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Indenumerability and Substitutional Quantification

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We here establish two theorems which refute a pair of what we believe to be plausible assumptions about differences between objectual and substitutional quantification. We first informally introduce a terminology which enables us to state these assumptions with reasonable clarity. Next we show that these assumptions have actually been made. Finally, in the remaining sections of the paper we prove the refuting theorems and explore the relations between the second and the Skolem submodel theorem.

1 L is any first-order language with countably many names and n-ary predicates as its nonlogical constants. An interpretation I of L is any triple $\langle D, p, d\rangle$ with nonempty set D, assignment p to the n-ary predicates of L of sets of n-tuples of elements of D, and assignment d to the names of L of elements of D. I is countable or indenumerable as D of I is countable or indenumerable. An *interpreted language* is a pair $\langle L, I\rangle$ with I an interpretation of L. $\langle L', I'\rangle$ is an *extension* of $\langle L, I\rangle$ iff L' results from adding countably many names to L and I' is like I except for its assignments to those new names. A definition of truth under an interpretation is *deviant* for interpreted language $\langle L, I\rangle$ iff some existential quantification of L is by that definition not true under I and yet some $x \in D$ of I satisfies its contained formula, and is *irreducibly deviant* for $\langle L, I\rangle$ iff it is deviant for $\langle L, I\rangle$ and for every extension of $\langle L, I\rangle$. I is a *complete* interpretation of L iff I is an interpretation of L and each $x \in D$ of I is by d of I assigned to some name of L. I is a *quantificationally complete* interpretation of L iff I is an interpretation of L, and for each degree-one formula of L, if some $x \in D$ of I satisfies that formula, then some such $x \in D$ of I is by d of I assigned to some name of L.

As propounded in [1], Mates' formal language $\mathcal{L}$ (less its zero-place letters) is one of our L. An interpretation of $\mathcal{L}$ (less the assignment to those

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letters) is one of our $\langle D, p, d\rangle$. Mates considers two definitions of truth under an interpretation: one objectual (the ODT), the other substitutional (the SDT). The ODT is deviant for no $\langle \mathcal{L}, I\rangle$. Not so, as Mates notes, for the SDT. On Mates' view, a definition of truth under an interpretation permits the application of $\mathcal{L}$ to a domain set D only if the definition of truth under an interpretation is nondeviant for some interpretation I of L of which D is the domain set. Thus, the SDT seems to incur the risk of restricting the application of $\mathcal{L}$. Mates also notes that the SDT is nondeviant for all $\langle L, I\rangle$ with complete interpretation I of $\mathcal{L}$. Thus, the application of $\mathcal{L}$ to countable domains is not restricted by the SDT, for every countable set D is the domain set of some complete interpretation of $\mathcal{L}$. In this manner Mates makes a case for the SDT which also, as he notes, is simpler than the ODT ([1], pp. 62-63). His determination of which truth definition to adopt is based on the following reasoning:

> . . . since in applying $\mathcal{L}$ we do not wish to be restricted to domains for which there are in $\mathcal{L}$ enough [names] to go around, we shall stick to a definition which does not presuppose that every element in the universe of discourse has a name. ([1], p. 63)

Since Mates sticks with the ODT and rejects the SDT, the above passage implies at least the following:

(I) The SDT would preclude the application of $\mathcal{L}$ to at least one indenumerable domain.

The main fact to which Mates appeals in connection with (I) is:

(II) If D is indenumerable, then there are not enough names to go around, i.e., then no $\langle D, p, d\rangle$ is a complete interpretation of $\mathcal{L}$.

To get from II to I Mates needs this assumption:

(M) At least one D which is the domain of no complete interpretation of $\mathcal{L}$ is also the domain of no quantificationally complete interpretation of $\mathcal{L}$.

This assumption is refuted by T1 below.

For the second assumption we dispute we go to the following passage from Quine:

> . . . the substitutional characterization of quantification is not coextensive with the characterization in terms of objects, or values of variables, if we assume a rich universe. An existential quantification could turn out false when substitutionally construed and true when objectually construed, because of there being objects of the purported kind but only nameless ones . . . And no lavishness with names can prevent there being nameless objects in a generous universe. Substitutional quantification is deviant if the universe is rich. ([2], p. 93)

By a rich or generous universe Quine means an indenumerable one. Also we capture his "lavishness with names" by our concept of an extension of a language. Now note that Quine is saying more than this:

(i) The SDT is deviant for some interpreted languages with indenumerable domains.

For the SDT is also deviant for some interpreted languages with countable domains. So if Quine meant (i) the ado about generous universes would be pointless. Now a fact implicitly noted by Quine is that in the case of countable domains deviance is always eliminated in some extension. So his point has to be that the SDT, for at least some interpreted langauges with indenumerable domains, is deviant and remains so for all extensions. That is to say, Quine is implicitly assuming this:

(Q) There exist languages with indenumerable domains for which substitutional quantification is irreducibly deviant.

This assumption is refuted by T2 below.[1]

2 In order to prove T1 and T2 we need to reintroduce some of the preceding terminology with a bit more precision. In the following, names and n-ary predicate letters are *nonlogical constants* and N, V, and the variables $x, x', \ldots$ are *logical constants.* A *language* L is any denumerable or finite set of nonlogical constants not void of predicate letters. The *formulas* of L are as follows: $\Psi\omega_1, \ldots, \omega_n$ if each ω_i is a name of L or a variable and Ψ is an n-ary predicate letter of L; $N\Psi_1\Psi_2$ if Ψ_1 and Ψ_2 are formulas of L; $V\alpha\Psi$ if α is a variable and Ψ is a formula of L and α has at least one occurrence in Ψ which is not a proper part of Ψ of the form $V\alpha\chi$ for any formula χ of L. The sentences of L are the formulas of L void of free occurrences of variables. $\Psi\alpha/\beta$ is the result of replacing each free occurrence of variable α in formula Ψ by a name β not in Ψ. Ψ^{α}_{β} is the result of replacing each free occurrence of variable α in formula Ψ by a name β.

An *interpretation* I of L is a triple $\langle D, p, d\rangle$, where D is a nonempty set, p is a function defined on the predicate letters of L assigning to each n-ary predicate letter of L some set of n-tuples of elements of D, and d is a function defined on the names of L assigning to each some element of D. If $I = \langle D, p, d\rangle$ is an interpretation of L then for each n-ary predicate Ψ of L, $I(\Psi) = p(\Psi)$ and for each name β of L, $I(\beta) = d(\beta)$. We also speak of what I assigns to predicate Ψ or name β. I is an *interpretation of a sentence* ϕ if and only if I is an interpretation of some language L of which ϕ is a sentence. Where d is a function defined on some set of names, $d^{\beta}_{a} = f$ is the function which differs from d at most in that f is defined for name β and $f(\beta) = a$. Where $I = \langle D, p, d\rangle$, $I|^{\beta}_{a} = \langle D, p, d^{\beta}_{a}\rangle$. We note that if I is an interpretation of $V\alpha\Psi$, then, for any name β and $a \in D$ of I, $I|^{\beta}_{a}$ is an interpretation of $\Psi\alpha/\beta$. For each sentence ϕ and interpretation I of ϕ, ϕ is true in I if and only if ϕ is an atomic sentence $\Psi\beta_1 \ldots \beta_n$ and $\langle I(\beta_1), \ldots, I(\beta_n)\rangle \in I(\Psi)$, or ϕ is a compound sentence $N\Psi_1\Psi_2$ and neither Ψ_1 nor Ψ_2 is true in I, or ϕ is a general sentence $V\alpha\Psi$ and $\Psi\alpha/\beta$ is true in some $I|^{\beta}_{a}$.

The main theorems we seek to establish are these:

T1 *For every language L with denumerable set Δ of names and D and p belonging to some interpretation of L, there is a d defined on Δ and into D such that $\langle D, p, d\rangle$ is an interpretation of L and for every sentence $V\alpha\Psi$ of L, $V\alpha\Psi$ is true in $\langle D, p, d\rangle$ iff, for some name $\beta \in \Delta$, Ψ^{α}_{β} is true in $\langle D, p, d\rangle$.*

T2 *For every language L and interpretation $\langle D, p, d\rangle$ of L there is a set Δ of names not in L and a d', $d \subset d'$, such that $\langle D, p, d'\rangle$ is an interpretation of $L' = L \cup \Delta$ and: (1) for every sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle$ if and only if ϕ is true in $\langle D, p, d'\rangle$, and (2) for every general sentence $V\alpha\Psi$ of L', $V\alpha\Psi$ is true in $\langle D, p, d'\rangle$ if and only if, for some name β of L', $\Psi\frac{\alpha}{\beta}$ is true in $\langle D, p, d'\rangle$.*

3 We begin by noting for future reference four lemmas of a relatively elementary character.

Lemma 1 *If I and I' are interpretations of sentence ϕ and do not differ in what they assign to the nonlogical constants in ϕ, then ϕ is true in I iff ϕ is true in I'.*

Lemma 2 *If a sentence ϕ is like ϕ' except for having an occurrence of a name γ wherever ϕ' has an occurrence of a name β and I is an interpretation of ϕ and I' is an interpretation of ϕ' and I' is just like I except that $I'(\beta) = I(\gamma)$, then ϕ is true in I iff ϕ' is true in I'.*

Definition Ψ is a *degree-one* formula iff Ψ is a formula and exactly one variable α has free occurrences in Ψ.

Note that Ψ is a degree-one formula iff for some variable α, $V\alpha\Psi$ is a sentence.

Definition *If* $V\alpha\Psi$ is a sentence and I an interpretation of $V\alpha\Psi$, then $E(\Psi, I) = \{a\colon \Psi\alpha/\beta$ is true in some $I|^{\beta}_{a}\}$.

Read '$E(\Psi, I)$' as 'the extension of Ψ in I'.

Lemma 3 *For every language L, interpretation I of L and sentence $V\alpha\Psi$ of L, $E(\Psi, I) \neq \wedge$ iff $V\alpha\Psi$ is true in I.*

Lemma 4 *For every language L, interpretation I of L and sentence $V\alpha\Psi$ of L, there is a name β of L such that $I(\beta) \in E(\Psi, I)$ iff there is a name β of L such that $\Psi\frac{\alpha}{\beta}$ is true in I.*

T1 is an immediate consequence of Lemmas 3 and 4 and

Lemma 5 *For every language L with denumerable set Δ of names and every D and p belonging to some interpretation of L, there is a d such that $\langle D, p, d\rangle$ is an interpretation of L such that for every sentence $V\alpha\Psi$ of L, $E(\Psi, \langle D, p, d\rangle) \neq \wedge$ iff there is a name β of L such that $d(\beta) \in E(\Psi, \langle D, p, d\rangle)$.*

The proof of Lemma 5 proceeds as follows:

Proof: Let L be any language with denumerable set Δ of names and let D and p be any set and function belonging to some interpretation of L. Then, there is a nonrepeating sequence $N = \langle \beta_1, \ldots, \beta_n, \ldots\rangle$ of all elements of Δ and a sequence $F = \langle \phi_1, \ldots, \phi_n, \ldots\rangle$ of all degree-one formulas of L such that: (1) $E(\phi_1, \langle D, p, \wedge\rangle) \neq \wedge$ and (2) for every i and j, $i, j > 0$, if β_i occurs in ϕ_j, then $i < j$. There also is a sequence $\langle A_1, \ldots, A_n, \ldots\rangle$ such that

$$A_1 = \{\{\langle \beta_1, x\rangle\}\colon x \in E(\phi_1, \langle D, p, \wedge\rangle)\};$$
$$A_n = \{f \cup \{\langle \beta_n, x\rangle\}\colon f \in A_{n-1} \text{ and } x \in E(\phi_n, \langle D, p, f\rangle) \text{ or } E(\phi_n, \langle D, p, f\rangle) = \wedge \text{ and } x = f(\beta_1)\}.$$

We set $A = \cup \{A_1, \ldots, A_n, \ldots\}$ and $P(A)$ = the power set of $A \sim \{\wedge\}$. By the axiom of choice, there is a function g defined on $P(A)$ such that for every x, $g(x) \in x$. Then for each $f_1 \in A_1$ there is a sequence $\langle d_1, \ldots, d_n, \ldots\rangle$ such that

$$d_1 = f_1;$$
$$d_n = g\{f: f \in A_n \text{ and } d_{n-1} \subset f\}.$$

We set $den = \cup \{d_1, \ldots, d_n, \ldots\}$.

From the construction of *den* each of the following sublemmas can be straightforwardly established in the indicated order.

Lemma 5.1 *For any interpretation I of sentence $V\alpha\Psi$, if $x \in E(\Psi, I)$, then $x \in D$ of I.*

Lemma 5.2 *If $f \in A_n$, then f is a function into D defined on just the first n names in N.*

Lemma 5.3 *For each $n > 0$, $A_n \neq \wedge$.*

Lemma 5.4 *For each $n > 1$, if $h \in A_{n-1}$, then for some $f \in A_n$, $h \subset f$.*

Lemma 5.5 *For each $n > 0$, $d_n \in A_n$.*

Lemma 5.6 *For each $n > 0$, $d_n \subset d_{n+1}$.*

We next turn to:

Lemma 5.7 *den is a function into D and defined on Δ.*

Proof: $den = \cup \{d_1, \ldots, d_n, \ldots\}$. By Lemmas 5.2 and 5.5 each d_i is a function. By Lemma 5.6 each $d_i \subset d_{i+1}$. Thus, *den* is a function. By Lemmas 5.2 and 5.5 each d_i is into D. Thus *den* is into D. Next note that if $x \in \Delta$, then for some $j > 0$, $x = \beta_j$. By Lemmas 5.2 and 5.5, for each j, d_j is defined for the first j names in N and thus for β_j. Thus, for all $j > 0$, *den* is defined for β_j. But by Lemma 5.2, each d_j is defined for just the first j names in N and thus is defined only for names in Δ. Thus *den* is defined on Δ.

Lemma 5.8 *$\langle D, p, den\rangle$ is an interpretation of L.*

Proof: Immediate from Lemma 5.7.

Lemma 5.9 *For each sentence $V\alpha\Psi$ of L, $E(\Psi, \langle D, p, den\rangle) \neq \wedge$ iff, for some $\beta \in \Delta$, $den\,(\beta) \in E(\Psi, \langle D, p, den\rangle)$.*

Proof: The right to left implication is trivial, and we consider now the converse implication. By assumption, $E(\Psi, \langle D, p, den\rangle) \neq \wedge$. We note the following: for some $n > 0$, $\Psi = \phi_n$; d_{n-1} is defined for the first $n - 1$ names in N; any name in $\Psi = \phi_n$ is one of the first $n - 1$ names in N. Since $d_{n-1} \subset den$, for each $0 < i < n$, $d_{n-1}(\beta_i) = den\,(\beta_i)$. Also, $d_{n-1}{}^{\beta}_{a}(\beta) = den^{\beta}_{a}(\beta)$, for any name β and $a \in D$. Thus, for any name β and $a \in D$, $\langle D, p, den^{\beta}_{a}\rangle$ and $\langle D, p, d_{n-1}{}^{\beta}_{a}\rangle$ are interpretations which do not differ in what they assign to the nonlogical constants in $\Psi\alpha/\beta$. So, by Lemma 1, $\Psi\alpha/\beta$ is true in $\langle D, p, den^{\beta}_{a}\rangle$ iff $\Psi\alpha/\beta$ is true in $\langle D, p, d_{n-1}{}^{\beta}_{a}\rangle$. Thus, $\{a: \Psi\alpha/\beta$ is true in some $\langle D, p, den^{\beta}_{a}\rangle\} = \{a: \Psi\alpha/\beta$ is true in $\langle D, p, d_{n-1}{}^{\beta}_{a}\rangle\}$. Thus, $E(\Psi, \langle D, p, den\rangle) = E(\Psi, \langle D, p, d_{n-1}\rangle)$. Thus, $E(\Psi, \langle D, p, d_{n-1}\rangle) \neq \wedge$. Thus, for some $x \in E(\Psi, \langle D, p, d_{n-1}\rangle)$, $\langle \beta_n, x\rangle \in d_n$. Thus, $\langle \beta_n, x\rangle \in den$ for some $x \in E(\Psi, \langle D, p, den\rangle)$. Thus, for some $\beta \in \Delta$, $den\ (\beta) \in E(\Psi, \langle D, p, den\rangle)$.

From Sublemmas 5.8 and 5.9 it follows that $\langle D, p, den\rangle$ is an interpretation of L such that for every sentence $V\alpha\Psi$ of L, $E(\Psi, \langle D, p, den\rangle) \neq \wedge$ iff there is a name β of L such that $den\ (\beta) \in E(\Psi, \langle D, p, den\rangle)$. This completes the proof of Lemma 5 and thus also of T1.

4 T2 is an immediate consequence of Lemmas 3 and 4 together with the following two lemmas.

Lemma 6 *If $\langle D, p, d\rangle$ is an interpretation of language L and $d \subset d'$ and $\langle D, p, d'\rangle$ is an interpretation of language $L' = L \cup \Delta$, for some denumerable set Δ of names not in L, then for any sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle$ iff ϕ is true in $\langle D, p, d'\rangle$.*

Proof: By hypothesis of the lemma, $\langle D, p, d\rangle$ and $\langle D, p, d'\rangle$ do not differ in what they assign to any of the nonlogical constants in any sentence ϕ of L. Thus, for each sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle$ iff ϕ is true in $\langle D, p, d'\rangle$, by Lemma 1.

Lemma 7 *If Δ is a denumerable set of names not in language L and language $L' = L \cup \Delta$ and $\langle D, p, d\rangle$ is an interpretation of L, then there is an interpretation $\langle D, p, d'\rangle$ of L' such that $d \subset d'$ and for every sentence $V\alpha\Psi$ of L', $E(\Psi, \langle D, p, d'\rangle) \neq \wedge$ iff there is a name β of L' such that $d'(\beta) \in E(\Psi, \langle D, p, d'\rangle)$.*

Proof: The proof proceeds along the lines of the proof of Lemma 5. $N = \langle \beta_1, \ldots, \beta_n, \ldots\rangle$ is a nonrepeating sequence of all names in Δ. $F = \langle \phi_1, \ldots, \phi_n, \ldots\rangle$ is a sequence of all degree-one formulas of L' such that (1) $E(\phi_1, \langle D, p, d\rangle) \neq \wedge$ and with condition (2) as before. The first element of sequence $\langle A_1, \ldots, A_n, \ldots\rangle = \{\langle \beta_1, x\rangle : x \in E(\phi_1, \langle D, p, d\rangle)\}$. A_n is defined as before. The same holds for A, $P(A)$, sequence $\langle d_1, \ldots, d_n, \ldots\rangle$ and *den*. Sublemmas 7.1-7.9 are the same as Sublemmas 5.1-5.9 except that '$E(\Psi, \langle D, p, \wedge\rangle)$' is everywhere replaced by '$E(\Psi, \langle D, p, d\rangle)$', 'the first n names in N' is everywhere replaced by 'the names in L and the first n names in N', 'L' is everywhere replaced by 'L'' and 'Δ' is replaced by 'the set of names in $L \cup \Delta$'. Sentences 7-9 of the proof of 5.7 are replaced by the following:

> Next note that if x is a name of L', then x is a name in L or for some $j > 0$, $x = \beta_j$. By Lemmas 7.2 and 7.5, for each j, d_j is defined for the names in L and the first j names in N and thus for β_j. Thus, *den* is defined for all names in L and, for all j, for β_j.

From Sublemmas 7.8 and 7.9 it follows that $\langle D, p, den\rangle$ is an interpretation of L' such that for every sentence $V\alpha\Psi$ of L', $E(\Psi, \langle D, p, den\rangle) \neq \wedge$ iff there is a name β of L' such that $den\ (\beta) \in E(\Psi, \langle D, p, den\rangle)$. This completes the proof of Lemma 7 and thus also of T2.

5 Skolem's strong submodel theorem says this:

SST *For each interpretation $\langle D, p, d\rangle$ of L, there is at least one interpretation $\langle D', p', d\rangle$ such that $D' \subseteq D$, D' is countable, p' is the limitation of p to D', and for each sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle$ iff ϕ is true in $\langle D', p', d\rangle$.*

In conversations about T2, we have invariably encountered this reaction: T2 and SST have essentially the same content and thus have essentially the same consequences for the relation between objectual and substitutional quantification over indenumerable domains. We think it is worthwhile to show why this reaction is in error.

Recall that an interpretation is countable or indenumerable as its domain is countable or indenumerable, and that an interpreted language is any pair $\langle L, I\rangle$ for interpretation I of L. Put in present terms, substitutional quantification is *deviant* for interpreted language $\langle L, I\rangle$ iff, for some $V\alpha\Psi$ of L, $V\alpha\Psi$ is true in I and for each name $\beta \in L$, Ψ^{α}_{β} is not true in I; substitutional quantification is *irreducibly deviant* for interpreted language $\langle L, I\rangle = \langle L, \langle D, p, d\rangle\rangle$ iff, for every d^* such that $d \subseteq d^*$ and the range of $d^* \subseteq D$, substitutional quantification is deviant for $\langle L^*, I^*\rangle = \langle L \cup$ the domain of $d^*(= \mathcal{D}(d^*)), \langle D, p, d^*\rangle\rangle$.

Since for each interpreted language $\langle L, I\rangle = \langle L, \langle D, p, d\rangle\rangle$ with countable I, there is a d^* such that $d \subseteq d^*$ and the range of $d^* = D$, there is a $\langle L^*, I^*\rangle$ for which substitutional quantification is not deviant; namely, for each "complete" d^*, the interpreted language $\langle L^*, I^*\rangle = \langle L \cup \mathcal{D}(d^*), \langle D, p, d^*\rangle\rangle$. Thus, substitutional quantification is irreducibly deviant for no interpreted language $\langle L, I\rangle$ with countable I.

Assumption (Q) noted at the start of this paper is that the class of interpreted languages $\langle L, I\rangle$ with *indenumerable I* does provide cases of irreducible deviance. T2 refutes this assumption, for T2 directly asserts that for each indenumerable interpretation $\langle D, p, d\rangle$ of any L there is a d^*, $d \subseteq d^*$, range of $d^* \subseteq D$, such that $\langle D, p, d^*\rangle$ is an interpretation of $L^* = L \cup \mathcal{D}(d^*)$ and substitutional quantification is *nondeviant* for $\langle L^*, \langle D, p, d^*\rangle\rangle$. SST does not directly assert the above result and thus does not refute assumption (Q), for it asserts only that for each indenumerable interpretation $\langle D, p, d\rangle$ of L there is a *countable* subinterpretation $\langle D', p', d\rangle$ such that for each sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle \equiv \phi$ is true in $\langle D', p', d\rangle$.

SST is a *reduction* theorem and T2 is not, and that is a fundamental difference between them relative to assumption (Q).

Is there yet some route from SST to the above T2 result? If there is, then it must lead from the existence of the countable submodel $\langle D', p', d\rangle$ established by SST to the appropriate d^* function. The whole problem, then, comes to this: whether there is some way to utilize $\langle D', p', d\rangle$ to construct the desired d^* function. The only plausible construction we can think of would be this: to so construct d^* from d as to make it complete with respect to D'. That construction yields $\langle D', p', d^*\rangle$. Two points plainly hold with respect to $\langle D', p', d^*\rangle$. First, $\langle D', p', d^*\rangle$ is an interpretation of L such that for every sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle \equiv \phi$ is true in $\langle D', p', d^*\rangle$. Second, substitutional quantification is *nondeviant* with respect to $\langle L^*, \langle D', p', d^*\rangle\rangle$, $L^* = L \cup \mathcal{D}(d^*)$. The hope would be that since this second point holds, substitutional quantification will also be nondeviant for $\langle L^*, \langle D, p, d^*\rangle\rangle$.

This hope comes to the following claim: *if* $\langle D', p', d\rangle$ is a Skolem subinterpretation of interpretation $\langle D, p, d\rangle$ of L and $d \subseteq d^*$ and substitutional quantification is not deviant for $\langle L^*, \langle D', p', d^*\rangle\rangle$, $L^* = L \cup \mathcal{D}(d^*)$, *then* substitutional quantification is also nondeviant for $\langle L^*, \langle D, p, d^*\rangle\rangle$.

But this claim is incorrect. Indeed, it fails to hold even for cases of countable I and thus, trivially, for cases of indenumerable I. A simple example establishes the point.

Let,

$L\ \ = \{\text{`}F\text{'}\}$
$h\ \ = \{\langle x, y\rangle$: for some prime p and $n \geqslant 1, x = p^n$ and $y = p^{n+1}\}$
$D\ \ =$ the field of h
$p\ \ = \{\langle\text{`}F\text{'}, h\rangle\}$
$d\ \ = \Lambda$
$g\ \ = \{\langle x, y\rangle$: for some prime p and $n > 1, x = p^n$ and $y = p^{n+1}\}$
$D'\ =$ the field of g
$p'\ = \{\langle\text{`}F\text{'}, g\rangle\}$
$d^* = \{\langle x, y\rangle$: $y \in$ the field of g and $x =$ the arabic numeral for $y\}$
$L^* = L \cup \mathcal{D}\,(d^*)$

then,

(1) $\langle D', p', d\rangle$ is a Skolem subinterpretation of $\langle D, p, d\rangle$
(2) $d \subset d^*$
(3) substitutional quantification is nondeviant for $\langle L^*, \langle D', p', d^*\rangle\rangle$
(4) substitutional quantification is deviant for $\langle L^*, \langle D, p, d^*\rangle\rangle$.

(3) is trivial since d^* is complete with respect to D'. By construction $D' \subseteq D$ and $p' =$ the limitation of p to D'. (1) may then be established by a simple inductive proof. (2) also is trivial since $d = \Lambda$. (4) is established by noting that '$VxFx4$' of L^* is true in $\langle D, p, d^*\rangle$ since $\langle 2, 4\rangle \in p(\text{`}F\text{'}) = h$ and yet for each name $\beta \in L^*$, $\ulcorner F\beta 4\urcorner$ is not true in $\langle D, p, d^*\rangle$ since $\langle x, 4\rangle \in p(\text{`}F\text{'})$ only if $x = 2$ and $2 \notin$ the range of d^*.

Thus, it is not only the case that T2 and SST sharply differ in content, it also appears that the T2 result with which we here are concerned just isn't a consequence of SST at all. If this is so, then T2 is stronger than SST, for SST is an elementary consequence of T2 (see the Appendix).

Appendix T2 can be used to derive SST. First we need to establish the following lemma:

Lemma 8 *For any language L and interpretations $\langle D, p, d\rangle$ and $\langle D', p', d\rangle$ such that $D' \subset D$ and p' is the limitation of p to D': if, for each sentence $V\alpha\Psi$ of L, $V\alpha\Psi$ is true in $\langle D, p, d\rangle$ iff, for some name β of L, $\Psi\frac{\alpha}{\beta}$ is true in $\langle D, p, d\rangle$, then, for each sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle$ iff ϕ is true in $\langle D', p', d\rangle$.*

Proof: By strong induction on $\ell(\phi) =$ the number of occurrences of N and V in ϕ. Cases 1 and 2 are obvious.

Case 3. $\ell(\phi) = n, n > 0$ and ϕ is $V\alpha\Psi$.

1. By antecedent of the lemma and hypothesis of induction, if $V\alpha\Psi$ is true in $\langle D, p, d\rangle$ then $V\alpha\Psi$ is true in $\langle D', p', d\rangle$.

2. Suppose $V\alpha\Psi$ is true in $\langle D', p', d\rangle$. Then $\Psi\alpha/\beta$ is true in some $\langle D', p', d^\beta_a\rangle$. Since d is into D', for some β^* of L, $d^\beta_a(\beta) = d(\beta^*) = a \in D'$. Thus, $\langle D', p', d^\beta_a\rangle$ differs from $\langle D', p', d\rangle$ at most in assigning to β what $\langle D', p', d\rangle$

assigns to β^*. Also $\Psi^{\alpha}_{\beta^*}$ has β^* wherever $\Psi\alpha/\beta$ has β. Thus, by Lemma 2, $\Psi^{\alpha}_{\beta^*}$ is true in $\langle D', p', d\rangle$. By hypothesis of induction, $\Psi^{\alpha}_{\beta^*}$ is true in $\langle d, p, d\rangle$. Thus $V\alpha\Psi$ is true in $\langle D, p, d\rangle$.

The proof of SST from T2 and Lemma 8 proceeds as follows:

Proof: Let $\langle D, p, d\rangle$ be an interpretation of L. (A) By T2, there is a set Δ of names not in L such that $L' = L \cup \Delta$; and there is a function d', $d \subset d'$, such that $\langle D, p, d'\rangle$ is an interpretation of L; and, for each sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle$ iff ϕ is true in $\langle D, p, d'\rangle$; and, for each sentence $V\alpha\Psi$ of L', $V\alpha\Psi$ is true in $\langle D, p, d'\rangle$ iff Ψ^{α}_{β} is true in $\langle D, p, d'\rangle$, for some name β of L'. (B) Now let D' be the range of d' and p' be the limitation of p to D'. (C) By Lemma 8, (1) if $\langle D, p, d'\rangle$ and $\langle D', p', d'\rangle$ are interpretations of L' such that $D' \subset D$ and p' is the limitation of p to D', and if (2) for each sentence $V\alpha\Psi$ of L', $V\alpha\Psi$ is true in $\langle D, p, d'\rangle$ iff, for some name β of L', Ψ^{α}_{β} is true in $\langle D, p, d'\rangle$, then (3) for each sentence ϕ of L', ϕ is true in $\langle D, p, d'\rangle$ iff ϕ is true in $\langle D', p', d'\rangle$. *(D) It follows* from (A), (B), and (C) that, for each sentence ϕ of L, ϕ is true in $\langle D, p, d\rangle$ iff ϕ is true in $\langle D', p', d'\rangle$ and D' is at most a denumerable subset of D and p' is the limitation of p to D'. SST follows from this by Lemma 1.

NOTE

1. As the referee pointed out to us, not all of Quine's remarks in connection with substitutional quantification are the same in character as the one we quote. Often enough his concern is not with purely formal logical considerations, as in the quoted remark, but turns instead to more pragmatically or epistemically oriented considerations as in [3]. In this paper we limit ourselves to the formal issues.

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