124
Notre Dame Journal of Formal Logic
Volume 27, Number 1, January 1986

A New Axiomatization of Belnap's Conditional Assertion

DANIEL H. COHEN

1 Introduction A *conditional assertion* is a statement which succeeds in making an assertion only if the supplied condition of assertion is met. Otherwise, it fails to assert; it is "nonassertive". It is comparable to a conditional bet. Although the notion of asserting something only conditionally has been around for several decades,[1] the definitive characterization is the presentation in [1]. That discussion will be presupposed.

In spite of attempted arguments to the effect that there can be no sense made of conditional assertions ([4], and pp. 338–347 in [5]), there have indeed been successful formalizations of languages with a conditional assertion connective. Most notable among these is Dunn [6].[2] However, the very success of that presentation raises a number of questions. First, there are the initial and explicit philosophical questions about the soundness of the motivation behind the enterprise. In particular, the two-logics structure presented needs scrutinizing. Secondly, there are implicit questions concerning some unfinished business. These are probably best read as challenges to other workers in the field. Finally, there is a serious unraised question about the completeness proofs themselves: there is a point in the proofs which is so susceptible to error that avoiding the pitfall without explicit mention of it could be misconstrued as fortuitous. The axiomatization offered here addresses all of these.

2 Ascertaining assertiveness Belnap [1] suggests that a formalization of conditional assertions might take a double-barreled approach, first axiomatizing the always-true formulas and then the never-false ones. This is what Dunn [6] does. In fact, he does more in that he also proves that success in one task guarantees success in the other – perhaps thereby proving he has done less. In either case,

Received July 9, 1984

the major obstacle to an axiomatization of conditional assertions is the problem of substitution ([6], pp. 383–384). Since nonassertive components of truth-functional compounds drop out, even so innocent a formula as

$$(A \vee \sim A) \vee B$$

can be falsified by substituting nonassertive A and false B.

Dunn's solution is to furnish axiom schemata that are "impure" in that they make explicit reference to a particular atomic formula, p. This exploits a particular feature of Belnap's semantics: ultimately, nonassertiveness only arises from failed conditional assertions so atomic formulas are always assertive.

For all its technical merits, the two-logics approach is probably inappropriate. What is novel about a conditional assertion connective is its syntactic embodiment of the essentially pragmatic notion of assertion. Nonassertive statements are too innocuous to violate any conversational conventions. Indeed, conditions of assertion that are themselves nonassertive count as having been satisfied.[3] Nonassertiveness, therefore, should count as a designated value; perhaps this lack of a truth value should count as a designated state.[4] Accordingly, it is the never-falses that should be the concern, recalling all the while that Dunn has shown the other set to be retrievable.

This narrowing of focus helps alleviate the problem of substitution. That problem arises only with the substitution of nonassertive components into otherwise unfalsifiable schemata. Perhaps axioms could be accompanied by a conditional rule of substitution. That is, letting / represent the conditional assertion connective, the rule could be:

$$\begin{array}{ll} \text{from} & \Phi(\ldots p \ldots) \\ \text{infer} & (A \text{ is assertive})/\Phi(\ldots A \ldots)\ . \end{array}$$

This requires some object language mechanism for expressing the assertiveness of a given formula. But, given that, one could do away with axioms and substitution in favor of conditioned axiom schemata.

That a given formula is assertive can be expressed in a number of ways, e.g., by a bivalent unary connective[5] or a predicate and sentential nominalizer. A third route, adopted here, is to take advantage of existing relations among sentences supplied by Belnap's semantics. This can be done by inductively defining a metalinguistic function taking sentences of that object language into others, which are true just when the first assert and false otherwise. Letting α represent that function, it is defined as follows:

1. αp is $p \vee \sim p$
2. $\alpha(\sim A)$ is αA
3. $\alpha(A \;\&\; B)$ is $\alpha A \vee \alpha B$
4. $\alpha(A \vee B)$ is $\alpha A \vee \alpha B$
5. $\alpha(A/B)$ is $A \;\&\; \alpha B$.

As defined, this function meets all the desiderata.

3 Axiomatization Dunn [6], pp. 387, 393, notes that the "tautologies$_{\mathrm{nf}}$", i.e., the never-false formulas, are recursively decidable. Since the concern there was axiomatizing a first-order logic of conditional assertion, all tautologies$_{nf}$ are simply taken as axioms. Axiomatizing the sentential fragment itself is left undone. That is the concern of this section.

The axioms of *CA*, the logic of conditional assertion, are given by the following (the slash is associated to the right):

CA1	$B \vee \sim B$
CA2	$\sim \alpha B/B$
CA3	$(B \,\&\, \sim B)/\sim \alpha B$
CA4	$\sim \alpha\alpha B/C$
CA5	$\sim(B/C)/(B/\sim C)$; $(B/\sim C)/\sim(B/C)$
CA6	$B/C/(B \,\&\, C)$
CA7	$(B \,\&\, C)/B$; $(B \,\&\, C)/C$
CA8	$(B/C)/(D/C)/((B \vee D)/C)$
CA9	$B/C/B$
CA10	$(B/C/D)/(B/C)/(B/D)$.

The sole rule of inference is detachment for the slash, (DET). Several points need to be made about this axiomatization. First, it is to be sure an axiomatization of the never-falses. Among the axioms themselves there are nonassertive formulas. Indeed, all instances of CA2, for example, are *never-true*! Second, while this is only an axiomatization of the sentential component, a first-order overlay will be supplied. Third, this axiomatization is not impure, in Dunn's sense: no particular formula is mentioned. There is, however, indirect reference to the atomic components of formulas through the assertivity function. Fourth, the function α is only needed for the metalinguistic presentation of the axioms. It occurs in the axiomatization but not in the axioms themselves. Rather, its object language sentential values do. Fifth, because of this, and the fact that α is not $1-1$, this is not a normal schematization. The schemata given do not determine logical form: the values of α are sometimes conjunctions but most often disjunctions.[6] Further, α is unusual in that it is not a homomorphism with respect to substitution:

$$\alpha(sub\ A \text{ for } p \text{ in } B) \neq sub(A \text{ for } p \text{ in } \alpha B).$$ [7]

For all this, the axiom set is decidable: each formula has finitely many subformulas, whether one formula is the value of α at another is determinable, and there are only finitely many schemata to check.[8] Finally, there is the following theorem:

Theorem *CA is sound and complete.*

The proofs are omitted here.[9] The soundness proof is a straightforward induction: the axioms are not falsifiable and the rule of inference preserves nonfalsity. The completeness proof involves the inductive construction of a maximal set of formulas, excluding some given nontheorem. This serves as the basis for the canonical model falsifying that formula. The only novelty is that the constructed maximal set must be partitioned into two—one part to be the true formulas and the other to be the nonassertive ones. Only the first is negation consistent.

4 Quantification The function α is extended to a first-order language as follows:

1. $\alpha(Fa_1 \ldots a_n)$ is $Fa_1 \ldots a_n \vee \sim Fa_1 \ldots a_n$
2. $\alpha \forall xA$ is $\exists x \alpha A$
3. $\alpha \exists xA$ is $\exists x \alpha A$.

The clauses for the quantifiers parallel those for conjunction and disjunction. Just as one assertive conjunct or disjunct suffices for the assertiveness of the whole, any assertive instance suffices for the assertiveness of the quantified formula; both existential and universal quantifications are nonassertive only when every instance is.

The semantic clause for the truth of the universal quantifier deserves special mention:

$\forall xA$ is true if some instance is true and no instance is false.

(This may be given either a substitutional or domain-and-values interpretation.) A true universal generalization may have some nonassertive, hence untrue, instances.

The axioms for the quantificational logic of conditional assertions, *QCA*, are:

QCA0 All formulas that are instances of CA1–CA10
QCA1 $\forall xCx/Ca$
QCA2 $\forall x(B/Cx)/B/\forall xCx$
QCA3 $\forall x(Cx/B)/\exists xCx/B$,

where the variable is not free in B. In addition to (DET), there is a generalizing rule of inference (GEN) with the usual provisions. We now state the following theorem:

Theorem *QCA is sound and complete.*

The soundness proof is again unproblematic, so omitted. The completeness proof will be sketched.

Let A be some given nontheorem to be falsified. The construction of a falsifying model will then establish, by contraposition, that all "valid" (i.e., nonfalsifiable) formulas are theorems. Begin by constructing a set, G, of formulas as follows:

Let $G_0 = \{B: \vdash B\}$, the set of theorems.
Let $b_1, b_2, \ldots$ be a denumerable list of new constant names.

The language of *QCA* is being enriched for the construction of the model, as is normal for Henkin-style completeness proofs. Then let $C_1, C_2, \ldots$ be an enumeration of the formulas of the enriched language and include all new instances of the axioms in G_0.

Let $G_{n+1} = G_n \cup \{C_n\}$, if $G_n \cup \{C_n\} \nvdash A$;
$G_{n+1} = G_n \cup \{{\sim}C_n\}$, if $G_n \cup \{C_n\} \vdash A$ and C_n is not of the form $\forall xB$;
and $G_{n+1} = G_n \cup \{{\sim}C_n, {\sim}Bb, \alpha \sim Bb\}$ where b is the alphabetically first new constant name from the above numeration, if $G_n \cup \{C_n\} \vdash A$ and C_n *is* of the form $\forall xB$.

As usual, G is defined as the union over the chain of G_i's.

The most important fact about G is this: $G \nvdash A$. The proof is by an induction and somewhat idiosyncratic. The base case, $G_0 \nvdash A$, is routine: the addition of new instances of the axioms does not give rise to any new theorems in the old vocabulary. The first two inductive cases are also familiar. The third case is the interesting one and the one worth investigating. When an existential sentence (more exactly, a negated universal one) is added to G, a "witness" is added at the same time to ensure that the existential sentence will be made true by the subsequent valuation. That much is standard fare. In addition to the corroborating witness, however, a "certificate of assertiveness" is added to the set. It is not enough simply to add an instance; an *assertive* instance must be added, since, as noted earlier, a nonfalse instance, by itself, cannot suffice to establish that an existential sentence is also nonfalse. The proof of this case is as follows: the inductive hypothesis is that $G_n \nvdash A$. For this third inductive case it is further assumed that $G_n \cup \{C_n\} \vdash A$. A provable fact about sets of formulas, G, is that if $G \cup \{C\} \vdash A$ but $G \nvdash A$, then $G \cup \{{\sim}C, \alpha \sim C\} \nvdash A$. In the present case this means that $G_n, {\sim}\forall xB, \alpha \sim \forall xb \nvdash A$. Suppose now that $G_n, {\sim}\forall xB, {\sim}Bb, \alpha \sim Bb \vdash A$, where b is new. Then $G_n, {\sim}\forall xB, \alpha \sim \forall xB \vdash ({\sim}Bb \,\&\, \alpha \sim Bb)/A$, by the deduction theorem (and importation). Then, by generalization (since b is new), $G_n, {\sim}\forall xB, \alpha \sim \forall xB \vdash \forall x(({\sim}B \,\&\, \alpha \sim B)/A)$. By confinement, QCA3, $G_n, {\sim}\forall xB, \alpha \sim \forall xB \vdash {\sim}\forall x \sim ({\sim}B \,\&\, \alpha \sim B)/A$, which is equivalent to $G_n, {\sim}\forall xB, \alpha \sim \forall xB \vdash {\sim}\forall x(B \vee {\sim}\alpha B)/A$. It is an easily proved theorem, using specification, CA8, and generalization, that $\forall x(B \vee {\sim}\alpha B)/\forall xB$. The presence of $\alpha \sim \forall xB$, which is the same as $\alpha\forall xB$, and the limited form of contraposition available establish that $G_n, {\sim}\forall xB, \alpha \sim \forall xB \vdash {\sim}\forall xB/{\sim}\forall x(B \vee {\sim}\alpha B)$. Two applications of detachment establish that $G_n, {\sim}\forall xB, \alpha \sim \forall xB \vdash A$, which contradicts what is known. So $G_n \cup \{{\sim}\forall xB, \alpha \sim \forall xB, {\sim}Bb, \alpha \sim Bb\} \nvdash A$. G_{n+1} is a subset of the premise set (G_{n+1} does not necessarily include $\alpha \sim \forall xB$ at this point), so $G_{n+1} \nvdash A$. Thus one cannot only add witnesses safely, supporting the claims of existential statements, one can safely add assertive witnesses—and certification to that effect. (This will be discussed in more detail in the next section.) Hence, by induction, $G \nvdash A$.

Other important facts about G are that it is *prime*—if $B \vee C \in G$, then $B \in G$ or $C \in G$; it is *negation complete*—at least one of B, ${\sim}B$ is in G; and it is *deductively closed*—if $G \vdash B$, then $B \in G$. Significantly, G is *not* negation consistent. Most importantly, G is *assertively E-complete*—if $\exists xB \in G$, then Bb and αBb are in G, for some constant b. This is trivial, given the construction of G. Simple E-completeness—if $\exists xB \in G$, then $Bb \in G$, for some b (which is what suffices for classical completeness proofs) would not be sufficient.

The set G is separated into those to be true and those to be nonassertive as follows:

Let $Tr = \{B\colon B \in G \text{ and } {\sim}B \notin G\}$
and $Na = \{B\colon B \in G \text{ and } {\sim}B \in G\}$.

Tr is the set that must be (simply) *E*-complete and making *G* assertively *E*-complete guarantees this. The valuation, *V*, is defined this way:

$V(B) = T$ iff $B \in Tr$,
$V(B) = N$ iff $B \in Na$,
and $V(B) = F$ iff $B \notin G$.

A domain-and-value semantics is easily recovered: let the domain be the set of constants in the language and the interpretation be the identity function for constants—$I(a) = a$—while for predicates $I(F) = \{a\colon Fa \in Tr\}$ (and similarly for predicates of higher degree). Since $G \nvdash A$, $A \notin G$ and $V(A) = F$. *A* is not, therefore valid; *QCA* is complete.

5 Comments on E-completeness In [6], two axiomatizations of versions of a first-order logic using conditional assertions are presented, one for the always true formulas and one for those which are never false. In the completeness proof for each, a set of formulas is constructed to serve as the truths and nonfalsities, respectively, of the recovered valuations, as in the proof presented here. There are a couple of points worth considering that concern the difference between making these good sets *E*-complete, as is done in [6], and making those sets assertively *E*-complete, as is done here. The α-function is new to this presentation and so was not available in the earlier paper, but the presence of the sentential constant, *t*, which is always true, and the defined connective, $\supset_t$, suffice to overcome the problems to which simply *E*-complete truth- (nonfalsity-) sets are vulnerable. $B \supset_t C$ is, by definition, $(t \supset B) \supset (t \supset C)$. It translates roughly as "*B* is not true or *C* is true".

When axiomatizing the never falses, the danger is this: in the construction of the good set, called "*G*" here, any time a formula of the form $\exists xB$ is added, a formula of the form *Bb* is also added, where *b* is one of the new constants. Nothing in this alone guarantees that ${\sim}Bb$ will not be added at a later stage of the construction, nullifying the value of *Bb* as a witness. If both *Bb* and ${\sim}Bb$ are present in *G*, they will get the value *N* by the valuation function in order that the given nontheorem can be excluded. It is only assertive contradictions that yield everything. Thus, even though the goal is a nonfalsity set, certain formulas in that set have to be *true*. In particular, if $\exists xB$ has any assertive instances, it can only be nonfalse if some of those instances are true. This is more than just a matter of fine tuning the proof. Without some true instances, $\exists xB$ cannot be true, but if even one of its instances is false, it cannot belong to the broader category of nonfalsities without true instances. If the only instance in hand is just nonfalse, $\exists xB$ might yet be false. A true witness ensures truth but a nonfalse witness does not ensure nonfalsity. One way to ensure the nonfalsity of $\exists xB$ when axiomatizing the never falses is to go beyond the call of duty, as it were, and make sure that it is true.

There is a companion problem that plagues the construction of a truth set in the completeness proof for any axiomatization of the always trues. Because the inference from Bb to $\exists xB$ is truth-preserving, albeit not nonfalsity-preserving, the presence of a witness to an existential formula in a truth set does suffice. Moreover, once a witness to an existential sentence does get added to a truth set, it will not be subject to forced revaluation as nonassertive because of the presence of its negation; truth sets, unlike nonfalsity sets, must remain negation consistent. The problem comes before this: it is not always possible to add *any* witness. Just as some formulas only have false instances, like $\exists x(Fx \,\&\, \sim Fx)$, others have only nonassertive ones. $\exists x((Fx \,\&\, \sim Fx)/Fx)$ is such a formula. In the case of those formulas which can have only false instances, the presence of their negations in the truth set excludes those instances. This is not the case for quantified formulas with only nonassertive instances. The negations of the instances of such formulas are equally nonassertive and equally bad to have in truth sets. Perhaps a formula like $\exists x((Fx \,\&\, \sim Fx)/Fx)$ can be added to a truth set without disturbing the (negation-) consistency of that set and without giving rise to the nontheorem to be excluded even though none of its instances can be.

The fine tuning becomes trickier at this point. In the proof offered here, $\sim\forall xB$ and a witness to $\sim\forall xB$ were added if $\forall xB$ *could not* be added, not merely if $\sim\forall xB$ could be added. It is easy to prove that if $\forall xB$ cannot be added, then its negation and a witness can be. Establishing that a witness can be added solely on the basis that $\sim\forall xB$ *can* be added requires much more from the proof theoretic machinery.

In [6], the whole problem of adding witnesses is fortuitously avoided. "Henkinizing" the set under construction (adding new witnesses whenever an existential claim is added) is only one way of making sure that a set is E-complete. The method employed in [6] is "Hasenjaegering"—adding a sentence of the form $\exists xB \supset_t Bb$, with b new, for each formula of the form $\exists xB$, to the *original* set, G_0. These sentences are added prior to the inductive expansion of the set. Thus, if an existential claim does get added, deductive closure will provide the witness. In addition, Dunn adds all "tautologies$_t$" to the original set, where these are (roughly) the truth-functionally decidable always-true formulas. The pitfall is avoided this way, letting f be a constantly false sentence: although $\exists x((Fx \,\&\, \sim Fx)/Fx) \supset_t f$ is always true, it is not a tautology$_t$ because the quantified antecedent is not subject to truth table analysis in the right way. However, $\exists x((Fx \,\&\, \sim Fx)/Fx) \supset_t (Fb \,\&\, \sim Fb)/Fb$ was previously added to the set, by the Hasenjaegering. These give rise to $(Fb \,\&\, \sim Fb)/Fb$, and $(Fb \,\&\, \sim Fb)/Fb \supset_t f$ *is* a tautology$_t$. Always nonassertive existential claims are thus excluded and even attempts to make the consequent instance of the Hasenjaegering nonassertive will be frustrated by a similar, if longer, chain of reasoning. The constructed set will indeed be E-complete and, in the terminology used here, it will be assertively E-complete.

There is a related, though ultimately trivial, problem for truth sets vis-à-vis universal formulas. Although either Hasenjaegering or (assertively) Henkinizing can guarantee that $\exists$-sentences have witnesses adequate to the task of ensuring truth, $\forall$-sentences also need witnesses in truth sets. The presence of $\forall xB$ does preclude any hostile, falsifying instances since $\sim Bb \supset_t \sim\forall xB$ is a theorem. The absence of all hostile witnesses, however, does not guarantee the presence of any supporting ones: $\forall xB \supset_t Bb$ is not a theorem. At least some instances of $\forall xB$ have to be true if it is to be true, as well as all instances having to be nonfalse. However, $\forall xB \supset_t \exists xB$ is a theorem so the "A-completeness" of truth sets can be made to ride piggy-back on their E-completeness.

NOTES

1. Quine [8] credits Rhinelander with the notion. Von Wright [10] suggests that conditionals in common discourse are conditional assertions.
2. Manor [7], van Fraassen [9], and Cohen [2] are also concerned with the formal development.
3. Manor [7] disagrees; van Fraassen [9] also considers what he calls the "quasi-Belnap conditional" which is nonassertive when its antecedent is not true.
4. Belnap's own semantics suggests gaps. This presentation exploits the isomorphism between a logic with gaps and a three-valued logic.
5. This would be similar to the role played by the assertion operator in Bochvar's three-valued logic or double complete negation in Reichenbach's.
6. An extension of *CA* to include an implication connective provides another nondisjunctive value for α. See [1], p. 71, and [2], pp. 111–117.
7. Belnap pointed this out in a letter.
8. This point is also made in [3].
9. See [2], Chapter 2.

REFERENCES

[1] Belnap, N. D., Jr., "Conditional assertion and restricted quantification," pp. 48–75 in *Truth, Syntax, and Modality*, ed., H. Leblanc, North-Holland Publishing Co., Amsterdam, 1973.

[2] Cohen, D. H., *The Logic of Conditional Assertions*, Doctoral Dissertation, Indiana University, 1983.

[3] Cohen, D. H., "Relevant implication and conditional assertion," forthcoming in Section 75 of *Entailment*, vol. II, ed., N. D. Belnap, Jr.

[4] Dummett, M., "Truth," *Proceedings of the Aristotelian Society*, vol. 59 (1959), pp. 141–162.

[5] Dummett, M., *Frege: Philosophy of Language*, Gerald Duckworth & Company, Limited, London, 1973.

[6] Dunn, J. M., "Axiomatizing Belnap's conditional assertion," *Journal of Philosophical Logic*, vol. 4 (1975), pp. 383–397.

[7] Manor, R., *Conditional Forms: Assertion, Necessity, Obligation and Commands*, Doctoral Dissertation, University of Pittsburgh, 1971.

[8] Quine, W. V. O., *Methods of Logic*, Holt, Rinehart and Winston, New York, 1950.

[9] van Fraasen, B. C., "Incomplete assertion and Belnap connectives," pp. 43-70 in *Contemporary Readings in Philosophical Logic and Linguistic Semantics*, ed., Hockney, et al., D. Reidel Publishing Co., Dordrecht, Holland, 1975.

[10] von Wright, G. H., "On conditionals," pp. 127-165 in *Logical Studies*, Routledge & Kegan Paul, London, 1957.

Department of Philosophy and Religion
Colby College
Waterville, Maine 04901