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Automorphisms of ω-Octahedral Graphs

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1 Preliminaries This paper is closely related to [2] which deals with automorphisms of the ω-graph Q_N associated with the ω-cube Q^N and [3] which deals with the ω-graph Oc_N associated with the ω-octahedron Oc^N. We use the notations, terminology, and results of [2]. The propositions of [2] are referred to as A1.1, A1.2, . . ., A2.1, A2.2, . . . etc., those of [3] as B1.1, B1.2, . . ., B2.1, B2.2, . . . etc.

For $n \geqslant 1$ the n-octahedral graph is defined as the complete n-partite graph $K(2, \ldots, 2)$ with two vertices in each of its partite sets ([4], p. 69). Let Oc_n have $\mu = (0, \ldots, 2n - 1)$ as set of vertices and $((0,1), \ldots, (2n - 2, 2n - 1))$ as class of its partite sets. Define f as the permutation of μ which for $0 \leqslant k \leqslant n - 1$ interchanges $2k$ and $2k + 1$. Call the vertices p and q of Oc_n *opposite,* if they correspond to each other under f, then p and q are adjacent, iff they are not opposite. Throughout this paper the symbols ν, ν_0, ν_1 denote nonempty sets, and μ and μ_ν stand for sets of cardinality $\geqslant 2$. An *involution without fixed points* (abbreviated: iwfp) of a set μ is a permutation f of μ such that $f^2 = i_\mu$ and $f(x) \neq x$, for $x \in \mu$. The iwfp f of μ is an *ω-iwfp,* if it has a partial recursive one-to-one extension. With every iwfp f of μ we associate a graph $G_f = \langle \mu, \theta \rangle$, where θ consists of all numbers $can(x,y) \in [\mu;2]$ such that $f(x) \neq y$. Note that the iwfp f is uniquely determined by G_f. The graph $G = \langle \mu, \theta \rangle$ is *octahedral,* if $G = G_f$, for some iwfp f of μ. The octahedral graph $G_f = \langle \mu, \theta \rangle$ is *ω-octahedral,* if f is an ω-iwfp of μ. The vertices p and q of the octahedral graph G_f are *opposite,* if $f(p) = q$; thus p and q are adjacent iff they are not opposite. According to B2.2 an ω-octahedral graph $G_f = \langle \mu, \theta \rangle$ is a uniform ω-graph for which there exists a nonzero RET N such that $Req\, \mu = 2N$ and $Req\, \theta = 2N(N - 1)$. Define the functions d_0 and d_1 by: $\delta d_0 = \delta d_1 = \varepsilon$, $d_0(x) = 2x$, $d_1(x) = 2x + 1$. With every set ν we associate the sets $\nu_0 = d_0(\nu)$, $\nu_1 = d_1(\nu)$, and $\mu_\nu = \nu_0 \cup \nu_1$. The *standard ω-iwfp associated with the set ν* is the ω-iwfp f of μ_ν such that $f(2x) = 2x + 1$ and $f(2x + 1) = 2x$, for $x \in \nu$. The *standard ω-octahedral graph Oc_ν associated with the set ν* is the ω-graph $G_f = \langle \mu_\nu, \theta_\nu \rangle$,

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where f is the standard ω-iwfp of μ_ν associated with the set ν. According to B2.3 a graph is ω-octahedral iff it is ω-isomorphic to some standard ω-octahedral graph. When studying the effective automorphisms of ω-octahedral graphs we may therefore restrict our attention to standard ω-octahedral graphs.

For nonempty sets α and β we have by B2.4: $\alpha \simeq \beta$ iff $Oc_\alpha \cong_\omega Oc_\beta$. For a nonzero RET N we define Oc_N as Oc_ν, for any $\nu \in N$. Thus Oc_N is uniquely determined by N up to ω-isomorphism.

2 ***Automorphisms of*** $\boldsymbol{Oc_\nu}$ An *automorphism* (*ω-automorphism*) of Oc_ν is an isomorphism (ω-isomorphism) from Oc_ν onto itself. We choose the notion of an ω-automorphism of Oc_ν as the formal equivalent of the intuitive notion of an effective automorphism of Oc_ν. We refer to [2] (p. 122) for the definitions of the groups $Per(\nu)$, $Per_\omega(\nu)$, P_ν of permutations. Let

$Aut\ Oc_\nu$ = the group of all automorphisms of Oc_ν,
$Aut_\omega\ Oc_\nu$ = the group of all ω-automorphisms of Oc_ν,

and put $\sigma_x = (2x, 2x+1)$, for $x \in \varepsilon$. An automorphism of $Oc_\nu = \langle \mu_\nu, \theta_\nu \rangle$ is a permutation of μ_ν which preserves adjacency or equivalently which preserves nonadjacency, i.e., which maps each pair of opposite vertices of Oc_ν onto a pair of opposite vertices. A permutation g of μ_ν is therefore an automorphism of $Oc_\nu = \langle \mu_\nu, \theta_\nu \rangle$ iff it permutes $\{\sigma_x | x \in \nu\}$. In symbols,

(1) $g \in Aut\ Oc_\nu \Longleftrightarrow (\exists f)[f \in Per(\nu)\ \&\ g(\sigma_x) = \sigma_{f(x)}, \textit{for } x \in \nu]$.

If the automorphism g of Oc_ν and the permutation f of ν are related by (1), then g is not uniquely determined by f. For given f, the function g can for each $x \in \nu$ still map σ_x onto $\sigma_{f(x)}$ in either of two ways, namely:

(i) $g(2x) = 2f(x) + 1, g(2x+1) = 2f(x)$ or
(ii) $g(2x) = 2f(x), g(2x+1) = 2f(x) + 1$.

Consider the case where ν is finite, say $\nu = (0, \ldots, n-1)$, for $n \geqslant 1$, hence $\mu_\nu = (0, \ldots, 2n-1)$. Then the function f such that $g(\sigma_x) = \sigma_{f(x)}$, for $x \in \nu$, can be chosen in $n!$ different ways. For each choice of f we can still choose g in 2^n different ways by choosing a subset α of ν such that: (i) holds for $x \in \alpha$ and (ii) for $x \notin \alpha$. Thus if ν is a finite set of cardinality n, the automorphism group of Oc_ν is a finite group of cardinality $2^n \cdot n!$ Let us now examine $Aut\ Oc_\nu$ for an arbitrary set ν, i.e., let us drop the condition that ν be finite. We define

(2) $H(\nu) =_{df} \{g \in Aut\ Oc_\nu | g(\sigma_x) = \sigma_x, \textit{for } x \in \nu\}$,
(3) $K(\nu) =_{df} \{g \in Aut\ Oc_\nu | g(x) \equiv x (\text{mod } 2), \textit{for } x \in \mu_\nu\}$.

Note that $H(\nu), K(\nu) \leqslant Aut\ Oc_\nu$. In order to characterize $H(\nu)$ and $K(\nu)$ in a different manner we define for $\alpha \subset \nu$, $h \in Per(\nu)$,

$$(4) \quad \delta\phi_\alpha = \mu_\nu, \begin{cases} \phi_\alpha(2x) = 2x+1,\ \phi_\alpha(2x+1) = 2x, \textit{ for } x \in \alpha, \\ \phi_\alpha(2x) = 2x,\ \phi_\alpha(2x+1) = 2x+1, \textit{ for } x \notin \alpha, \end{cases}$$

$$(5) \quad \delta\psi_h = \mu_\nu,\ \psi_h(2x) = 2h(x),\ \psi_h(2x+1) = 2h(x)+1, \textit{ for } x \in \nu.$$

We write $S(\nu)$ for the class of all subsets of ν and $\alpha \oplus \beta$ for the symmetric difference of α and β.

Proposition C2.1 *For every set* ν,

(a) $H(\nu) = \{\phi_\alpha \in Aut\ Oc_\nu | \alpha \in S(\nu)\}$,
(b) $K(\nu) = \{\psi_h \in Aut\ Oc_\nu | h \in Per(\nu)\}$,
(c) $H(\nu) \cong \langle S(\nu), \oplus\rangle$ *and* $K(\nu) \cong Per(\nu)$.

Proof: Denote the right sides of (a) and (b) by $H^*(\nu)$ and $K^*(\nu)$ respectively. Relations (4) and (5) imply that $H^*(\nu) \subset H(\nu)$ and $K^*(\nu) \subset K(\nu)$. Now assume $f \in H(\nu)$ and $g \in K(\nu)$. Put $\alpha = \{x \in \nu | f(2x) = 2x + 1\}$, $h(x)$ = the number y such that $g(\sigma_x) = \sigma_y$. Then $f = \phi_\alpha$, $g = \psi_h$, hence $H(\nu) = H^*(\nu)$ and $K(\nu) = K^*(\nu)$. This proves (a) and (b). As far as (c) is concerned, $\psi_h \psi_k = \psi_{hk}$, for $h, k \in Per(\nu)$, so that $K(\nu) \cong Per(\nu)$. The mapping $\alpha \to \phi_\alpha$ maps $S(\nu)$ onto $H(\nu)$ by (4) and C2.1(a). This mapping is one-to-one, since $\alpha = \{x \in \nu | \phi_\alpha(2x) = 2x + 1\}$, for $\alpha \in S(\nu)$. Now assume $\alpha, \beta \in S(\nu)$. Then $\phi_\alpha\phi_\beta(2x) = 2x + 1$, for $x \in \alpha \oplus \beta$, while $\phi_\alpha\phi_\beta(2x) = 2x$, for $x \notin \alpha \oplus \beta$. Thus $\phi_\alpha\phi_\beta(2x) = \phi_{\alpha\oplus\beta}(2x)$ and similarly we see that $\phi_\alpha\phi_\beta(2x + 1) = \phi_{\alpha\oplus\beta}(2x + 1)$. Hence $\phi_\alpha\phi_\beta = \phi_{\alpha\oplus\beta}$ and $\langle S(\nu), \oplus\rangle \cong H(\nu)$. This completes the proof of (c).

Let $H, K \leqslant G$, where G is a group with unit element i. We write $G = H \times K$, if G is the *semidirect product of H by K*, i.e., ([5], p. 212), if

(6) $HK = G$,
(7) $H \cap K = (i)$,
(8) $H \lhd G$.

If we also have $K \lhd G$ we call G the *direct* product of H and K. For a set ν we define

(9) $H_\omega(\nu) = \{g \in H(\nu) | g$ *has a partial recursive 1–1 extension*$\}$,
(10) $K_\omega(\nu) = \{g \in K(\nu) | g$ *has a partial recursive 1–1 extension*$\}$,

so that $H_\omega(\nu) \leqslant H(\nu)$, $K_\omega(\nu) \leqslant K(\nu)$ and $H_\omega(\nu)$, $K_\omega(\nu) \leqslant Aut_\omega\ Oc_\nu$. We also see that $H_\omega(\nu) = H(\nu)$, $K_\omega(\nu) = K(\nu)$, if ν is finite, while $H_\omega(\nu) < H(\nu)$, $K_\omega(\nu) < K(\nu)$, if ν is infinite. For in the latter case, $H_\omega(\nu)$ and $K_\omega(\nu)$ are denumerable, while $H(\nu)$ and $K(\nu)$ have cardinality c.

Proposition C2.2 *For every set* ν,

(a) $Aut\ Oc_\nu = H(\nu) \times K(\nu)$,
(b) $Aut_\omega\ Oc_\nu = H_\omega(\nu) \times K_\omega(\nu)$.

Proof: To prove (a) we shall verify (6), (7), and (8) for $H = H(\nu)$, $K = K(\nu)$, and $G = Aut\ Oc_\nu$.

Re (6). Since $H(\nu), K(\nu) \leqslant Aut\ Oc_\nu$, it suffices to prove

$$g \in Aut\ Oc_\nu \Rightarrow (\exists\alpha)(\exists h)[\alpha \in S(\nu)\ \&\ h \in Per(\nu)\ \&\ g = \phi_\alpha\psi_h].$$

Assume the hypothesis. By (1) there is an $f \in Per(\nu)$ such that $g(\sigma_x) = \sigma_{f(x)}$, for $x \in \nu$. Then ψ_f is an automorphism of Oc_ν by C2.1(b), hence so are ψ_f^{-1} and $g\psi_f^{-1}$. However,

$$g\psi_f^{-1}(\sigma_x) = g\psi_{f^{-1}}(\sigma_x) = g(\sigma_{f^{-1}(x)}) = \sigma_{ff^{-1}(x)} = \sigma_x\ ,$$

so that $g\psi_f^{-1} \in H(\nu)$, say $g\psi_f^{-1} = \phi_\alpha$, where $\alpha \in S(\nu)$. Then $g = \phi_\alpha\psi_f$ and $g \in H(\nu)K(\nu)$.

Re(7). Immediate by (4) and (5).

Re(8). We only need to prove $\psi_f^{-1}H(\nu)\psi_f \subset H(\nu)$, for $f \in Per(\nu)$, i.e.,

$$h \in H(\nu) \;\&\; f \in Per(\nu) \Rightarrow \psi_f^{-1}h\psi_f \in H(\nu).$$

Assume the hypothesis. Then $\psi_f^{-1}h\psi_f \in H(\nu)$, since

$$\psi_f^{-1}h\psi_f(\sigma_x) = \psi_{f^{-1}}h\psi_f(\sigma_x) = \psi_{f^{-1}}h(\sigma_{f(x)}) = \psi_{f^{-1}}(\sigma_{f(x)}) = \sigma_x.$$

This proves (a). To verify (b) we need to show that (6), (7), and (8) hold for $H = H_\omega(\nu)$, $K = K_\omega(\nu)$, and $G = Aut_\omega Oc_\nu$.

Re(6). Since $H_\omega(\nu)$, $K_\omega(\nu) \leqslant Aut_\omega Oc_\nu$, it suffices to prove

$$g \in Aut_\omega Oc_\nu \Rightarrow (\exists h)(\exists k)[h \in H_\omega(\nu) \;\&\; k \in K_\omega(\nu) \;\&\; g = hk].$$

Assume the hypothesis. By (a) there exists a unique ordered pair $\langle h, k\rangle$ of functions such that $h \in H(\nu)$, $k \in K(\nu)$ and $g = hk$. Let $\bar{g}$ be a partial recursive one-to-one extension of g. Put $\bar{\nu} = \{x \mid 2x \in \delta\bar{g} \;\&\; 2x+1 \in \delta\bar{g}\}$, then $\nu \subset \bar{\nu}$, where $\bar{\nu}$ is r.e. Define $\bar{\nu}_0 = \{2x \mid x \in \bar{\nu}\}$, $\bar{\nu}_1 = \{2x+1 \mid x \in \bar{\nu}\}$, then $\bar{\nu}_0 \cup \bar{\nu}_1$ is a r.e. superset of $\nu_0 \cup \nu_1$. Define the function $\bar{h}$ by: $\delta\bar{h} = \bar{\nu}_0 \cup \bar{\nu}_1$ and

$$\bar{h}(2x) = 2x \;\&\; \bar{h}(2x+1) = 2x+1, \text{ if } \bar{g}(2x) \text{ is even and } \bar{g}(2x+1) \text{ odd},$$
$$\bar{h}(2x) = 2x+1 \;\&\; \bar{h}(2x+1) = 2x, \text{ if } \bar{g}(2x) \text{ is odd and } \bar{g}(2x+1) \text{ even}.$$

Then $\bar{h}$ is a partial recursive extension of h. Let $p, q \in \delta\bar{h}$, $p \neq q$, say $p \in \sigma_x$, $q \in \sigma_y$, for $x, y \in \bar{\nu}$. If $x = y$ we have $\sigma_x = \sigma_y = (p, q)$; then $\bar{h}(p) \neq \bar{h}(q)$, since $\bar{h}$ is one-to-one on σ_x. If $x \neq y$ we have $\bar{h}(p) \in \sigma_x$, $\bar{h}(q) \in \sigma_y$, where σ_x and σ_y are disjoint, hence $\bar{h}(p) \neq \bar{h}(q)$. Thus the partial recursive function $\bar{h}$ is one-to-one and $h \in H_\omega(\nu)$. Since g and h have partial recursive one-to-one extensions, so has $h^{-1}g = k$; thus $k \in K_\omega(\nu)$.

Re(7). From $H_\omega(\nu) \leqslant H(\nu)$, $K_\omega(\nu) \leqslant K(\nu)$ and $H(\nu) \cap K(\nu) = (i)$.

Re(8). We only need to prove

$$h \in H_\omega(\nu) \;\&\; k \in Per_\omega(\nu) \Rightarrow \psi_k^{-1}h\psi_k \in H_\omega(\nu).$$

Assume the hypothesis. Then ψ_k has a partial recursive one-to-one extension (since k has one), hence so has $\psi_k^{-1}h\psi_k$. However, $\psi_k^{-1}h\psi_k \in H(\nu)$ by (a), hence $\psi_k^{-1}h\psi_k \in H_\omega(\nu)$.

Remark: If card $\nu \geqslant 2$ the two semidirect products are not direct. For let $p, q \in \nu$, $p \neq q$ and h be the permutation of ν which interchanges p and q, then $\psi_h \in K(\nu)$. Put $\alpha = (p)$, then

$$\phi_\alpha\psi_h\phi_\alpha^{-1}(2p) = \phi_\alpha\psi_h\phi_\alpha(2p) = \phi_\alpha\psi_h(2p+1) = \phi_\alpha(2q+1) = 2q+1,$$

so that $\phi_\alpha\psi_h\phi_\alpha^{-1}(2p) \not\equiv 2p (mod\ 2)$ and $\phi_\alpha\psi_h\phi_\alpha^{-1} \notin K(\nu)$. Hence $K(\nu) \lhd Aut\ Oc_\nu$ is false. The functions ϕ_α and ψ_h can also be used to show that $K_\omega(\nu) \lhd Aut_\omega\ Oc_\nu$ is false.

3 ***Representation by ω-groups*** We define the following subclasses of the class $S(\nu)$ of all subsets of ν:

$S_{fin}(\nu) = \{\alpha \subset \nu \mid \alpha \textit{ is finite}\}$, $S_{cof}(\nu) = \{\alpha \subset \nu \mid \nu - \alpha \textit{ is finite}\}$,
$S_{fcf}(\nu) = S_{fin}(\nu) \cup S_{cof}(\nu)$, $S_\omega(\nu) = \{\alpha \subset \nu \mid \alpha \textit{ is separable from } \nu - \alpha\}$.

The classes $S_{fin}(\nu)$ and $S_{cof}(\nu)$ are equal iff ν is finite, disjoint iff ν is infinite. Moreover,

(11) $S_{fin}(\nu) \subset S_{fcf}(\nu) \subset S_\omega(\nu) \subset S(\nu)$, *for all* ν,
(12) $S_{fin}(\nu) = S_{fcf}(\nu) = S_\omega(\nu) = S(\nu)$, *if* ν *is finite*,
(13) $S_{fcf}(\nu) \subset S_\omega(\nu) \subset_+ S(\nu)$, *if* ν *is infinite.*

The proper inclusion in (13) follows from: if ν is infinite, then card $S_\omega(\nu) = \aleph_0$ and card $S(\nu) = c$. We need a characterization of the sets ν for which $S_{fcf}(\nu) = S_\omega(\nu)$. This clearly depends only on $N = Req\ \nu$. Recall that an RET N is *indecomposable,* if $A + B = N$ implies that A or B is finite. Thus every finite RET is indecomposable and every indecomposable RET is an isol. It is known that there are c infinite, indecomposable isols. Note that for $N = Req\ \nu$,

(14) $(\exists\alpha)[\alpha \subset \nu\ \&\ \alpha | \nu - \alpha\ \&\ \alpha \notin S_{fcf}(\nu)] \iff N$ *decomposable,*
(15) $S_{fcf}(\nu) = S_\omega(\nu) \iff N$ *indecomposable.*

We define

$$(16)\quad \begin{cases} D_{fin}(\nu), D_{fcf}(\nu), D_\omega(\nu), D(\nu) \textit{ are the groups under } \oplus \textit{ formed} \\ \textit{by the classes } S_{fin}(\nu), S_{fcf}(\nu), S_\omega(\nu), S(\nu) \textit{ respectively.} \end{cases}$$

It follows from (11), (12), (13), (15), and (16) that for $N = Req\ \nu$,

(17) $D_{fin}(\nu) \leqslant D_{fcf}(\nu) \leqslant D_\omega(\nu) \leqslant D(\nu)$, *for all* ν,
(18) $D_{fin}(\nu) = D_{fcf}(\nu) = D_\omega(\nu) = D(\nu)$, *if* ν *is finite*,
(19) $D_{fcf}(\nu) \leqslant D_\omega(\nu) < D(\nu)$, *if* ν *is infinite,*
(20) $D_{fcf}(\nu) = D_\omega(\nu) \iff N$ *is indecomposable.*

In the proof of C2.1(c) we noted that the mapping $\alpha \to \phi_\alpha$, for $\alpha \in S(\nu)$ is an isomorphism from $D(\nu)$ onto $H(\nu)$.

Proposition C3.1 *The mapping* $\alpha \to \phi_\alpha$, *for* $\alpha \in D_\omega(\nu)$ *is an isomorphism from* $D_\omega(\nu)$ *onto* $H_\omega(\nu)$.

Proof: Let $H'(\nu)$ be the image of $D_\omega(\nu)$ under the mapping $\alpha \to \phi_\alpha$, for $\alpha \in D(\nu)$. Suppose $\alpha \in D_\omega(\nu)$, say $\alpha = \nu \cap \bar{\alpha}$, $\nu - \alpha = \nu \cap \bar{\beta}$, for disjoint r.e. sets $\bar{\alpha}$ and $\bar{\beta}$. Put $\bar{\nu} = \bar{\alpha} \cup \bar{\beta}$ and let $\phi_{\bar{\alpha}}$ be defined in terms of $\bar{\alpha}$ and $\bar{\nu}$ as ϕ_α is defined by (4) in terms of α and ν. Then $\phi_{\bar{\alpha}}$ is a partial recursive one-to-one extension of ϕ_α so that $\phi_\alpha \in H_\omega(\nu)$; hence $H'(\nu) \subset H_\omega(\nu)$. Now suppose $\phi_\alpha \in H_\omega(\nu)$ and $\bar{g}$ is a partial recursive extension of ϕ_α. Then

$$\alpha = \{x \in \nu | \phi_\alpha(2x) = 2x + 1\}, \quad \nu - \alpha = \{x \in \nu | \phi_\alpha(2x) = 2x\},$$
$$\alpha \subset \{x | 2x \in \delta\bar{g}\ \&\ \bar{g}(2x) = 2x + 1\}, \quad \nu - \alpha \subset \{x | 2x \in \delta\bar{g}\ \&\ \bar{g}(2x) = 2x\},$$

where the sets on the right sides of the inclusions are r.e. and disjoint. Thus $\alpha \in D_\omega(\nu)$ and $\phi_\alpha \in H'(\nu)$; hence $H_\omega(\nu) \subset H'(\nu)$. We conclude that $H'(\nu) = H_\omega(\nu)$.

Let $N = Req\ \nu$. We know ([2], Sections 4 and 5) that the group P_ν of all finite permutations of ν can be represented by (i.e., is isomorphic to) the uniform ω-group P_N of order $N!$ In order to represent the group $D_{fcf}(\nu)$ by an ω-group we need an effective enumeration without repetitions of the class $S_{fcf}(\varepsilon)$. We choose the enumeration $\langle\sigma_n\rangle$, where

(21) $\sigma_{2n} = \rho_n, \quad \sigma_{2n+1} = \varepsilon - \rho_n,$ *for* $n \in \varepsilon$.

Henceforth "σ_x" will only be used as defined in (21). Define for $\nu \subset \varepsilon$ and $x, y \in \varepsilon$,

$$\delta_\nu = \begin{cases} \{2n \in \varepsilon | \sigma_{2n} \subset \nu\}, & \text{if } \nu \text{ is finite,} \\ \{2n \in \varepsilon | \sigma_{2n} \subset \nu\} \cup \{2n+1 \in \varepsilon | \sigma_{2n} \subset \nu\}, & \text{if } \nu \text{ is infinite,} \end{cases}$$
$$\bar{d}(x,y) = can(\sigma_x \oplus \sigma_y), \mathbf{D}_{fcf}(\nu) = \langle \delta_\nu, d_\nu \rangle, \text{ where } d_\nu = \bar{d} | \delta_\nu \times \delta_\nu,$$

then it is readily seen that

$$\alpha \cong \beta \Rightarrow \mathbf{D}_{fcf}(\alpha) \cong_\omega \mathbf{D}_{fcf}(\beta), \textit{ for nonempty sets } \alpha \textit{ and } \beta.$$

For a nonzero RET N we define $\mathbf{D}_{fcf}(N) = \mathbf{D}_{fcf}(\nu)$, for any $\nu \in N$. Thus $\mathbf{D}_{fcf}(N)$ is uniquely determined by N up to ω-isomorphism.

Proposition C3.2 *Let $N = Req\ \nu$. Then the group $D_{fcf}(\nu)$ is isomorphic to the uniform ω-group $\mathbf{D}_{fcf}(N)$. Moreover, $\mathbf{D}_{fcf}(N)$ has order 2^N, if N is finite, but 2^{N+1}, if N is infinite.*

Proof: Let $N = Req\ \nu$. The function $\bar{d}$ is recursive, hence $\mathbf{D}_{fcf}(\varepsilon)$ is a r.e. group. Also, $\mathbf{D}_{fcf}(\nu)$ is a finite group if ν is finite, while $\mathbf{D}_{fcf}(\nu) \leqslant \mathbf{D}_{fcf}(\varepsilon)$ if ν is infinite. Thus $\mathbf{D}_{fcf}(\nu)$ is a uniform ω-group for every ν. Clearly,

$$\{2n \in \varepsilon | \sigma_{2n} \subset \nu\} \simeq \{2n+1 \in \varepsilon | \sigma_{2n} \subset \nu\} \simeq 2^\nu,$$

for every set ν, so that $Req\ \delta_\nu$ equals 2^N, if N is finite, but 2^{N+1} if N is infinite.

4 The main result

Theorem *Let $\nu \in N$ and $N \in \Omega_0$. Then*

(a) *$Aut_\omega\ Oc_\nu = H_\omega(\nu) \times K_\omega(\nu)$, i.e., $Aut_\omega\ Oc_\nu$ is the semidirect product of $H_\omega(\nu)$ by $K_\omega(\nu)$,*
(b) *if N is an indecomposable isol, the group $H_\omega(\nu)$ can be represented by the uniform ω-group $\mathbf{D}_{fcf}(N)$ whose order is 2^N, if N is finite, but 2^{N+1}, if N is infinite,*
(c) *if N is a multiple-free isol, the group $K_\omega(\nu)$ can be represented by the uniform ω-group $\mathbf{P}_N$ of order $N!$,*
(d) *if N is an indecomposable isol, the group $Aut_\omega\ Oc_\nu$ can be represented by a uniform ω-group whose order is $2^N \cdot N!$, if N is finite, but $2^{N+1} \cdot N!$, if N is infinite.*

Proof: Part (a) holds by C2.2(b), part (b) by (20) and C3.1, and part (c) holds by [2], section 3. Now consider part (d). The statement is trivial, if N is finite, for then $Aut_\omega\ Oc_\nu$ is a finite group. Assume that N is an infinite, indecomposable isol. Then $H_\omega(\nu)$ and $K_\omega(\nu)$ can be represented by the uniform ω-groups $\mathbf{D}_{fcf}(\nu)$ and $\mathbf{P}_\nu$, respectively, where $\mathbf{D}_{fcf}(\nu) \leqslant \mathbf{D}_{fcf}(\varepsilon)$, $\mathbf{P}_\nu \leqslant \mathbf{P}_\varepsilon$. By C2.2(b) we have $Aut_\omega\ Oc_\varepsilon = H_\omega(\varepsilon) \times K_\omega(\varepsilon)$, where $H_\omega(\varepsilon) \cap K_\omega(\varepsilon) = (i)$. Define

$$\beta_\varepsilon = \{j(a, \tilde{f}) | a \in \delta_\varepsilon \ \&\ \tilde{f} \in \mathbf{P}_\varepsilon\},$$
$$\delta h_\varepsilon = \beta_\varepsilon, \quad h_\varepsilon j(a, \tilde{f}) = \phi_\alpha f, \text{ where } \alpha = \sigma_a,$$

Let for $x, y \in \beta_\varepsilon$, say $x = j(a,\tilde{f})$, $y = j(b,\tilde{g})$, $\alpha = \sigma_a$, $\beta = \sigma_b$,

$$t_\varepsilon(x,y) = \text{the unique number } z \text{ such that } h_\varepsilon(z) = \phi_\alpha f \phi_\beta g.$$

Now consider the group $\boldsymbol{G}_\varepsilon = \langle \beta_\varepsilon, t_\varepsilon \rangle$. The set β_ε is r.e. We claim that the function t_ε is partial recursive. For given the numbers $x, y \in \beta_\varepsilon$, we can compute the numbers $a, b, \tilde{f}, \tilde{g}$ such that $x = j(a,\tilde{f})$, $y = j(b,\tilde{g})$, hence also the finite or cofinite sets α and β such that $h_\varepsilon(x) = \phi_\alpha f$, $h_\varepsilon(y) = \phi_\beta g$ and the number $t_\varepsilon(x,y) = z$ such that $h_\varepsilon(z) = \phi_\alpha f \phi_\beta g$. Thus the group $\boldsymbol{G}_\varepsilon$ is r.e. Define

$$\beta_\nu = \{ j(a,\tilde{f}) \mid a \in \delta_\nu \;\&\; \tilde{f} \in \boldsymbol{P}_\nu \},\; t_\nu = t_\varepsilon | \beta_\nu \times \beta_\nu,$$

and $\boldsymbol{G}_\nu = \langle \beta_\nu, t_\nu \rangle$. Then $\boldsymbol{G}_\nu \leqslant \boldsymbol{G}_\varepsilon$, hence $\boldsymbol{G}_\nu$ is a uniform ω-group. Put

$$\boldsymbol{H}_\omega(\nu) = \{ j(a,\tilde{i}) \mid a \in \delta_\nu \},\; \boldsymbol{K}_\omega(\nu) = \{ j(0,\tilde{f}) \mid \tilde{f} \in \boldsymbol{P}_\nu \},$$

where i is the identity permutation on ε, hence $\tilde{i} = 1$. Then $\boldsymbol{H}_\omega(\nu)$ and $\boldsymbol{K}_\omega(\nu)$ are uniform ω-groups and since $N = Req\ \nu$ is indecomposable, $\boldsymbol{H}_\omega(\nu) \cong_\omega \boldsymbol{D}_{fcf}(\nu)$ and $\boldsymbol{K}_\omega(\nu) \cong_\omega \boldsymbol{P}_\nu$. We conclude that

$$o\boldsymbol{G}_\nu = o\boldsymbol{H}_\omega(\nu) \cdot o\boldsymbol{K}_\omega(\nu) = o\boldsymbol{D}_{fcf}(\nu) \cdot o\boldsymbol{P}_\nu = 2^{N+1} \cdot N!$$

5 Concluding remarks (A) Comparison with Q_ν. Let $N = Req\ \nu$ be an indecomposable isol, then N is also multiple-free. Comparing the group $Aut_\omega\ Q_\nu$ discussed in [2] with the group $Aut_\omega\ Oc_\nu$ discussed in the present paper, we notice an essential difference:

(1) $Aut_\omega\ Q_\nu$ can be represented by a uniform ω-group of order $2^N \cdot N!$,

(2) $Aut_\omega\ Oc_\nu$ can be represented by a uniform ω-group which has order $2^N \cdot N!$, if N is finite, but $2^{N+1} \cdot N!$, if N is infinite.

This essential difference between the ω-graphs Q_ν and Oc_ν is related to the fact that $Q_\nu = \langle 2^\nu, \eta \rangle$ has opposite vertices, i.e., vertices p and q such that $\rho_p = \nu - \rho_q$, iff the set ν is finite, while $Oc_\nu = Oc_f = \langle \mu_\nu, \theta_\nu \rangle$ has opposite vertices, i.e., vertices p and q such that $f(p) = q$, for every set ν. Thus, if ν is infinite, every permutation of μ_ν which maps almost all vertices of Oc_ν onto their opposites (and the others onto themselves) is an ω-automorphism of Oc_ν which has no analogue in Q_ν.

(B) Effective duality. In [2] we used "Q^ν" for the *directed* ω-cube on the set ν, i.e., for $\langle 2^\nu, \leqslant \rangle$, where $x \leqslant y \iff \rho_x \subset \rho_y$, for $x, y \in 2^\nu$. In [3] we used "Q^ν" for the *undirected* ω-cube on the set ν, i.e., for $\langle 2^\nu, F_\nu \rangle$, where F_ν is the class of all faces, i.e., of all subsets σ of 2^ν such that $\sigma = \{ x \in 2^\nu | \beta \subset \rho_x \subset \beta \cup \gamma \}$, for two disjoint finite subsets β and γ of ν. In both cases the (undirected) ω-graph corresponding to the ω-cube Q^ν is the ω-graph Q_ν. Similarly, Oc_ν is the ω-graph corresponding to the ω-octahedron Oc_ν discussed in [3]. We showed in [3] that for an indecomposable $N = Req\ \nu$, the undirected ω-cube Q^ν is effectively dual to the ω-octahedron Oc^ν iff N is finite. Thus if N is an infinite, indecomposable isol, Q^ν and Oc^ν are not effectively dual and one should therefore not be surprised that the ω-groups we used to represent $Aut_\omega\ Q_\nu$ and $Aut_\omega\ Oc_\nu$ have different orders.

(C) The group $D_{fcf}(\nu)$. In this remark "Q^ν" denotes the directed cube on the set ν. Let $N = Req\ \nu$. We have

(22) $Aut_\omega\, Q_\nu = C_\nu \times Aut_\omega\, Q_\nu, \quad Aut_\omega\, Oc_\nu = H_\omega(\nu) \times K_\omega(\nu)$.

Both $Aut_\omega\, Q^\nu$ and $K_\omega(\nu)$ are isomorphic to the group $Per_\omega(\nu)$. The difference between $Aut_\omega\, Q_\nu$ and $Aut_\omega\, Oc_\nu$ is therefore due to the difference between C_ν and $H_\omega(\nu)$. Note that

(23) $C_\nu \cong D_{fin}(\nu)$, $H_\omega(\nu) \cong D_\omega(\nu)$, *for every* N,
(24) $D_\omega(\nu) = D_{fcf}(\nu)$, *if* N *is indecomposable.*

From now on we assume that N is indecomposable. According to (22), (23) and (24) the difference between $Aut_\omega\, Q_\nu$ and $Aut_\omega\, Oc_\nu$ is due to the difference between the groups $D_{fin}(\nu)$ and $D_{fcf}(\nu)$, hence between the ω-groups representing them, namely $\boldsymbol{D}_{fin}(\nu)$ [or $\boldsymbol{Z}_2(\nu)$] and $\boldsymbol{D}_{fcf}(\nu)$. We have

$$\boldsymbol{D}_{fcf}(\nu) \cong_\omega \boldsymbol{D}_{fin}(\nu) \Longleftrightarrow \nu \textit{ is finite},$$
$$o\boldsymbol{D}_{fcf}(\nu) = 2 \cdot o\boldsymbol{D}_{fin}(\nu) \Longleftrightarrow \nu \textit{ is infinite}.$$

This is a direct consequence of the trivial observation that $S_{fin}(\nu)$ and $S_{cof}(\nu)$ are equal iff ν is finite, but disjoint iff ν is infinite.

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