, x_1 + x_6)$.

In [3] van Benthem proved that under VAR_1 there are no euclidean determiners. But his proof actually shows that VAR_1 can be replaced by NONTRIV:

Theorem 3.6   *There are no euclidean determiners.*

Proof: (The following proof is essentially a translation of van Benthem's proof.) First let $x_2 = x_3 = x_5 = 0$ in (2). Then we get

(3) $D(x, y) \to D(0, y + z)$,

i.e., weak reflexivity of D. Next, putting $x_2 = x_3 = x_4 = x_6 = 0$ and $x_1 = x_5$ in (2) we get

(4) $D(0, x) \to D(x, x)$.

Next, with $x_1 = x_4 = x_5 = x_6 = 0$ and $x_2 = x_3$ we get

(5) $\cdot D(x, x) \to D(x, 0)$.

Also, from (3) with $y = z = 0$,

(6) $D(x, 0) \to D(0, 0)$.

Finally, letting $x_1 = x_2 = x_3 = x_4 = 0$ in (2) we get

(7) $D(0, 0) \to D(x, y)$.

But it is easily seen that (3)-(7) are inconsistent with NONTRIV.

As a bonus we get the following

Corollary 3.7

(i) *There are no circular determiners.*
(ii) *There are no symmetric and transitive determiners.*

Proof: (i) Circularity is the property that

(8) $D(x_3 + x_4, x_1 + x_2) \wedge D(x_2 + x_5, x_1 + x_6) \to D(x_6 + x_7, x_1 + x_3)$.

Letting $x_2 = x_3 = x_6 = 0$ and $x_4 = x_5$ in (8) we get

(9) $D(x, y) \to D(z, y)$,

i.e., D is symmetric. Further, applying symmetry to circularity in their set-theoretic formulations (for once) gives us

(10) $DBA \wedge DBC \to DAC$.

But this is just euclidity, which is impossible by Theorem 3.6.

(ii) Under symmetry, transitivity clearly implies circularity, so the result follows from (i).

Corollary 3.7 is also proved in [3], under the additional hypothesis of VAR_1.

Now we turn to antieuclidity, i.e.,

(11) $\quad D(x_3 + x_4, x_1 + x_2) \wedge D(x_3 + x_7, x_1 + x_6) \rightarrow D(x_2 + x_4, x_1 + x_3)$.

Theorem 3.8 *Antieuclidity is equivalent to the property*

$$D(x, y) \leftrightarrow D(0, x + y),$$

or, in the set-theoretic framework,

$$DAB \leftrightarrow DAA.$$

Proof: First suppose that D is antieuclidean. Putting $x_2 = x_4 = x_6 = x_7 = 0$ in (11) we get

(12) $\quad D(x, y) \rightarrow D(0, x + y)$.

Likewise, with $x_3 = x_4 = x_7 = 0$ and $x_2 = x_6$ we get

(13) $\quad D(0, x + y) \rightarrow D(x, y)$.

Now suppose that $D(x, y) \leftrightarrow D(0, x + y)$ holds. Assume $D(x_3 + x_4, x_1 + x_2)$. Then, by the hypothesis, $D(0, x_1 + x_2 + x_3 + x_4)$, and so, again by the hypothesis, $D(x_2 + x_4, x_1 + x_3)$. This proves (11) (and also that the second conjunct in (11) is redundant).

The theorem tells us that D is antieuclidean iff whenever (x, y) is in D, so is the whole diagonal $x + y$. This gives a characterization of the antieuclidean determiners:

Corollary 3.9 *D is antieuclidean iff there is a set $X \subseteq \omega$ such that $D(x, y) \leftrightarrow x + y \in X$ (i.e., $DAB \leftrightarrow |A| \in X$).*

Corollary 3.10 (Zwarts) *Under VAR_2 there are no antieuclidean determiners.*

VAR_1 does not exclude antieuclidity; an example is the determiner defined by $DAB \leftrightarrow |A| = 0 \vee |A| = 2$.

By the proof of Theorem 3.8, antieuclidity can also be written as

$$DAB \rightarrow DAC.$$

Thus antieuclidity implies transitivity, and from Corollary 3.7 we get

Corollary 3.11 *There are no antieuclidean and symmetric determiners.*

Finally, we look at the most interesting of the four properties under consideration, viz., transitivity:

(14) $\quad D(x_3 + x_4, x_1 + x_2) \wedge D(x_2 + x_5, x_1 + x_6) \rightarrow D(x_2 + x_4, x_1 + x_3)$.

We prove a series of lemmas which lead up to a characterization of transitivity.

Lemma 3.12 *If D is transitive, then*

$$x, y > 0 \wedge D(x, y) \rightarrow D(x + 1, y - 1).$$

Proof: Let $x_2 = x_6 = 1$, $x_4 = x_5 + 1$, and $x_3 = 0$ in (14), and the lemma follows.

Corollary 3.13 (van Benthem)   *If D is transitive, then*

$$D(x, y) \rightarrow x = 0 \vee D(x + y, 0),$$

or, equivalently,

$$DAB \rightarrow A \subseteq B \vee DA\phi.$$

Proof: Use Lemma 3.12 y times.

Lemma 3.14   *If D is transitive, then*

$$D(x + y, 0) \rightarrow D(x, y),$$

or, equivalently,

$$DA\phi \rightarrow DAB.$$

Proof: Let $x_1 = x_2 = x_6 = 0$ and $x_5 = x_3 + x_4$ in (14).

From Corollary 3.13 and Lemma 3.14 we see that if D is transitive, and one point in a diagonal which is not its rightmost point is in D, then the whole diagonal is in D. But what if the rightmost point *is* in D? The next lemma gives an answer.

Lemma 3.15   *Let D be a transitive determiner. If $D(0, x + y)$, and if $D(u, v)$ for some u, v such that $u \neq 0$ and $u + v > x + y$, then $D(x, y)$.*

Proof: Since $u \neq 0$ we may in fact suppose that $u \geqslant x$ and $v \geqslant y$. For we can clearly assume that $x \neq 0$, and if we put $u' = x$ and $v' = u + v - x$ we get $u' \geqslant x$, $v' \geqslant y$, and $u' + v' = u + v$. Furthermore, by Corollary 3.13 and Lemma 3.14 it follows that $D(u', v')$. This shows that the assumption that $u \geqslant x$ and $v \geqslant y$ is legitimate. Now put $x_3 = x_4 = 0$ in (14), and the lemma follows.

Now we can characterize the transitive determiners. If $X, Y \subseteq \omega$, we let $X < Y$ mean that all elements of X are smaller than all elements of Y. In particular, if $X < Y$ and X is infinite, then $Y = \phi$. Also, $X < \phi$ for all X.

Theorem 3.16   *D is transitive iff there are subsets X, Y of ω such that $X < Y$ and*

$$D(x, y) \Leftrightarrow x + y \in X \vee (x = 0 \wedge y \in Y)$$

(in other words, $DAB \Leftrightarrow |A| \in X \vee (A \subseteq B \wedge |A| \in Y)$).

Proof: For the "if" part we use the set-theoretic formulation. Suppose that there are $X, Y \subseteq \omega$ such that the condition of the theorem holds, and that DAB and DBC. If $|A| \in X$, then DAC. So suppose that $A \subseteq B$ and $|A| \in Y$. Then $|A| \leqslant |B|$, so $|B| \notin X$ (since $X < Y$). Thus $B \subseteq C$ and $|B| \in Y$. It follows that $A \subseteq C$ and hence that DAC, and we have shown that D is transitive.

Now suppose that D is transitive. Let $X = \{x : D(x, 0)\}$ and let $Y = \{y : X < y$ and $D(0, y)\}$. Then $X < Y$, and if $x + y \in X$, then, by Lemma 3.14, $D(x, y)$. Suppose now that $D(x, y)$ holds. If $x \neq 0$, then $x + y \in X$, by Corollary

3.13, so suppose that $x = 0$. We show that $y \in Y$ or $x + y = y \in X$. If $X < y$ then $y \in Y$ by definition of Y. On the other hand, if *not* $X < y$, i.e., if $y \leqslant x'$ for some $x' \in X$, then $D(y, 0)$, so $y \in X$. This is trivial if $y = x'$, and follows from Lemma 3.15 if $y < x'$. Thus we have shown that the desired condition holds.

This theorem gives a rather good idea of what transitive determiners look like in the tree (draw a picture!). Also, we get the following corollaries:

Corollary 3.17   *Under VAR*, transitivity and antisymmetry are* equivalent *properties. Thus, under VAR*, the only transitive determiners are* all *and* all$_e$.

Proof: Suppose that D is transitive. VAR* implies that, in Theorem 3.16, $X = \phi$ or $X = \{0\}$. Thus $D(x, y)$ implies that $x = 0$, which, by Lemma 3.1(c), means that D is antisymmetric. Conversely, if D is antisymmetric, then, by Theorem 3.3(d), D is either *all* or *all$_e$*, both of which are transitive.

Corollary 3.18 (van Benthem)   *The only transitive and reflexive determiners are* allAB or $|A| < n$ *(for any natural number n). In particular, under VAR$_1$ the only transitive and reflexive determiner is* all.

Proof: Clearly these determiners are all transitive and reflexive. Conversely, if D is transitive and reflexive, let n be the least x such that *not* $D(x, 0)$. n exists by Lemma 3.14 and NONTRIV. Since reflexivity means that $D(0, x)$ for all x we easily see that, in Theorem 3.16, $X = \{0, \ldots, n - 1\}$ and $Y = \{x : x \geqslant n\}$, which gives the desired result.

The last statement of Corollary 3.18 does not hold if we assume VAR$_2$ instead of VAR$_1$. We may also note that transitivity can be expressed by a universal statement with *four* number variables. For it is clear from the proofs of Corollary 3.13–Theorem 3.16 that transitivity is equivalent to the conjunction of the three (universal) expressions in Corollary 3.13, Lemma 3.14, and Lemma 3.15, respectively.

van Benthem notes in [3] that the present type of properties of determiners can be regarded as *inference patterns,* and that results such as those of this section answer questions which, in a sense, reverse Aristotle's approach to logic: instead of asking which inference patterns are satisfied by certain given logical constants, we ask which logical constants realize certain given inference patterns. For example, for the four inference patterns

(15)   *DAB*   *DAB*   *DAB*   *DAB*
         DBC   *DBC*   *DAC*   *DCB*
         ————   ————   ————   ————
         DAC   *DCA*   *DBC*   *DAC*
      (transitivity)   (circularity)   (euclidity)   (antieuclidity)

we have already obtained definite answers. Are there other interesting inference patterns? We end this section by considering the case of inference patterns with *three* set variables 'A', 'B', and 'C', leaving the cases of one or two set variables to the reader. We also assume that there are *two* premises, and that in each premise and conclusion two *different* set variables occur. The patterns (15) comprise (up to notational variants (including changes of the order between the

premises)) all inference patterns where the two premises share a "middle term" which does not occur in the conclusion. If the premises do not share a middle term, there are (up to notational variants) only two patterns, namely,

(16) DAB DAB
       DBA DBA
       ─────   ─────
       DAC DCA.

However, it is easy to check that the second premise in (16) is redundant, so that the patterns reduce to $DAB \rightarrow DAC$ and $DAB \rightarrow DCA$. The second of these implies circularity, and is thus impossible by Corollary 3.7. The first is, as was noted in connection with Corollary 3.11, equivalent to antieuclidity.

An inventory of the remaining possibilities shows that (up to notational variants) there are only four more inference patterns of the present type, namely,

(17) DAB DAB DAB DAB
       DAC DCA DCB DBC
       ─────   ─────   ─────   ─────
       DBA DBA DBA DBA

(these are the patterns where the middle term occurs in the conclusion). These patterns resemble symmetry, and it is easy to check that, in fact, the first three of them are all equivalent to symmetry. The fourth, however, which we may call *weak symmetry*, is strictly weaker than symmetry, as is shown by the determiner *noAB and A ≠ φ* (note that VAR* holds for this determiner). In the number-theoretic framework, weak symmetry becomes (with $x = x_3 + x_4$)

(18) $D(x, x_1 + x_2) \wedge D(x_2 + x_5, x_1 + x_6) \rightarrow D(x_5 + x_6, x_1 + x_2)$.

This can be simplified as follows.

Theorem 3.19   *Weak symmetry is the property that*

$$D(x, y) \wedge D(y + z - u, u) \rightarrow D(z, y).$$

Proof: If the property in the theorem holds, then we obtain (18) by letting $y = x_1 + x_2$, $z = x_5 + x_6$, and $u = x_1 + x_6$. Conversely, suppose that (18) holds, and assume $D(x, y)$ and $D(y + z - u, u)$. Let x_1 be the smallest of y, u, and let $x_2 = y - x_1$, $x_6 = u - x_1$. Now $y + z \geqslant u$ by assumption, i.e., $x_1 + x_2 + z \geqslant x_1 + x_6$, and so $x_2 + z \geqslant x_6$. We let $x_5 = z - x_6$. Then $x_5 \geqslant 0$, for if $x_6 \neq 0$, then $x_2 = 0$ (by the definition of x_1, x_2, x_6), and thus $z \geqslant x_6$. Substitution in (18) now yields the property in the theorem.

By this theorem, weak symmetry means that if (x, y) is in D, and if some point on the diagonal $y + z$ is in D, then (z, y) is in D. The following corollary is immediate.

Corollary 3.20   *Under VAR*, weak symmetry is the property that*

$$D(x, y) \wedge y + z \neq \phi \rightarrow D(z, y),$$

or, equivalently,

$$DAB \land B \neq \phi \rightarrow DBA.$$

4 Monotonicity and continuity   In [3] van Benthem gives a model-theoretic proof that all *doubly monotone* determiners are first-order definable (in the sense that there is a sentence ψ of first-order predicate logic in two unary predicate symbols and identity such that $B \in D^E(A) \leftrightarrow \langle E, A, B \rangle \models \psi$), provided that only finite models are considered. In this section we use the number-theoretic representation of determiners to show a stronger result, namely, that all *left continuous* determiners are first-order definable (definitions of mono-tonicity and continuity follow presently). This will also give us a characteriza-tion of the first-order definable determiners.

We begin with a few definitions. The following *continuity* properties are from [3]:

(RIGHT) CONT $B \subseteq B' \subseteq B'' \land DAB \land DAB'' \rightarrow DAB'.$
LEFT CONT $A \subseteq A' \subseteq A'' \land DAB \land DA''B \rightarrow DA'B.$
DOUBLE CONT $A \subseteq A' \subseteq A'' \land B \subseteq B' \subseteq B'' \land DAB \land DA''B'' \rightarrow DA'B'.$

It is straightforward to verify that the corresponding number-theoretic versions are

CONT $x + y = x' + y' = x'' + y'' \land y \leqslant y' \leqslant y'' \land D(x, y) \land D(x'', y'')$
$\rightarrow D(x', y').$
LEFT CONT $x \leqslant x' \leqslant x'' \land y \leqslant y' \leqslant y'' \land D(x, y) \land D(x'', y'') \rightarrow D(x', y').$
DOUBLE CONT $x + y \leqslant x' + y' \leqslant x'' + y'' \land y \leqslant y' \leqslant y'' \land D(x, y) \land D(x'', y'')$
$\rightarrow D(x', y').$

These express various *convexity* properties in the tree. Thus, (right) continuity means that if two points on a diagonal are in D, then so are all points in between. The meaning of left and double continuity is conveniently illustrated in Figure 2.
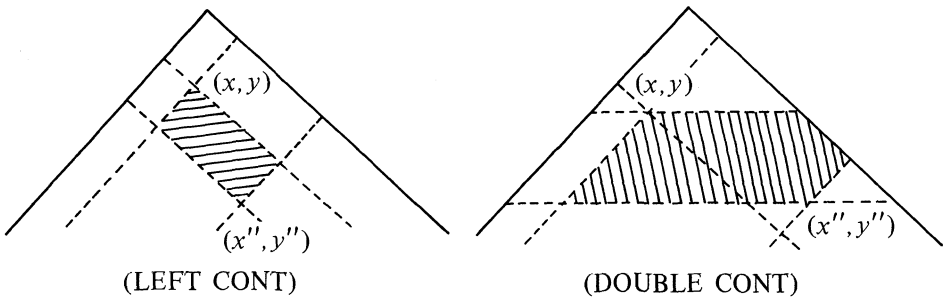

(LEFT CONT)   (DOUBLE CONT)

Figure 2.

(If (x, y) and (x'', y'') are in D, then so are all points in the shaded area.)

Corresponding to the three continuity properties there are three types of *monotonicity* properties. We exemplify with *downward* ($\downarrow$) monotonicity, in both versions:

MON↓ $B' \subseteq B \wedge DAB \rightarrow DAB'$

$x + y = x' + y' \wedge x \leqslant x' \wedge D(x, y) \rightarrow D(x', y').$

↓MON $A' \subseteq A \wedge DAB \rightarrow DA'B$

$x' \leqslant x \wedge y' \leqslant y \wedge D(x, y) \rightarrow D(x', y').$

↓MON↓ $A' \subseteq A \wedge B' \subseteq B \wedge DAB \rightarrow DA'B'$

$x' + y' \leqslant x + y \wedge x' \leqslant x \wedge D(x, y) \rightarrow D(x', y').$

MON↓ means that if a point is in D, then so are all points to the left on the same diagonal. ↓MON and ↓MON↓ can be illustrated in the tree by letting (x, y) in the above figure be $(0, 0)$.

The reader can check the usefulness of these "tree representations" right away by verifying that ↓MON↓ is equivalent to MON↓ *and* ↓MON, whereas DOUBLE CONT implies, but is *not* implied by, CONT *and* LEFT CONT. Also, LEFT (DOUBLE) CONT is implied by ↓MON (↓MON↓), and, if $(0, 0)$ is in D, the two are equivalent.

We also define *upward* monotonicity, MON↑ and ↑MON, as well as ↑MON↑, ↑MON↓, and ↓MON↑, in the obvious way.

In [1], right monotonicity is called simply *monotonicity,* and left monotonicity is called (*anti-*)*persistence.* These properties are well known from model theory and recursion theory, and Barwise and Cooper apply them in [1] to the semantics of natural language. They are important in the logical theory of determiners as well, as we shall see (cf. also [3]).

The determiner *most* is MON↑ but not ↑MON. Clearly *most* is not first-order definable (for a proof, cf. [1]). This is no accident, since, as we shall show, all left monotone determiners are first-order definable.

Theorem 4.1   *If D is ↓MON, then D is first-order definable. More precisely, D is definable by a* universal *first-order sentence, namely, a conjunction of sentences of the form*

$$|A - B| \leqslant n \vee |A \cap B| \leqslant k$$

(here one of the disjuncts may be missing).

Proof: Suppose that D is ↓MON. Then, by NONTRIV, there are x_0, y_0 such that for all $x > x_0$ and all $y > y_0$ not $D(x, y)$. In fact, we may assume that $D(x_0, y_0)$. To see this, let P be the set of pairs (x, y) with the desired property. Take (x_0, y_0) in P with $x_0 + y_0$ minimal. Then $(x_0 - 1, y_0)$ is not in P, so there are $x_1 \geqslant x_0$ and $y_1 > y_0$ such that $D(x_1, y_1)$. But then, by ↓MON, $D(x_0, y_0)$.

Now consider the *columns* $0, 1, \ldots, y_0, \ldots$. Define, for all i,

$$k_i = \begin{cases} \omega, \text{ if there are arbitrarily large } x \text{ such that } D(x, i) \\ \text{the largest } x \text{ such that } D(x, i), \text{ if there is such a largest } x \\ -1, \text{ if there are no } x \text{ such that } D(x, i). \end{cases}$$

↓MON now implies that the following holds: (a) if $k_i = \omega$ then the whole column i is in D, and $k_j = \omega$ for $j < i$; (b) if k_i is a natural number then $D(x, i)$ for all $x \leqslant k_i$; (c) if $k_i = -1$ then $k_j = -1$ for $j > i$; (d) if $i > j$ then $k_i \geqslant k_j$.

Now let i_0 be the smallest i such that $k_i \neq \omega$. i_0 exists by NONTRIV (in fact $i_0 \leqslant y_0 + 1$ by our assumptions), and it follows that k_i has a smallest value, k, say. Let i_1 be the smallest i such that $k_i = k$. The numbers $k_{i_0} \geqslant k_{i_0+1} \geqslant$

$\ldots \geqslant k_{i_1}$ completely specify D, as shown in Figure 3 (where we have assumed that $i_0 > 0$ and $k > -1$):
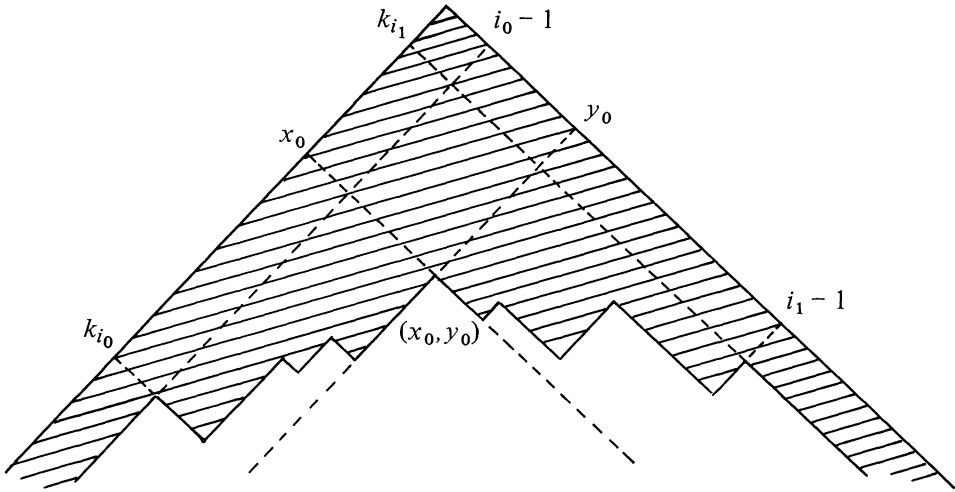

Figure 3.

Also, it is clear that D can be represented as the *intersection* of a finite number of determiners of the form shown in Figure 4; i.e., $|A - B| \leqslant n \lor |A \cap B| \leqslant k$. This completes the proof of the theorem.
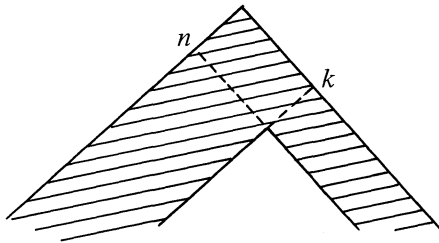

Figure 4.

Corollary 4.2 *If D is $\uparrow$MON, then D is definable by an* existential *first-order sentence, namely, a disjunction of sentences of the type*

$$|A - B| > n \land |A \cap B| > k$$

(where one of the conjuncts may be missing).

Proof: If D is $\uparrow$MON then $\sim D$ is $\downarrow$MON, so the result follows from Theorem 4.1.

Note that no VAR constraints are used in these results. Indeed, it is easily verified that, under VAR*, the only $\downarrow$MON determiners are *no* and *all*, and that the only $\uparrow$MON determiners are *some* and *not all*.

The next result strengthens Theorem 4.1.

Theorem 4.3   *Every LEFT CONT determiner is first-order definable.*

Proof: The proof is a modification of the proof of Theorem 4.1. Suppose that D is LEFT CONT, and let

$$x_0 = \text{the least } x \text{ such that for some } y, D(x, y)$$
$$y_0 = \text{the least } y \text{ such that } D(x_0, y)$$
$$y_1 = \text{the least } y \text{ such that for some } x, D(x, y)$$
$$x_1 = \text{the least } x \text{ such that } D(x, y_1).$$

Thus $x_0 \leqslant x_1$ and $y_1 \leqslant y_0$. Now define a new determiner D' by

$$D'(x, y) \Leftrightarrow D(x, y) \vee (x_0 \leqslant x \leqslant x_1 \wedge y_1 \leqslant y \leqslant y_0).$$

Then, *in the subtree* $[x \geqslant x_0, y \geqslant y_1]$, D' is in fact $\downarrow$MON. There are two cases:

Case 1: For all $x \geqslant x_0$ and all $y \geqslant y_1$ there are $x' > x$ and $y' > y$ such that $D'(x', y')$. Then D' is the whole subtree $[x \geqslant x_0, y \geqslant y_1]$.

Case 2: There are $x_2 \geqslant x_0$ and $y_2 \geqslant y_1$ such that for all $x' > x_2$ and all $y' > y_2$ *not* $D'(x', y')$. Then we argue exactly as in the proof of Theorem 4.1 to find, in the subtree under consideration, the familiar pattern for D'.

This completes the description of D' (in the whole tree), and it is clear that D' is first-order definable. But then, so is D, since D differs from D' at most in the finite region $[x_0 \leqslant x \leqslant x_1, y_1 \leqslant y \leqslant y_0]$. The pattern for D may be depicted (in Case 2) as shown in Figure 5. This completes the proof.
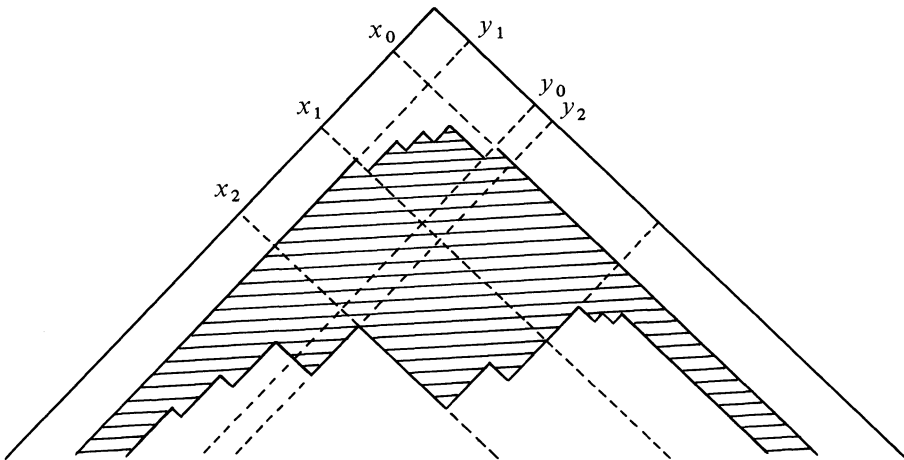

Figure 5.

From the proof of Theorem 4.3 one can find a general form of the first-order definition of LEFT CONT determiners, but we content ourselves with stating it in the case that VAR* holds.

Theorem 4.4   *Under VAR*, each LEFT CONT determiner is either* all *or* no, *or definable as a conjunction where the first conjunct is either* $|A - B| \geqslant 1$ *or* $|A \cap B| \geqslant 1$, *and the other conjuncts are of the form*

$$|A - B| \leqslant n \vee |A \cap B| \leqslant k$$

(where one of the disjuncts may be missing).

Proof: Suppose first that $D(0,0)$ holds. If $D(1,0)$, then, by VAR* and LEFT CONT, D is *no*. Similarly, if $D(0,1)$, D is *all*. Now suppose that *not* $D(0,0)$. Then the cases $D(1,0)$ and $D(0,1)$ can be treated by considering subtrees as in the proof of Theorem 4.3, which gives the desired result.

Finally, we show that Theorem 4.3 can be used to characterize the first-order definable determiners.

Theorem 4.5   *D is first-order definable iff D is a disjunction of LEFT CONT determiners.*

Proof: (outline) If D is a disjunction of LEFT CONT determiners, then D is first-order definable by Theorem 4.3. For the converse we can use a well-known description of monadic first-order logic with identity. It can be seen, if we bear in mind the conditions on determiners from Section 1, that this description (which is mentioned in [3]) yields that any first-order definable determiner can be written as a Boolean combination of determiners of the types

$$|A - B| \leqslant n,$$
$$|A \cap B| \leqslant n,$$
$$|A| \leqslant n.$$

In fact, the last form is superfluous, since, if $DAB \leftrightarrow |A| \leqslant n$, then

$$DAB \leftrightarrow \bigvee_{i+k \leqslant n} (|A - B| = i \wedge |A \cap B| = k).$$

Thus, after transformation into disjunctive normal form, we get disjuncts of the form (disregarding some "degenerate" cases)

$$n \leqslant |A - B| \leqslant n + k \wedge m \leqslant |A \cap B| \leqslant m + 1.$$

Clearly all determiners of these forms are LEFT CONT.

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