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A Topos-Theoretic Approach to Reference and Modality*

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Abstract This paper presents an approach to modal logic based on the notion of *kind* or interpretation of a count noun as a prerequisite for reference. It gives a mathematical formalization of this notion in the context of a locally connected topos $\mathcal{E}$ (thought of as a universe of variable sets) over a topos $\mathcal{S}$ (thought of as a universe of constant sets). In this context, modal operators are intrinsically definable and the resulting formal system is described in some detail.

Introduction This paper is an essay on reference and generality in natural languages (to borrow from the title of Geach [5]). More precisely, it is concerned with the semantics of pronouns, proper names, and count nouns.

It is a remarkable fact of natural languages that a proper name picks up its reference uniquely any time that it is uttered, whether or not its reference is present at the moment of utterance, whether or not we know the reference's whereabouts at that moment, whether or not we are able to recognize its reference, and whether or not we are referring to events that took place in the past or may take place in the future. My main concern will be with the following *problem*: What semantical structure should be postulated for this relation (between a name and its reference) to accomplish the formidable tasks just described? I shall propose an answer based on the notion of *kind* or *sortal* viewed as the semantical interpretation of count nouns.

The essay is divided into two parts. In the first, "Count nouns and kinds", taken from an unpublished paper in collaboration with Marie Reyes, I state and give arguments for a series of *theses on reference and generality* involving proper names, count nouns, and kinds. Although this medieval practice has long gone out of fashion, I believe it useful to understand the issues involved. Some of these

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theses have been already argued by Gupta [6] and Macnamara [13], following the lead of Geach [5], Bressan [3] and others. Nevertheless, the most important of the theses for the whole development, namely the modal constancy of *all kinds*, is not to be found in the writings of these authors and, in fact, it contradicts further theses of some of these authors.

The second part of this essay, "Topos semantics", is an attempt to give a mathematical formalization of the notion of kind in the context of topos theory. It is my conviction (to rephrase Montague [17]) that philosophy, at this stage in history, has as its proper theoretical framework category theory (which includes set theory with "urelements") rather than set theory alone, as Montague believed in 1970. I believe that topos theory provides an adequate context to formalize the basic notion of modal constancy. Furthermore, the very possibility of using topos theory imposes very strong and, I believe, fruitful constraints on the logic of reference and generality as we will show in some detail. To mention just two of them: the logic of quantification and identity of a topos is the standard one; furthermore, under the natural assumption that kinds are exponentiable (so that higher-order logic is interpretable), *modal operators are intrinsically definable in* a topos and need not be introduced as a further structure. Thus this approach differs from others (such as in [17]) which change the logic of quantifiers to accommodate the modal operators.

This concludes the description of the contents of this paper. I have left out some developments of this approach to reference and modality which have taken place after the first draft of this paper. Connections between modal operators and locale theory have been studied in some detail in Reyes and Zawadowski [19]; questions of soundness and completeness of the formal systems obtained via this topos-theoretic approach are studied in Lavendhomme, Lucas and Reyes [11]. Finally, this approach was developed with a view to applications to cognition and in close contact with the work on language learning of Macnamara and the work of M. Reyes on the semantics of literary texts. The interested reader may consult [13], Macnamara and Reyes [14], and M. Reyes [20] for these applications.

1 Count nouns and kinds The first of our theses on reference concerns proper names.

(1) The denotations of proper names are rigid.

This thesis asserts that a proper name, say "Nixon" has the property of denoting its reference throughout all actual as well as possible situations, past, present, and future in which Nixon appears or may appear. A biographer of Nixon would use the single proper name "Nixon" to refer to the boy who grew up in California, to the young politician who won his first election, and to the president who was forced to resign from office in spite of the different times and situations. He may be unsure of whether Nixon won his first election by fraudulent means and he may discuss both the possibility that he used such means and the possibility that he did not to arrive at the truth of the matter. In other words, he must consider counterfactual situations (since clearly one of the possibilities cannot be realized) about Nixon, i.e., the reference of "Nixon", in order to describe the actual course of events. Similarly, to explain Nixon's actions, the biographer should entertain a series of possibilities as to Nixon's motives and evaluate them critically. The point is once more that we are forced to consider counterfactual situations about the reference of the proper name "Nixon". The thesis of rigidity has been forcefully argued by Kripke [9].

As an aside, let me remark that the fact that counterfactual situations have to be considered to describe real situations is beautifully exemplified in Classical Mechanics and in the Calculus of Variations: to describe the real trajectory of a body, we compute the Lagrangian of all its possible trajectories (most of which are not physically possible!) and we take that trajectory for which the Lagrangian has a minimum (or a stationary value).

(2) Rigidity presupposes count nouns.

This thesis, which is characteristic of the approach we are developing, asserts that the only way of tracing the identity of the reference of the proper name "Nixon" throughout all real and possible situations, past or present, is by means of a count noun such as "person" or "man". Indeed, the boy in California, the young politician, and the president remained one and the same *person*, although he successively stopped being a boy, a young politician, and a president. We cannot, for instance, trace Nixon's identity through the molecules that compose his body, since those that constituted his body at the time of his youth were different from those that constituted his body at the time that he was a president. As Aristotle pointed out, change requires *something* to change. There must be a constancy that underlies the change of the boy into the president. The thesis guarantees such a constancy through the count nouns to the interpretation of which the individual experiencing the change belongs.

To make this thesis more precise, I shall state claims on the interpretations of count nouns, the kinds. Kinds are the interpretations of count nouns such as "person", "dog", "atom of hydrogen". There are other nouns besides count nouns (e.g., mass nouns such as "money", "clay", and "oxygen"), but we associate kinds with count nouns only. Members of a kind may be individuated and counted. We can interpret expressions such as "three dogs", "two drops of water", and "three atoms of oxygen", as well as expressions of generality such as "every dog", but we cannot interpret "three oxygens" or "two moneys". I consider the notion of kind as being so basic that it cannot be analysed in terms of more basic notions. All that we can assert is that a kind has members which are individuated and that it makes sense to say that two members are identical. Kinds are the constitutive domains which allow us to use quantifiers and the equality symbol correctly. I shall not assume that equality between members of a given kind is decidable.

An important consequence of this postulate for kinds is that members of a kind can be counted (at least under some limitations) and that they are subject to the logic of quantification. In fact, counting does not apply to heaps or conglomerates of objects, but to kinds: the same "heap" which makes up an army may be counted as 1 army, 6 divisions, 18 brigades, or half a million men. This point was made quite forcefully by Frege and I have nothing to add.

(3) Kinds are modally constant.

This thesis is in fact a further development of the previous one. In fact, as we saw in (2), kinds are what remains "constant" from situation to situation and thus kinds are precisely the required standard against which we can understand changes and modalities. In other words, kinds should be modally constant "by definition". A consequence of this thesis is that modalities can be genuinely applied to predicates only, not to kinds themselves. A member of the kind PASSENGER is necessarily a passenger *qua* passenger. On the other hand, a particular person who happens to be a passenger need not be necessarily a passenger *qua* person. In this case, we are saying something about the predicate "to be a passenger" of the kind PERSON, namely that it is not necessary. In the first, we are saying something about the predicate "to be a passenger" of the kind PASSENGER, namely that it is necessary. It follows that attempts to apply modalities to kinds themselves to form new "kinds" such as POSSIBLE APPLE, POSSIBLE CAR, or POSSIBLE MAN (as used by Gupta [6] to define "modal constancy") are wrongheaded. And in fact, there are serious difficulties with any attempt to view possible apple or possible man as kinds: neither membership in the kind nor the relation of identity are well-defined. As regard the first: Does a portion of jelly apple count as a possible apple? Or again, does a piece of junk metal count as a possible car? As regards identity, Quine [18] has already asked the relevant question: How many possible fat men are there in the door? On the other hand, we may ask of a given *fruit* whether it is possibly an apple and the intuition that "apples are necessarily apples" that Gupta tried to express as "POSSIBLE APPLE = APPLE" could be expressed rather as "the predicate 'to be an apple' of the kind FRUIT is a necessary predicate". We cannot eliminate the kind FRUIT in this formulation, since the statement "the predicate 'to be an apple' of the kind INGREDIENT IN A RECIPE is a necessary predicate" should be false for cooking to be possible at all!

This way of considering "possibility" and "necessity" agrees with the grammar of these notions. Suppose that we find an archeological site with skeletons of some anthropoids. If we are asked whether some are humanoids, we could naturally reply that "three of these are possibly humanoids's skeletons" or "it is possible that three of these skeletons are humanoid's", but we would not say "there are three possible humanoid's skeletons". Similarly, we do not say that Mr. and Mrs. X have twelve possible children, but rather that it is possible for Mr. and Mrs. X to have twelve children.

I shall distinguish sharply between kinds and predicates of a kind. Whereas the first are modally constant and independent of any particular situation, the second are not and whether a member of a kind falls under a predicate or not will depend on the situation envisaged. I believe that this context does better justice to the dialectics of change, constancy, and modalities. As I said in the introduction, this thesis is basic for the whole approach to the semantics of proper names and count nouns that I develop in this essay.

2 *Topos semantics*

2.1 Constant sets vs variable sets in a topos On several occasions, Lawvere has pointed out that the dialectics of **variation vs constancy** (which we hinted upon in Section 1) may be given an explicit formulation in topos theory via the notion of a geometric morphism $\mathcal{E} \to \mathbb{S}$. In this case $\mathcal{E}$ may be thought of as a universe of variable sets, whereas $\mathbb{S}$ may be thought of as the universe of constant sets (possibly with "urelements").

I recall that a geometric morphism $\mathcal{E} \to \mathbb{S}$ is given by a couple of adjoint functors

$$\mathbb{S} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathcal{E}$$

such that $\Delta \dashv \Gamma$ and Δ preserves finite limits. It is customary to call Δ "the constant functor" and Γ "the global sections functor" or "points".

Recalling the thesis that kinds are modally constant, whereas predicates of kinds carry the burden of change, its seems natural to identify a kind with a (constant) set S of $\mathbb{S}$ and a predicate of that kind S with a morphism $\Delta S \to \Omega_{\mathcal{E}}$ of $\mathcal{E}$, or, what amounts to the same, with a variable subset of ΔS. Indeed $=$ in S is given by the relation of identity of the kind in question. Basic kinds such as PERSON, DOG will be identified with sets of "urelements".

It is natural, therefore, to define the category $\mathcal{C}$ of constant sets as the full subcategory of $\mathcal{E}$ where objects are of the form ΔS (with $S \in \mathbb{S}$).

Remark This identification amounts to viewing kinds and their predicates as the hyperdoctrine $(\mathbb{S}, P_{\mathcal{E}})$ (in the sense of Lawvere [12]) where $P_{\mathcal{E}} : \mathbb{S}^{op} \to \text{Sets}$ is the functor defined by

$$P_{\mathcal{E}}(S) = \mathrm{Sub}_{\mathcal{E}}(\Delta S) \simeq \mathcal{E}(\Delta S, \Omega)$$

$$P_{\mathcal{E}}(S \xrightarrow{f} T) = \mathrm{Sub}_{\mathcal{E}}(\Delta T) \xrightarrow{(\Delta f)^{-1}} \mathrm{Sub}_{\mathcal{E}}(\Delta S)$$

where $(\Delta f)^{-1}$ is the pull-back along Δf.

It is well-known that $(\Delta f)^{-1}$ has both a left and a right adjoint: $\exists_{\Delta f} \dashv (\Delta f)^{-1} \dashv \forall_{\Delta f}$.

We may now express modal constancy of kinds as follows:

Assumption 1 *Kinds are constant sets of the geometric morphism* $\mathcal{E} \to \mathbb{S}$.

2.2 The $\Box$ operator In this section we show from Assumption 1 that the modal operator of necessity $\Box$ is definable for predicates of kinds, essentially in terms of "global sections".

In fact, from the adjunction $\mathbb{S} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathcal{E}$, $\Delta \dashv \Gamma$, we derive the following adjoint maps

$$\Omega_{\mathbb{S}} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathcal{E}}),\ \delta \dashv \gamma$$

essentially "by restricting Δ, Γ to subobjects of 1".

In more detail, given p, we define $\delta(p)$ as follows:

$$\frac{X \xrightarrow{p} \Omega_{\mathbb{S}}}{P \rightarrowtail X} \quad (\Omega_{\mathbb{S}} \text{ classifies subobjects}).$$

Applying Δ we obtain the equivalences

$$\frac{\dfrac{\Delta P \rightarrowtail \Delta X}{\Delta X \longrightarrow \Omega_{\mathcal{E}}}\quad (\Omega_{\mathcal{E}} \text{ classifies subobjects})}{X \xrightarrow{\delta(p)} \Gamma(\Omega_{\mathcal{E}})}\quad (\Delta \dashv \Gamma).$$

We remark that δ may be defined, alternatively, as the transpose of the classifying map $\tau : \Delta\Omega_{\mathbb{S}} \to \Omega_{\mathcal{E}}$ of $1 \simeq \Delta\ 1 \rightarrowtail \Delta\Omega_{\mathbb{S}}$.

Similarly, $\gamma(K)$ is defined for a given K as follows:

$$\frac{\dfrac{X \xrightarrow{K} \Gamma(\Omega_{\mathcal{E}})}{\Delta X \longrightarrow \Omega_{\mathcal{E}}}\quad (\Delta \dashv \Gamma)}{K \rightarrowtail \Delta X}\quad (\Omega_{\mathcal{E}} \text{ classifies subobjects}).$$

We apply Γ and form the pull-back

$$\begin{tikzcd}
\Gamma(K) \arrow[r, tail] & \Gamma\Delta X \\
N(K) \arrow[u] \arrow[r, tail] & X \arrow[u, "\eta_X"']
\end{tikzcd}$$

where $\mathrm{Id} \xrightarrow{\eta} \Gamma\Delta$ is the unit of the adjunction $\Delta \dashv \Gamma$. The lower horizontal map in the pull-back diagram is classified by $X \xrightarrow{\gamma(K)} \Omega_{\mathbb{S}}$.

An easy computation gives that $\delta \dashv \gamma$. Furthermore, since Δ preserves finite limits

$$\delta(\top) = \top \quad \text{and} \quad \delta(p \wedge q) = \delta(p) \wedge \delta(q).$$

We may now define the modal operator

$$\Box = \delta\gamma : \Gamma(\Omega_{\mathcal{E}}) \to \Gamma(\Omega_{\mathcal{E}}).$$

The following can be proved easily (or invoke the fact that $\Box$ is a lex cotriple)

1. $\Box \leq \mathrm{Id}$
2. $\Box^2 = \Box$
3. $\Box T = T$
4. $\Box(K_1 \wedge K_2) = \Box K_1 \wedge \Box K_2$.

From here we can define an operator $\Box_S$ on predicates of ΔS as follows: if $\Delta S \xrightarrow{\varphi} \Omega_{\mathcal{E}}$ is an arbitrary predicate of ΔS, we let $\Delta S \xrightarrow{\Box_S\varphi} \Omega_{\mathcal{E}}$ be the transpose of $S \xrightarrow{\Box\circ\mathrm{tr}(\varphi)} \Gamma(\Omega_{\mathcal{E}})$, where $\mathrm{tr}(\varphi) : S \to \Gamma(\Omega_{\mathcal{E}})$ is the transpose of φ.

Example 1

$$\mathbb{S} = \mathrm{Sets} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathrm{Sets}^I = \mathcal{E},$$

where I is a set. In this case, $\Delta(S) = (S)_{i\in I}$ and

$$\Gamma((X_i)_{i\in I}) = \prod_{i\in I} X_i.$$

From these functors we derive

$$\Omega_{\mathbb{S}} = 2 \underset{\gamma}{\overset{\delta}{\rightleftarrows}} 2^I = \Gamma(\Omega_{\mathcal{E}})$$

which are easily checked to be

$$\delta(p) = \begin{cases} I & \text{if } p = \top \\ \phi & \text{if } p = \bot \end{cases}$$

$$\gamma(K) = \begin{cases} \top & \text{if } K = I \\ \bot & \text{if } K \neq I. \end{cases}$$

We shall write

$$\delta(p) = \{i \in I \mid p\}, \gamma(K) = \|\forall i \in I(i \in K)\|,$$

where $\| \ldots \|$ stands for "truth-value of . . .".

The action of $\Box_S$ on predicates of ΔS may be described simply as

$$i \Vdash \Box_S \varphi[s] \quad \text{iff} \quad \forall j \in I\, j \Vdash \varphi[s]$$

for all $s \in S$.

Example 2

$$\mathbb{S} = \text{Sets} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \text{Sets}^{\mathbb{P}^{op}} = \mathcal{E},$$

where $\mathbb{P} = (P, \leq)$ is a pre-ordered set. In this case, ΔS is the constant presheaf $\Delta S(U) = S\ \forall U \in P$ and $\Gamma(F) = \varprojlim_{\mathbb{P}^{op}} F$.

The maps

$$\Omega_{\mathbb{S}} = 2 \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Omega(1) = \Gamma(\Omega_{\mathcal{E}})$$

where $\Omega(1) = \{K \subseteq P \mid K \text{ is downwards closed}\}$ are easily seen to be given by

$$\delta(p) = \{V \in P \mid p\}, \quad \gamma(K) = \|\forall U \in P(U \in K)\|.$$

The action of $\Box_S$ on predicates of ΔS is easily described:

$$U \Vdash \Box_S \varphi[s] \quad \text{iff} \quad \forall V \in P\, V \Vdash \varphi[s]$$

for all $s \in S$.

Example 3 $\mathbb{S} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathcal{E}$, where Γ is bounded. This means that $\mathcal{E}$ may be presented as the category of sheaves over a site $\mathbb{C}$ in $\mathbb{S}$.

The maps

$$\Omega_{\mathbb{S}} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathcal{E}}),$$

where $\Gamma(\Omega_{\mathcal{E}})$ = set of closed sieves, are given by:

$$\delta(p) = \text{closure of } \{C \in \mathbb{C} \mid p\}$$

$$\gamma(K) = \|\forall C \in \mathbb{C}(C \in K)\|.$$

The action of $\Box_S$ on predicates of ΔS is given by $C \Vdash \Box\varphi[S]$ iff $\exists\{C_i \to C\}_{i \in I} \in \text{Cov}(C) \forall i \in I \forall C' \in \mathbb{C}\ C' \Vdash \varphi[S]$, where we have identified $s \in S$ with its image under the canonical map $S \to i\, \Delta S(C)$. For proofs of these assertions and further details, see the Appendix.

2.3 Locally connected topos In several examples of a topos $\mathcal{E}$ defined over $\mathbb{S}$, $\mathcal{E} \to \mathbb{S}$, which include the case that $\mathcal{E} \to \mathbb{S}$ is given as the category of presheaves over a category in $\mathbb{S}$, the functor $\Delta : \mathbb{S} \to \mathcal{E}$ has a left adjoint $\pi_0 \dashv \Delta$. When this is so, π_0 is called the "connected components functor" for reasons to be discussed later. I shall be especially interested in cases that π_0 satisfies some Frobenius type conditions.

The following theorems are due to Barr and Paré [1]:

Theorem 2.3.1 *Let $\mathcal{E} \to \mathbb{S}$ be a geometric morphism. The following are equivalent*
(a) *Δ is a cartesian closed functor*
(b) *Δ has a left adjoint π_0 satisfying the Frobenius reciprocity condition*

$$\pi_0(\Delta S \times \mathsf{E}) \cong S \times \pi_0 \mathsf{E}.$$

Theorem 2.3.2 *Let $\mathcal{E} \to \mathbb{S}$ be a geometric morphism. The following are equivalent*
(a) *Δ is a locally cartesian closed functor*
(b) *Δ has a left adjoint satisfying the generalized Frobenius condition*:

$$\pi_0\begin{pmatrix} \Delta S \times \mathsf{E} \\ \Delta T \end{pmatrix} \simeq S \underset{T}{\times} \pi_0 \mathsf{E}.$$

Remarks

(1) The first condition of Theorem 2.3.1 may be reformulated as "Δ preserves exponentials", whereas the first condition of Theorem 2.3.2 may be reformulated as "Δ preserves the operations Π_f's".

(2) The second condition of Theorem 2.3.2 is equivalent to the statement that π_0 is $\mathbb{S}$-indexed, which according to [1] is the right generalization of the notion of a locally connected topos over sets due to Grothendieck. The reason for this terminology comes from the particular case $\mathrm{Sh}(X) \to$ Sets, where X is a topological space. In this case, all the conditions of both theorems are equivalent to the statement that X is locally connected. Furthermore, by identifying $\mathrm{Sh}(X)$ with the category of étale spaces over X, π_0 turns out to be the functor which sends an étale space over X into the set of its connected components.

I now state a second assumption on kinds.

Assumption 2 *The geometric morphism $\mathcal{E} \to \mathbb{S}$ is locally connected.*

This assumption may be stated in the equivalent form (cf. Theorem 2.3.2): Δ has a left adjoint $\pi_0 \dashv \Delta$ satisfying the generalized Frobenius condition.

A consequence of this assumption is that kinds (= constant sets, by Assumption 1) are exponentiable. Another consequence is the fact that the possibility operator $\Diamond$ is definable for predicates of kinds.

2.4 MAO operators In this section we show how to define a couple of operators $(\Diamond, \Box)$ on predicates of kinds from Assumptions 1 and 2. Whereas $\Box$ was defined in terms of global sections, $\Diamond$ will be defined in terms of connected components.

We start from

$$\Omega_{\mathbb{S}} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathcal{E}})$$

and we define a left adjoint $\lambda \dashv \delta$ as follows:

$$\begin{array}{ll} \dfrac{X \xrightarrow{K} \Gamma(\Omega_{\mathcal{E}})}{\dfrac{\Delta X \longrightarrow \Omega_{\mathcal{E}}}{K \rightarrowtail \Delta X.}} & \begin{array}{l}(\Delta \dashv \Gamma)\\[1ex] (\Omega_{\mathcal{E}} \text{ classifies subobjects})\end{array} \end{array}$$

We apply π_0 and take the image factorization

$$\begin{array}{ccc} \pi_0(K) & \longrightarrow & \pi_0(\Delta X) \\ \downarrow & & \downarrow{\scriptstyle \epsilon_X} \\ P(X) & \hookrightarrow & X \end{array}$$

The lower horizontal map is classified by $X \xrightarrow{\lambda(K)} \Omega_{\mathbb{S}}$ (here $\epsilon_X : \pi_0\, \Delta \to \mathrm{Id}$ is the counit of $\pi_0 \dashv \Delta$). Easy computations give

$$\lambda \dashv \delta$$

and hence $\lambda(K \wedge \delta p) = \lambda K \wedge p$. We now define $\Diamond = \delta\lambda : \Gamma(\Omega_{\mathcal{E}}) \to \Gamma(\Omega_{\mathcal{E}})$ and check, easily, that the couple $(\Diamond, \Box)$ with $\Box = \delta\gamma$ satisfies the following conditions:

(1) $\Diamond \dashv \Box$
(2) $\Box \leq \mathrm{Id} \leq \Diamond$
(3) $\Box^2 = \Box$, $\Diamond^2 = \Diamond$
(4) $\Diamond(K_1 \wedge \Box K_2) = \Diamond K_1 \wedge \Box K_2$.

A couple of operators $(\Diamond, \Box)$ satisfying (1)–(4) will be called a MAO couple (MAO for "modal adjoint operators").

It is clear that we can define an operation $\Diamond_S$ on predicates of ΔS, just as we did for $\Box$: If $\Delta S \xrightarrow{\varphi} \Omega_{\mathcal{E}}$ is a predicate of ΔS, we let $\Delta S \xrightarrow{\Diamond_S \varphi} \Omega_{\mathcal{E}}$ be the transpose of $S \xrightarrow{\Diamond \circ \mathrm{tr}(\varphi)} \Gamma(\Omega_{\mathcal{E}})$, where $S \xrightarrow{\mathrm{tr}(\varphi)} \Gamma(\Omega_{\mathcal{E}})$ is the transpose of φ.

Remark In the case that Δ has a left adjoint π_0 satisfying Frobenius condition we can simplify the expression of the unit $\eta_S : S \to \Gamma\Delta S$ as follows:

$$\begin{array}{ll} \dfrac{S \xrightarrow{\eta_S} \Gamma\Delta S}{\Delta S \xrightarrow{1_{\Delta S}} \Delta S} & (\Delta \dashv \Gamma) \\ \hline \dfrac{1 \times \Delta S \longrightarrow \Delta S}{\dfrac{\pi_0(1 \times \Delta S) \longrightarrow S}{\dfrac{\pi_0(1) \times S \xrightarrow{\pi_2} S}{S \xrightarrow{C} S^{\pi_0(1)}.}}} & \begin{array}{l}(\pi_0 \dashv \Delta)\\[1ex] (\pi_0 \text{ satisfies Frobenius})\end{array} \end{array}$$

In other words, η_S is just the constant map. From here, we conclude that $\Delta\eta_S : \Delta S \to \Delta S^{\Delta\pi_0(1)}$ is again the constant map (since Δ preserves exponentials).

Similarly, $\epsilon_{\Delta S}: \Delta\Gamma\Delta S \to \Delta S$ is just "evaluation at the unit $1 \to \Delta\pi_0(1)$", since $\Delta\Gamma\Delta S \simeq \Delta S^{\Delta\pi_0(1)}$.

These maps $\Delta\eta_S$ and $\epsilon_{\Delta S}$ play an important role in [17], where Montague calls them "intension" and "extension", respectively.

Example 1 (bis)

$$\mathbb{S} = \text{Sets} \underset{\Gamma}{\overset{\pi_0}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\Delta}}}}} \text{Sets}^I = \mathcal{E},$$

where I is a set. In this case $\pi_0((X_i)_{i\in I}) = \coprod_{i\in I} X_i$ is the disjoint union of the members of the family.

Furthermore, the map λ in the diagram

$$\Omega_{\mathbb{S}} = 2 \underset{\gamma}{\overset{\lambda}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\delta}}}}} 2^I = \Gamma(\Omega_{\mathcal{E}})$$

is given by

$$\lambda(K) = \begin{cases} \top & \text{if } K \neq \varnothing \\ \bot & \text{if } K = \varnothing; \end{cases}$$

i.e., $\lambda(K) = \| \exists i \in I (i \in K) \|$.

The action of $\Diamond_S$ on predicates may be described as follows: $i \Vdash \Diamond_S \varphi[s]$ iff $\exists j \in I j \Vdash \varphi[s]$ for all $s \in S$.

We see that we obtain the usual "possible worlds" semantics for modal logic, with $I =$ the set of possible worlds.

Example 2 (bis)

$$\mathbb{S} = \text{Sets} \underset{\Gamma}{\overset{\pi_0}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\Delta}}}}} \text{Sets}^{\mathbb{P}^{op}} = \mathcal{E}$$

where $\mathbb{P} = \langle P, \leq \rangle$ is a pre-ordered set in $\mathbb{S}$. In this case, $\pi_0(F) = \varinjlim_{\mathbb{C}^{op}} F$ and the map λ in the diagram

$$\Omega_{\mathbb{S}} = 2 \underset{\gamma}{\overset{\lambda}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\delta}}}}} \Omega(1) = \Gamma(\Omega_{\mathcal{E}})$$

is given by $\lambda(K) = \| \exists U \in P (U \in K) \|$.

Furthermore, the action of $\Diamond_S$ on predicates ΔS may be described as follows:

$$U \Vdash \Diamond_S \varphi[s] \text{ iff } \exists V \in P\ V \Vdash \varphi[s],$$

for all $s \in S$.

The semantics thus obtained may be called the "possible situations" semantics. In this case $\mathbb{P}$ may be thought of as the set of (partial) possible situations pre-ordered by the relation of "containing whatever goes on in". A thorough study of this semantics with applications to literary texts may be found in [20].

Example 3 (bis) $\mathcal{E} \to \mathbb{S}$ is bounded and locally connected. By a theorem from [1] we can present $\mathcal{E}$ as the category of sheaves over a site $\mathbb{C}$ of $\mathbb{S}$ such that the constant presheaves are sheaves. In this case, we have the diagram

$$\mathbb{S} \underset{\Gamma}{\overset{\pi_0}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\Delta}}}}} \mathrm{Sh}_{\mathbb{S}}(\mathbb{C})$$

with $\Delta S(C) = S$ for all $C \in \mathbb{C}$, $\pi_0(F) = \varinjlim_{\mathbb{C}^{op}} F$, $\Gamma(F) = \varprojlim_{\mathbb{C}^{op}} F$. The maps λ, δ, γ are given, in our case by

$$\Omega_{\mathbb{S}} \underset{\gamma}{\overset{\lambda}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\delta}}}}} \Gamma(\Omega_{\mathcal{E}}),$$

where

$$\begin{aligned} \delta(p) &= \{C \in \mathbb{C} \mid p\} \\ \gamma(K) &= \| \forall C \in \mathbb{C}(C \in K) \| \\ \lambda(K) &= \| \exists C \in \mathbb{C}(C \in K) \| . \end{aligned}$$

In this case, the action of $\Diamond_S, \Box_S$ on predicates of ΔS may be described as follows:

$$\begin{aligned} C \Vdash \Box_S \varphi[s] \quad &\text{iff} \quad \forall C' \in \mathbb{C}\ C' \Vdash \varphi[s] \\ C \Vdash \Diamond_S \varphi[s] \quad &\text{iff} \quad \exists C' \in \mathbb{C}\ C' \Vdash \varphi[s] . \end{aligned}$$

2.5 Locally connected and connected toposes As the reader has probably noticed, we had a choice in our definition of the category of kinds: either as $\mathbb{S}(\Gamma(\Omega_{\mathcal{E}}))$ or as the full subcategory $\mathcal{C}$ of $\mathcal{E}(\Omega)$ consisting of objects of the form $(\Delta S, \delta_{\Delta S})$.

In this section we shall study a condition which says, roughly, that this choice is irrelevant. We just start with the following:

Proposition 2.5.1 *Let $\mathcal{E} \to \mathbb{S}$ be a locally connected geometric morphism. Then the following conditions are equivalent*:
(a) Δ *is full and faithful*
(b) $\Gamma\Delta \simeq \mathrm{Id}_{\mathbb{S}}$
(c) $\pi_0\Delta \simeq \mathrm{Id}_{\mathbb{S}}$
(d) $\pi_0(1) = 1$.

Proof:
(a) $\Leftrightarrow$ (b) is well known and easy to check
(b) $\Leftrightarrow$ (d) follows from $\Gamma\Delta S \cong S^{\pi_0(1)}$, whereas
(c) $\Leftrightarrow$ (d) follows from the Frobenius condition on π_0.

Notice that only the (usual) Frobenius condition is used in this proof, so we can weaken the hypothesis to require the existence of π_0 satisfying this condition.

We may now state the irrelevance of the choice:

Proposition 2.5.2 *Assume that $\mathcal{E} \to \mathbb{S}$ is locally connected and connected. Then the lifting*

$$\bar{\Delta} : \mathbb{S}(\Gamma(\Omega_{\mathcal{E}})) \to \mathcal{E}(\Omega)$$

induces an equivalence of categories between $\mathbb{S}(\Gamma(\Omega_{\mathcal{E}}))$ and $\mathcal{C}$.

Proof: Simple computations using Proposition 2.5.1(b).

In the presence of Assumption 2, connectivity of Γ leads to further conditions on the maps

$$\Omega_{\mathbb{S}} \overset{\lambda}{\underset{\gamma}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\delta}}}}} \Gamma(\Omega_{\mathcal{E}}).$$

In fact, $\gamma\delta = 1_{\Omega_{\mathbb{S}}} = \lambda\delta = 1_{\Omega_{\mathbb{S}}}$.

Finally, we shall look again at the example of Section 2.1 to see what this condition on connectivity means in these particular cases:

Example 1

$$\mathbb{S} = \text{Sets} \overset{\pi_0}{\underset{\Gamma}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\Delta}}}}} \text{Sets}^I = \mathcal{E}.$$

In this case $\pi_0(1) = \pi_0((1)_{i \in I}) = \sum_{i \in I} 1 = I$ and the connectivity means that $I = 1$. Therefore, connectivity brings about a collapse of the "possible worlds" semantics to just one world. Correspondingly, $\Diamond = \Box = \mathrm{Id}$.

Example 2

$$\mathbb{S} = \text{Sets} \overset{\pi_0}{\underset{\Gamma}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\Delta}}}}} \text{Sets}^{\mathbb{P}^{op}} = \mathcal{E}.$$

In this case $\pi_0(1)$ = set of connected components of the graph $\mathbb{P}$. Therefore, $\pi_0(1) = 1$ iff given $U, V \in \mathbb{P}$ there is a finite chain of elements of $\mathbb{P}$:

$$U \geq W_1 \leq W_2 \geq \ldots \leq W_{n-1} \geq V.$$

Therefore, it is possible to impose a condition of connectivity on "possible situations" semantics without collapsing modal operators to the identity.

Example 3 This example is delicate and I shall deal with it elsewhere.

In Section 3 we shall see what further logical rules connectivity imposes on $\Diamond$ and $\Box$.

2.6 Sets with coincidence relation and kinds Members of a kind such as DOG, PERSON, etc., may come in and go out of existence, contrary to members of the set of natural numbers, say. On the other hand, kinds such as PROPOSITION or PREDICATE OF PERSON have members that may happen to coincide at a given situation, although they are not identical.

It may well happen, for instance, that singing and working may coincide at a situation in which precisely those who are singing are those who are working. Nevertheless, these predicates are not identical.

This notion of coincidence is essential to understand problems of opacity which are ubiquitous in natural languages. Keenan and Faltz [8] have given numerous examples of lack of substitutivity of "equals for equals" which do not result from epistemic contexts of the kind "believes that", "wonders whether", etc. For instance, in the situation envisaged above, those who are singing with Fred need not coincide with those who are working with Fred. Of course modal operators create contexts for which we cannot substitute "equals for equals" without altering truth values.

It is not cogent to say that we are dealing with identical members of a kind which have different properties! Nevertheless, it is quite cogent to consider members which happen to coincide but are different (and hence they may have different properties).

In this section, we shall provide means to represent these facts in our theory, by defining, for any topos $\mathcal{E}$, and any complete Heyting algebra H in $\mathcal{E}$, the *category of sets with coincidence relation* ($\mathcal{E}(H)$).

The *objects* of $\mathcal{E}(H)$ are couples $(\mathsf{E},\delta_{\mathsf{E}})$ where $\delta_{\mathsf{E}}: \mathsf{E} \times \mathsf{E} \to H$ satisfies

(i) $\delta_{\mathsf{E}}(e,e') = \delta_{\mathsf{E}}(e',e)$
(ii) $\delta_{\mathsf{E}}(e,e') \wedge \delta_{\mathsf{E}}(e',e'') \leq \delta_{\mathsf{E}}(e,e'')$.

On the other hand, a morphism $(\mathsf{E},\delta_{\mathsf{E}}) \xrightarrow{f} (F,\delta_F)$ is a map $\mathsf{E} \xrightarrow{f} F$ of $\mathcal{E}$ which satisfies

$$\delta_{\mathsf{E}}(e,e') \leq \delta_F(f(e),f(e')).$$

Note in spite of the similarity, this is *not* the category of H-valued sets of Higgs [7] in the case of $\mathcal{E} = \text{Sets}$. In fact, $\text{Sets}(H)$ is *not* a topos, although it is locally cartesian closed (and even a quasi-topos): the map $(\mathbb{1},\delta_\phi) \rightarrowtail (\mathbb{1},\delta_1)$ where $\delta_\phi(*,*) = 0 \in H$ is both a monomorphism and an epimorphism, but *not* an isomorphism.

Theorem 2.6.1 *The category $\mathcal{E}(H)$ is locally cartesian closed.*

Proof: The terminal object is $(\mathbb{1},\delta_1)$, where $\mathbb{1} = \{*\}$ is the terminal object of $\mathcal{E}$ and $\delta_1(*,*) = 1 \in H$. The product of $(\mathsf{E},\delta_{\mathsf{E}})$ with (F,δ_F) is the diagram

$$(\mathsf{E},\delta_{\mathsf{E}}) \xleftarrow{\pi_{\mathsf{E}}} (\mathsf{E} \times F,\delta_{\mathsf{E}\times F}) \xrightarrow{\pi_F} (F,\delta_F)$$

where $\mathsf{E} \leftarrow \mathsf{E} \times F \to F$ is the product in $\mathcal{E}$ and $\delta_{\mathsf{E}\times F}((e,f),(e',f')) = \delta_{\mathsf{E}}(e,e') \wedge \delta_F(f,f')$. The equalizer of $(\mathsf{E},\delta_{\mathsf{E}}) \underset{g}{\overset{f}{\rightrightarrows}} (F,\delta_F)$ is the diagram

$$(H,\delta_H) \xrightarrow{e} (\mathsf{E},\delta_{\mathsf{E}}) \underset{g}{\overset{f}{\rightrightarrows}} (F,\delta_F),$$

where $H \xrightarrow{e} \mathsf{E} \underset{g}{\overset{f}{\rightrightarrows}} F$ is the equalizer of $\mathcal{E}$ and $\delta_H(h,h') = \delta_{\mathsf{E}}(e(h),e(h'))$.

The exponential of (F,δ_F) to $(\mathsf{E},\delta_{\mathsf{E}})$ is the diagram

$$(F^{\mathsf{E}},\delta_{F^{\mathsf{E}}}) \times (\mathsf{E},\delta_{\mathsf{E}}) \to (F,\delta_F),$$

where $F^{\mathsf{E}} \times \mathsf{E} \to F$ is the exponential in $\mathcal{E}$ and

$$\delta_{F^{\mathsf{E}}}(f,f') = \bigcap_{e,e' \in \mathsf{E}} [\delta_{\mathsf{E}}(e,e') \Rightarrow \delta_{\mathsf{E}}(fe,f'e')].$$

Finally, the Π operators may be described as follows: if $(\mathsf{E},\delta_{\mathsf{E}}) \xrightarrow{f} (F,\delta_F) \in \mathcal{E}(H)$,

$$\Pi_f\begin{pmatrix}(G,\delta_G)\\ \varphi\big\downarrow\\ (\mathsf{E},\delta_{\mathsf{E}})\end{pmatrix} = \begin{pmatrix}(P,\delta_p)\\ \big\downarrow\\ (F,\delta_F)\end{pmatrix}, \quad \text{where} \quad \begin{pmatrix}P\\ \big\downarrow\\ F\end{pmatrix} = \Pi_f\begin{pmatrix}G\\ \varphi\big\downarrow\\ E\end{pmatrix}.$$

To describe δ_p we need some preliminaries. Recall that in the category of sets, $P = \sum_{b\in F}\prod_{a\in f^{-1}b}\varphi^{-1}(a)$. Using the representation of partial functions into G as total functions into $\tilde{G}$, we may describe P in set theoretical terms as

$$P = \{(b_1,\gamma) \in F \times \tilde{G}^{\mathsf{E}} \mid \forall a \in \mathsf{E}(\exists x \in G\gamma(a) = \{x\} \leftrightarrow f(a) = b)$$
$$\wedge\ \forall a \in \mathsf{E}\forall x \in G(\gamma(a) = \{x\} \rightarrow \varphi(x) = a)\}$$

and the map $P \rightarrow F$ is just the (restriction) of the first projection. We now define

$$\delta_p((b_1,\gamma_1),(b_2,\gamma_2)) = \delta_F(b_1,b_2) \wedge \bigcap \{\delta_{\mathsf{E}}(a_1,a_2) \rightarrow \delta_G(x_1,x_2) \mid \gamma_1(a_1) = \{x_1\} \wedge \gamma_2(a_2) = \{x_2\}\}.$$

We check that δ_p is a coincidence relation:

$$\begin{aligned}
&\delta_p((b_1,\gamma_1),(b_2,\gamma_2)) \wedge \delta_p((b_2,\gamma_2),(b_3,\gamma_3))\\
&\quad\leq \delta_F(b_1,b_2) \wedge \delta_F(b_2,b_3) \wedge (\delta_{\mathsf{E}}(a_1,a_2) \rightarrow \delta_G(x_1,x_2))\\
&\qquad\qquad \wedge (\delta_{\mathsf{E}}(a_2,a_3) \rightarrow \delta_G(x_2,x_3))\\
&\quad\leq \delta_F(b_1,b_3) \wedge (\delta_{\mathsf{E}}(a_1,a_2) \wedge \delta_{\mathsf{E}}(a_2,a_3) \rightarrow \delta_G(x_1,x_2) \wedge \delta_G(x_2,x_3))\\
&\quad\leq \delta_F(b_1,b_3) \wedge (\delta_{\mathsf{E}}(a_1,a_2) \wedge \delta_{\mathsf{E}}(a_2,a_3) \rightarrow \delta_G(x_1,x_3)),
\end{aligned}$$

for every a_1,a_2,a_3,x_1,x_2,x_3 such that $\gamma_i(a_i) = \{x_i\}$.
Hence,

$$\begin{aligned}
&\delta_p((b_1,\gamma_1),(b_2,\gamma_2) \wedge \delta_p(b_2,\gamma_2),(b_3,\gamma_3))\\
&\quad\leq \delta_F(b_1,b_3) \wedge \left(\bigvee \{\delta_{\mathsf{E}}(a_1,a_2) \wedge \delta_{\mathsf{E}}(a_2,a_3) \mid \gamma_2(a_2) = \{x_2\}\} \rightarrow \delta_G(x_1,x_3)\right)\\
&\quad\leq \delta_F(b_1,b_3) \wedge (\delta_{\mathsf{E}}(a_1,a_3) \rightarrow \delta_G(x_1,x_3)) \text{ for all } a_1,a_3,x_1,x_3\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{such that } \gamma_i(a_i) = \{x_i\}\\
&\quad\leq \delta_F(b_1,b_3) \wedge \bigwedge \{\delta_{\mathsf{E}}(a_1,a_3) \rightarrow \delta_G(x_1,x_3) \mid \gamma_1(a_1) = \{x_1\}, \gamma_3(a_3) = \{x_3\}\}\\
&\quad\leq \delta_p((b_1,\gamma_1),(b_3,\gamma_3)).
\end{aligned}$$

The proof that $f^* \dashv \Pi_f$ is long but straightforward.

Remarks

(i) $\mathcal{E}(H)$ is in fact a quasi-topos, since $(\Omega,\delta_{\mathsf{T}})$ classifies regular monos, where δ_{T} is the coincidence relation whose value is always $\top$.

(ii) $\mathcal{E}(H)$ has images: if $(\mathsf{E},\delta_{\mathsf{E}}) \xrightarrow{f} (F,\delta_F) \in \mathcal{E}(H)$, one can easily check that $\mathrm{Im}(f) = (f(\mathsf{E}),\delta)$, where δ is the smallest coincidence relation containing the map $\delta_0 : f(\mathsf{E}) \times f(\mathsf{E}) \rightarrow H$ defined by $\delta_0(y,y') = \bigvee_{f(x)=y,f(x')=y'}\delta_{\mathsf{E}}(x,x')$.

Images, however, are not stable under pull-backs, as the following example shows in Sets (Ω_0), where $\Omega_0 = \{0,\frac{1}{2},1\}$ with obvious operations.

Let $\mathsf{E} = \{0,1,2,3\}$, $F = \{0,1,2\}$, $X = \{0,1\}$ and let $\delta_F = \delta_\top$, $\delta_X = \delta_\top$, where $\delta_\top$ is the coincidence relation whose value is always equal to $\top \in \Omega$, $\delta_{\mathsf{E}}(0,2) = \delta_{\mathsf{E}}(1,2) = \delta_{\mathsf{E}}(1,3) = \delta_{\mathsf{E}}(0,3) = \frac{1}{2}$ and $\top$ elsewhere and $\delta(0,1) = \delta(1,0) = \frac{1}{2}$ and $\top$ elsewhere. Then the diagram

$$\begin{tikzcd}
(\mathsf{E},\delta_{\mathsf{E}}) \arrow[r, "\alpha"] & (F,\delta_F) \\
(X,\delta) \arrow[u, "v"] \arrow[r, "\mathrm{Id}"'] & (X,\delta_X) \arrow[u, "u"']
\end{tikzcd}$$

is a pull-back, where $\alpha(0) = 0$, $\alpha(3) = 2$, $\alpha(1) = \alpha(2) = 1$, $u(0) = 0$, $u(1) = 2$, $v(0) = 0$, $v(1) = 3$. Clearly α is an image, but Id is not! Similarly, $\mathcal{E}(H)$ has suprema of subobjects which are, in general, *not* stable.

From our definition of morphisms in $\mathcal{E}(H)$ we obtain the forgetful functor $U: \mathcal{E}(H) \to \mathcal{E}$ defined by

$$U(\mathsf{E},\delta_{\mathsf{E}}) = \mathsf{E} \quad \text{and} \quad U(f) = f.$$

Proposition 2.6.2 *The forgetful functor U is locally cartesian closed and has both a left and a right adjoint*: $L \dashv U \dashv R$.

Proof: The first statement is obvious from the description of the operations of the category. As for the second,

$$L(\mathsf{E}) = (\mathsf{E},\delta_0), \quad R(\mathsf{E}) = (\mathsf{E},\delta_1), \quad \text{and } L(f) = R(f) = f,$$
$$\text{where } \delta_1(e,e') = 1 \in H,\ \delta_0(e,e') = 0 \in H.$$

A particular case which is of great importance for us is $H = \Omega \in \mathcal{E}$.

In this case $\mathcal{E}(\Omega)$ can be described as the category PER $(\mathcal{E})$ of partial equivalence relations, i.e., an object of PER $(\mathcal{E})$ is a couple (E,R) where $R \hookrightarrow \mathsf{E} \times \mathsf{E}$ is a symmetric and transitive relation. A morphism f: $(\mathsf{E},R) \to (F,S)$ of PER $(\mathcal{E})$ is a map $f: \mathsf{E} \to F$ of $\mathcal{E}$ such that $(e,e') \in R \to (f(e),f(e')) \in S$. In fact, this follows immediately from the bijection

$$\frac{\mathsf{E} \times \mathsf{E} \xrightarrow{\delta_{\mathsf{E}}} \Omega}{R \hookrightarrow \mathsf{E} \times \mathsf{E}}$$

given by definition of Ω.

3 The language of many-sorted modal theory and its interpretations

In our topos theoretic semantics we have variable sets (arbitrary objects of $\mathcal{E}$) and constant sets (objects of $\mathcal{E}$ of the form ΔS, with $S \in \mathbb{S}$). It is natural to introduce *types* and *sorts* to be interpreted as variable and constant sets, respectively. However, to simplify our exposition and bring forth what is new in our approach, we shall concentrate on sorts only. Furthermore the theory of types has already been dealt with at great length by several authors.

We define *sorts* and *terms* by recursion:

Sorts

(a) Basic sorts are sorts: passenger, person, boy, bachelor, reading, river, etc.
(b) 1, PROP are sorts
(c) If X, Y are sorts, so are $X \times Y$ and Y^X
(d) Nothing else is a sort.

Terms In what follows, we write "$t : X$" for "t is a term of sort X".

(a) Basic constant terms $c \in \mathrm{Con}_X$ are terms of sort X: $* \in \mathrm{Con}_1$; John $\in \mathrm{Con}_{\mathrm{person}}$; run $\in \mathrm{Con}_{\mathrm{PROP}^{\mathrm{person}}}$; meet $\in \mathrm{Con}_{\mathrm{PROP}^{\mathrm{person}\times\mathrm{person}}}$, etc.
(b) If $x \in \mathrm{Var}_X$, then x is a term of sort X, where Var_X is an infinite set of variables, for each sort X.
(c) If $t : X$ and $s : Y$, then $\langle t, s\rangle : X \times Y$.
(d) If $x \in \mathrm{Var}_X$ and $t : Y$, then $\lambda x t : Y^X$.
(e) If $t : Y^X$ and $s : X$, then $t(s) : Y$.
(f) $\top, \bot$: PROP.
(g) If $t, s : X$, then $t = s$ and $t \asymp s$: PROP
(h) If φ, ψ : PROP, then $\varphi * \psi$: PROP, where $* \in \{\wedge, \vee, \rightarrow\}$
(i) If φ : PROP and $x \in \mathrm{Var}_X$, then $\exists x\varphi, \forall x\varphi$: PROP
(j) If φ : PROP, then $\Box\varphi, \Diamond\varphi$: PROP
(k) Nothing else is a term.

A *formula* is a term of sort PROP. We use the following abbreviations:

If $t : X$, we let $\mathsf{E}(t) \equiv t \asymp t$
If φ is a formula, we let $\neg\varphi \equiv \varphi \rightarrow \bot$.

We shall assume that we have defined the usual notions like "substitution of a variable by a term", "free variable of a term or a formula", "a term is free for a variable in a term or formula", etc.

We now interpret this language in a topos $\mathcal{E} \rightarrow \mathbb{S}$. The main idea, already discussed, is that sorts are interpreted as couples consisting of a constant set of $\mathcal{E}$ and a coincidence relation.

Our interpretation proceeds in two steps: we first interpret sorts as kinds i.e., as objects of the category $\mathcal{E}(\Omega)$ of the form $(\Delta S, \delta_{\Delta S})$.

Via this interpretation, we shall interpret terms and formulas in the topos $\mathcal{E}$ in the usual way (cf. Makkai and Reyes [15] and Lambek and Scott [10]).

To interpret sorts in $\mathcal{E}(\Omega)$, it is enough to interpret basic sorts. In fact, once these sorts have been interpreted, we extend the interpretation I to all sorts as follows:

$$I(1) = (\Delta 1, \delta_T)$$

$$I(\mathrm{PROP}) = (\Delta\Gamma\Omega, \delta_{\Delta\Gamma\Omega})$$

where $\delta_{\Gamma(\Omega)} : \Gamma(\Omega) \times \Gamma(\Omega) \rightarrow \Gamma(\Omega)$ is defined as $\Gamma(\leftrightarrow)$.

Furthermore, if X and Y have been interpreted, then

$$I(X \times Y) = I(X) \times I(Y) \text{ and}$$

$$I(Y^X) = I(Y)^{I(X)},$$

where products and exponentials in the right-hand sides refer to the cartesian closed structure of $\mathcal{E}(\Omega)$. Furthermore, constant sets of $\mathcal{E}$ are exponentiable by our assumption on $\mathcal{E} \to \mathbb{S}$.

We recall that we have a forgetful functor,

$$\mathcal{E}(\Omega) \xrightarrow{\cup} \mathcal{E},$$

in terms of which we define the interpretation of sorts as follows: if X is a sort, then $\|X\| = \bigcup I(X)$. Notice that $\|X\|$ is a constant set of the form ΔA. In the sequel, we shall let tr$(\ldots)$ be the transpose of $(\ldots)$ given by the adjunction $\Delta \dashv \Gamma$.

For each term $t:X$ and each sequence $\vec{x} = \langle x, \ldots, x_n \rangle$ of distinct variables such that the free variables of t are among the elements of $\vec{x}$, we define by recursion $\|\vec{x}:t\| : \|X_1\| \times \ldots \times \|X_n\| \to \|X\| \in \mathcal{E}$ as follows:

a. Basic constant terms $c \in \mathrm{Con}_X$ are interpreted as global sections $\|c\|: 1 \to \|X\| \in \mathcal{E}$. If $\vec{x}$ is any sequence of (distinct) variables of sorts $X_1, \ldots, X_n$, we let $\|\vec{x}:c\| : \|X_1\| \times \ldots \times \|X_n\| \to 1 \xrightarrow{\|c\|} \|X\|$ be the composite of the unique morphism $\|X_1\| \times \ldots \times \|X_n\| \to 1$ with $\|c\|$.
b. If $x_i \in \mathrm{Var}_X$, then $\|\vec{x}:x_i\| = \pi_i$, the i^{th} projection.
c. If $t:X$ and $s:Y$, then

$$\|\vec{x}:\langle t,s\rangle\| = \langle \|\vec{x}:t\|, \|\vec{x}:s\| \rangle.$$

d. If $x \in \mathrm{Var}_X$ and $t:Y$, then $\|\vec{x}:\lambda x t\| : \|X_1\| \times \ldots \times \|X_n\| \to \|Y\|^{\|X\|}$ is defined to be exponential transpose of the map

$$\|\vec{x}x:t\| : \|X_1\| \times \ldots \times \|X_n\| \times \|X\| \to \|Y\|$$

(we assume that $\vec{x}x$ consists of distinct variables).
e. If $t: Y^X$ and $s:X$, then $\|\vec{x}:t(s)\| = \mathrm{ev} \circ \langle \|\vec{x}:t\|, \|\vec{x}\cdot s\| \rangle$, where ev: $\|Y\|^{\|X\|} x \|X\| \to \|Y\|$.
f.

$$\|\vec{x}:\top\| = \|X_1\| \times \ldots \times \|X_n\| \to 1 \simeq \Delta 1 \xrightarrow{\Delta\top} \Delta\Gamma\Omega$$
$$\|\vec{x}:\bot\| = \|X_1\| \times \ldots \times \|X_n\| \to 1 \simeq \Delta 1 \xrightarrow{\Delta\bot} \Delta\Gamma\Omega.$$

g. $\|\vec{x}:t = s\|$ is the composite

$$\|X_1\| \times \ldots \times \|X_n\| \xrightarrow{\langle \|\vec{x}:t\|, \|\vec{x}:s\| \rangle} \|X\| \times \|X\| \xrightarrow{\Delta \mathrm{tr}(\Delta_{\|X\|})} \Delta\Gamma\Omega,$$

where $\Delta_{\|X\|}$ classifies the diagonal $\|X\| \rightarrowtail \|X\| \times \|X\|$

$\|\vec{x}:t \asymp s\|$ is the composite

$$\|X_1\| \times \ldots \times \|X_n\| \xrightarrow{\langle \|\vec{x}:t\|, \|\vec{x}:s\| \rangle} \|X\| \times \|X\| \xrightarrow{\Delta \mathrm{tr}(\delta_{\|X\|})} \Delta\Gamma\Omega,$$

where $(\|X\|, \delta_{\|X\|}) = I(X)$.
h. $\|\vec{x}:\varphi * \psi\| : \|X_1\| \times \ldots \times \|X_n\| \to \Delta\Gamma\Omega$ is the composite

$$\|X_1\| \times \ldots \times \|X_n\| \xrightarrow{\langle \|\vec{x}:\varphi\|, \|\vec{x}:\psi\| \rangle} \Delta\Gamma\Omega \times \Delta\Gamma\Omega \xrightarrow{\Delta\Gamma(*)} \Delta\Gamma\Omega$$

where $* : \Omega \times \Omega \to \Omega$ is one of the operations $\vee, \wedge, \to$ on Ω.
i. $\|\vec{x}:\exists x\varphi\| : \|X_1\| \times \ldots \times \|X_n\| \to \Delta\Gamma\Omega$ is defined to be $\Delta\,\mathrm{tr}(\exists_\pi(\epsilon_\Omega \circ \|\vec{x}x:\varphi\|))$
$\|\vec{x}:\forall x\varphi\| : \|X_1\| \times \ldots \times \|X_n\| \to \Delta\Gamma\Omega$ is defined to be $\Delta\,\mathrm{tr}(\forall_\pi(\epsilon_\Omega \circ \|\vec{x}x:\varphi\|))$,
where $\pi: \|X_1\| \times \ldots \times \|X_n\| \times \|X\| \to \|X_1\| \times \ldots \times \|X_n\|$ is the canonical projection.

j. $\|\vec{x}:\Box\varphi\|:\|X_1\|\times\ldots\times\|X_n\|\to\Delta\Gamma\Omega$ is the composite

$$\|X_1\|\times\ldots\times\|X_n\| \xrightarrow{\|\vec{x}:\varphi\|} \Delta\Gamma\Omega \xrightarrow{\Delta\Box} \Delta\Gamma\Omega$$

$\|\vec{x}:\Diamond\varphi\|:\|X_1\|\times\ldots\times\|X_n\|\to\Delta\Gamma\Omega$ is the composite

$$\|X_1\|\times\ldots\times\|X_n\| \xrightarrow{\|\vec{x}:\varphi\|} \Delta\Gamma\Omega \xrightarrow{\Delta\Diamond} \Delta\Gamma\Omega.$$

Remark The reader has certainly noticed that we have defined a nonstandard interpretation of formulas as maps into $\Delta\Gamma\Omega$, rather than Ω. However, given such a nonstandard interpretations of a formula φ, $\|X_1\|\times\ldots\times\|X_n\| \xrightarrow{\|\vec{x}:\varphi\|} \Delta\Gamma\Omega$ we obtain a standard interpretation of φ, namely, $\|X_1\|\times\ldots\times\|X_n\| \xrightarrow{\epsilon_\Omega\circ\|\vec{x}:\varphi\|} \Omega$ simply by composing $\|\vec{x}:\varphi\|$ with the counit $\epsilon_\Omega:\Delta\Gamma\Omega\to\Omega$.

To finish this section, we shall describe a formal system MAO (for "modal adjoint operators") based on Gentzen's sequents. These sequents, following Boileau and Joyal [2] will be expressions of the form $\Gamma\vdash_X\varphi$, where Γ is a finite set of formulas (of the language already described), φ a single formula and X a finite set of variables containing all the free variables of Γ and φ. We shall assume that sequents satisfy 1–8 below. This system partly follows [10], p. 134:

1. Structural rules:

1.1 $p\vdash_X p$

1.2 $\dfrac{\Gamma\vdash_X p \quad \Gamma\cup\{p\}\vdash_X q}{\Gamma\vdash_X q}$

1.3 $\dfrac{\Gamma\vdash_X q}{\Gamma\cup\{p\}\vdash_X q}$

1.4 $\dfrac{\Gamma\vdash_X q}{\Gamma\vdash_{X\cup\{y\}} q}$

1.5 $\dfrac{\Gamma\vdash_{X\cup\{y\}}\varphi}{\Gamma[t/y]\vdash_X\varphi[t/y]}$,

where t is free for y in φ and Γ.

2. Logical rules:

2.1 $p\vdash_X\top$ and $\bot\vdash_X p$.
2.2 $r\vdash_X p\wedge q$ iff $r\vdash_X p$ and $r\vdash_X q$
$p\vee q\vdash_X r$ iff $p\vdash_X r$ and $q\vdash_X r$
2.3 $p\vdash_X q\to r$ iff $p\wedge q\vdash_X r$
2.4 $p\vdash_X\forall x\varphi$ iff $p\vdash_{X\cup\{x\}}\varphi$
$\exists x\varphi\vdash_X p$ iff $\varphi\vdash_{X\cup\{x\}} p$,
provided that $x\notin X$.

3. Identity rules:

3.1 $\vdash_X t=t$
3.2 $t=s,\varphi[t/x]\vdash_X\varphi[s/x]$,
provided that t and s are free for x in φ.

4. Rules on special symbols:

4.1 $\vdash_{\{x\}} x = * \ (x \in \mathrm{Var}_1)$
4.2 $\langle a, b\rangle = \langle c, d\rangle \vdash_X a = c$
$\langle a, b\rangle = \langle c, d\rangle \vdash_X b = d$

4.3 $$\frac{\Gamma, z = \langle x, y\rangle \vdash_{X \cup \{x,y,z\}} \varphi}{\Gamma \vdash_{X \cup \{z\}} \varphi},$$
provided that x and y are not free in Γ or φ.

5. Coincidence rules:

5.1 $\vdash_X * \asymp *$
5.2 $\langle a, b\rangle \asymp \langle c, d\rangle \vdash_X a \asymp c$
$\langle a, b\rangle \asymp \langle c, d\rangle \vdash_X b \asymp d$
$a \asymp c, b \asymp d \vdash_X \langle a, b\rangle \asymp \langle c, d\rangle$
5.3 $x \asymp y \vdash_X y \asymp x$
$x \asymp y, y \asymp z \vdash_X x \asymp z$
$f \asymp g, x \asymp y \vdash_X f(x) \asymp g(y)$
$$\frac{\Gamma, x \asymp y \vdash_{X \cup \{x,y\}} f(x) \asymp g(y)}{\Gamma \vdash_X f \asymp g},$$
provided that x and y are not free in Γ.

6. Rules for the λ-calculus:

6.1 $\vdash_X \lambda x t(x) = t$,
provided that $x \notin X$.
6.2 $\vdash \lambda x \varphi(\tau) = \varphi[t/x]$,
provided that t is free for x in φ.

6.3 $$\frac{\Gamma \vdash_{X \cup \{x\}} t = s}{\Gamma \vdash_X \lambda x t = \lambda x s}.$$

7. Rules for modal operators:

7.1 $\Box\varphi \vdash_X \varphi$ $\qquad \varphi \vdash_X \Diamond\varphi$
7.2 $\Box\varphi \vdash_X \Box\Box\varphi$ $\qquad \Diamond\Diamond\varphi \vdash_X \Diamond\varphi$
7.3 $\varphi \vdash_X \Box\Diamond\varphi$ $\qquad \Diamond\Box\varphi \vdash_X \varphi$

7.4 $$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$
$$\frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}.$$

This completes our system. We have not tried to describe it in the simplest or most economical way.

To state a soundness theorem, we first need a definition. Let $\|.\|$ be an interpretation of the language already discussed in a locally connected topos $\mathcal{E} \to \mathbb{S}$. We say that a sequent $\Gamma \vdash_X q$ is *valid* under $\|.\|$ iff $\epsilon_\Omega \circ \|\vec{x} : \wedge\Gamma\| \leq \epsilon_\Omega \circ \|\vec{x} : q\|$, where $\vec{x}$ is a sequence of distinct variables of X. Similarly, we say that a rule of inference is *sound* under $\|.\|$ if the conclusion is valid whenever the premises are.

Theorem 3.1 (Soundness of MAO) *Assume that the geometric morphism* $\mathcal{E} \to \mathbb{S}$ *is locally connected. Then all sequents of* MAO *are valid and all rules of inference of* MAO *are sound, under any interpretation.*

Proof: To simplify our computations, we first define a forcing relation as follows: $C \Vdash \varphi[a_1, \ldots, a_n]$ iff $C \in \mathrm{tr}(\epsilon_\Omega \circ \|\vec{x}:\varphi\|)(a_1, \ldots, a_n)$ where $\mathrm{tr}(\ldots)$ is the transpose of $(\ldots)$ for the adjunction $\Delta \dashv \Gamma$.

Lemma 3.2 $\Vdash$ *satisfies the usual clauses for the forcing of Beth–Kripke–Joyal.*

Proof: We shall do only the clause $\forall$, the others being similar. Let $\Delta A \times \Delta B \xrightarrow{\|(x,y):\varphi\|} \Delta\Gamma\Omega$. We have the following equivalences:

$$\frac{\dfrac{\dfrac{C \Vdash \forall y\varphi[a]}{C \in \mathrm{tr}(\epsilon_\Omega \circ \|x:\forall y\varphi\|)(a)}\ (\text{definition of } \Vdash)}{C \in \mathrm{tr}(\forall_\pi(\epsilon_\Omega \circ \|(x,y):\varphi\|))(a)}\ (\text{definition of } \|x:\forall y\varphi\|)}{(\forall_\pi(\epsilon_\Omega \circ \|(x,y):\varphi\|))_C(a) = \top_C}\ (\text{Proposition A.4, Appendix})$$

By definition of $\forall_\pi$, the last line is equivalent to $\forall C' \to C \in \mathbb{C}\ \forall b \in B\ (\epsilon_\Omega \circ \|(x,y):\varphi\|)_{C'}(a,b) = \top_{C'}$.

Using Proposition A.4 of the Appendix once again, this is equivalent to

$$\forall C' \to C \in \mathbb{C}\ \forall b \in B\ C' \in \mathrm{tr}(\epsilon_\Omega \circ \|(x,y):\varphi\|)_{C'}(a,b)$$

which, in turn, is equivalent to

$$\forall C' \to C \in \mathbb{C}\ \forall b \in B\ C' \Vdash \varphi[a,b]$$

by definition of $\Vdash$.

By the argument used in Proposition 4.1 (§4), one can easily conclude that

$$\epsilon_\Omega \circ \|\vec{x}\cdot\wedge\Gamma\| \leq \epsilon_\Omega \circ \|\vec{x}:\phi\|$$

precisely when their transposes satisfy

$$\mathrm{tr}(\epsilon_\Omega \circ \|\vec{x}:\wedge\Gamma\|) \leq \mathrm{tr}(\epsilon_\Omega \circ \|\vec{x}:\phi\|).$$

Therefore, validity of the sequent $\Gamma \vdash_X q$ under $\|\ldots\|$ is equivalent to $\forall C \in \mathbb{C}(C \Vdash \wedge\Gamma[a_1, \ldots, a_n] \Rightarrow C \Vdash q[a_1, \ldots, a_n])$.

But this is the usual notion of validity under which sequents and rules which do not involve $\Diamond, \Box, \rightarrowtail\!\!\!\!\leftarrowtail$ may be proved valid and sound, respectively. For other sequents and rules, validity is straightforward: for instance, the first two axioms of 5.3 state that $\delta_{\|X\|}$ is symmetric and transitive, whereas the rule of inference of 5.3 expresses the coincidence relation of an exponential. Finally, the sequents in Group 7 just assert that $(\Diamond, \Box)$ constitutes an MAO couple.

If we impose connectivity, a new sequent, the connectivity axiom below, is validated.

Connectivity axiom:

$$\square(\varphi \vee \psi), \Diamond\varphi, \Diamond\psi \vdash_X \Diamond(\varphi \wedge \psi).$$

Indeed, we have the following result whose proof will appear elsewhere:

Theorem 3.3 (Soundness of MAO with Connectivity axiom) *Assume that the geometric morphism $\mathcal{E} \to \mathbb{S}$ is locally connected and connected. Then all sequences of* MAO *as well as the Connectivity Axiom are valid and all rules of inference are sound, under any interpretation.*

4 Kinds and variable sets with coincidence relation The main result of this section concerns the relation between $\mathbb{S}(\Gamma(\Omega))$ and $\mathcal{E}(\Omega)$. We first "lift" the functor Δ to a functor

$$\bar{\Delta}: \mathbb{S}(\Gamma(\Omega)) \to \mathcal{E}(\Omega)$$

which sends (X, δ_x) into $(\Delta X, \delta_{\Delta X})$, where $\delta_{\Delta X}$ is the transpose of δ_X via the adjunction $\Delta \dashv \Gamma$, i.e., $\delta_{\Delta X} = \delta_\Omega \circ \Delta\delta_X$ and sends $(X, \delta_X) \xrightarrow{f} (Y, \delta_y)$ into Δf.

Proposition 4.1
(a) *δ_X is a coincidence relation iff $\delta_{\Delta X}$ is a coincidence relation.*
(b) *f is a morphism of $\mathbb{S}(\Gamma(\Omega))$ iff Δf is a morphism of $\mathcal{E}(\Omega)$.*

Proof: Easy exercise in transposition. As an example, the assertion that $\delta_{\Delta X}(\xi_1, \xi_2) \wedge \delta_{\Delta X}(\xi_2, \xi_3) \leq \delta_{\Delta X}(\xi_1, \xi_3)$ is equivalent to the existence of the dotted arrow in the commutative diagram:

$$\begin{tikzcd}
\Delta X \times \Delta X \times \Delta X \arrow[r, "\langle\langle \delta_{\Delta X}\circ\langle \pi_1, \pi_2\rangle, \delta_{\Delta X}\circ\langle \pi_2, \pi_3\rangle, \delta_{\Delta X}\circ\langle \pi_1, \pi_3\rangle\rangle"] \arrow[d, dotted] & \Omega \times \Omega \times \Omega \arrow[d, "\wedge \times \Omega"] \\
\leq \arrow[r, tail] & \Omega \times \Omega,
\end{tikzcd}$$

which in turn is equivalent (by taking transposes) to the existence of the dotted arrow of the commutative diagram

$$\begin{tikzcd}
X \times X \times X \arrow[r, "\langle\langle \delta_X\circ\langle \pi_1, \pi_2\rangle, \delta_X\circ\langle \pi_2, \pi_3\rangle, \delta_X\circ\langle \pi_1, \pi_3\rangle\rangle"] \arrow[d, dotted] & \Gamma(\Omega) \times \Gamma(\Omega) \times \Gamma(\Omega) \arrow[d, "\Gamma(\wedge)\times\Gamma(\Omega)"] \\
\Gamma(\leq) \arrow[r, tail] & \Gamma\Omega \times \Gamma\Omega.
\end{tikzcd}$$

But this last statement is equivalent to $\delta_X(x_1, x_2) \wedge \delta_X(x_2, x_3) \leq \delta_X(x_1, x_3)$.

Similarly, the assertion that Δf is a morphism is equivalent to the existence of the dotted arrow in the commutative diagram:

$$\begin{tikzcd}
\Delta X \times \Delta X \arrow[r, "\langle \delta_{\Delta X}, \delta_{\Delta Y}\circ \Delta f \times \Delta f\rangle"] \arrow[dr, dashed] & \Omega \times \Omega \\
 & \leq \arrow[u, tail]
\end{tikzcd}$$

By taking transposes, this is equivalent to the distance of the dotted arrow in

$$\begin{tikzcd} X \times X \arrow[r, "\langle \delta_X, \delta_Y \circ f \times f \rangle"] \arrow[dr, dotted] & \Gamma(\Omega) \times \Gamma(\Omega) \\ & \Gamma(\leq). \arrow[u, tail] \end{tikzcd}$$

But this last condition says that f is a morphism.

Proposition 4.2 *Assume that* $\mathcal{E} \to \mathbb{S}$ *is a geometric morphism* $(\Delta \dashv \Gamma)$. *Then* $\bar{\Delta} : \mathbb{S}(\Gamma(\Omega) \to \mathcal{E}(\Omega)$ *preserves finite limits and has a right adjoint* $\bar{\Delta} \dashv \bar{\Gamma}$ *such that the following diagram is commutative*:

$$\begin{tikzcd} S(\Gamma(\Omega)) \arrow[r, shift left, "\bar{\Delta}"] \arrow[d, "U"] & \mathcal{E}(\Omega) \arrow[l, shift left, "\bar{\Gamma}"] \arrow[d, "U"] \\ \mathbb{S} \arrow[u, shift left=2, "L"] \arrow[u, shift right=2, "R"'] \arrow[r, shift left, "\Delta"] & \mathcal{E} \arrow[l, shift left, "\Gamma"] \arrow[u, shift left=2, "L"] \arrow[u, shift right=2, "R"'] \end{tikzcd}$$

Proof: The fact that $\bar{\Delta}$ preserves finite limits is easy and left to the reader.

We define $\bar{\Gamma}$ as the functor which sends $(\mathsf{E}, \delta_{\mathsf{E}})$ into $(\Gamma(\mathsf{E}), \delta_{\Gamma(\mathsf{E})})$, where $\delta_{\Gamma(\mathsf{E})} = \Gamma(\delta_{\mathsf{E}})$, and which sends a morphism $(\mathsf{E}, \delta_{\mathsf{E}}) \xrightarrow{\alpha} (F, \delta_F)$ into $\Gamma(\alpha)$: $\Gamma(\mathsf{E}) \to \Gamma(F)$.

Proposition 4.3
(a) $\delta_{\Gamma(\mathsf{E})}$ *is a coincidence relation*
(b) $\Gamma(\alpha)$ *is a morphism of* $\mathbb{S}(\Gamma(\Omega))$.

Proof: Let us prove (b) only, since (a) is quite similar.

The assertion that α is a morphism is equivalent to the factorization of the horizontal map as indicated:

$$\begin{tikzcd} \mathsf{E} \times \mathsf{E} \arrow[r, "\langle \delta_{\mathsf{E}}, \delta_F \circ \alpha \times \alpha \rangle"] \arrow[dr, dotted] & \Omega \times \Omega \\ & \leq \arrow[u, tail] \end{tikzcd}$$

Applying Γ to this diagram, we obtain the factorization through $\Gamma(\leq) \rightarrowtail \Gamma\Omega \times \Gamma\Omega$ of the map $\langle \delta_{\Gamma(\mathsf{E})}, \delta_{\Gamma(F)} \circ \Gamma(\alpha) \times \Gamma(\alpha) \rangle$, i.e., we obtain that $\Gamma(\alpha)$ is a morphism.

We now check that $\bar{\Delta} \dashv \bar{\Gamma}$. In other words, we have to check that we have a natural bijection:

$$\frac{(\Delta X, \delta_{\Delta X}) \xrightarrow{\alpha} (\mathsf{E}, \delta_{\mathsf{E}}) \in \mathcal{E}(\Omega)}{(X, \delta_X) \xrightarrow[\mathrm{tr}(\alpha)]{} (\Gamma(\mathsf{E}), \delta_{\Gamma(\mathsf{E})}) \in \mathbb{S}(\Gamma(\Omega))}$$

where $\mathrm{tr}(\alpha)$ is the transpose of α via $\Delta \dashv \Gamma$. Since this bijection exists at the level of maps of $\mathcal{E}$ and maps of $\mathbb{S}$, it is enough to show that α is a morphism iff $\mathrm{tr}(\alpha)$ is a morphism. Once again we have the following equivalences:

$$\alpha \text{ is a morphism}$$
$$\overline{\hspace{10em}}$$
$$\begin{tikzcd}
\Delta X \times \Delta X \arrow[r, "\langle \delta_{\Delta X}, \delta_{\mathsf{E}} \circ \alpha \times \alpha \rangle"] \arrow[dr, dashed] & \Omega \times \Omega \\
& \geq \arrow[u, tail]
\end{tikzcd}$$
$$\overline{\hspace{10em}}$$
$$\begin{tikzcd}
X \times X \arrow[r, "\langle \delta_X, \mathrm{tr}(\delta_{\mathsf{E}} \circ \alpha \times \alpha) \rangle"] \arrow[dr, dashed] & \Gamma(\Omega) \times \Gamma(\Omega) \\
& \geq \arrow[u, tail]
\end{tikzcd}$$
$$\overline{\hspace{10em}}$$
$$\mathrm{tr}(\alpha) \text{ is a morphism}$$

In fact, the last equivalence follows from

$$\mathrm{tr}(\delta_{\mathsf{E}} \circ \alpha \times \alpha) = \delta_{\Gamma(\mathsf{E})} \circ \mathrm{tr}(\alpha) \times \mathrm{tr}(\alpha),$$

as can be easily checked.

Theorem 4.4 *Let* $\mathcal{E} \to \mathbb{S}$ *be a geometric morphism* $\Delta \dashv \Gamma$. *If* $\Delta : \mathbb{S} \to \mathcal{E}$ *has a left adjoint* $\pi_0 \dashv \Delta$, *then* $\bar{\Delta} : \mathbb{S}(\Gamma(\Omega)) \to \mathcal{E}(\Omega)$ *has also a left adjoint* $\bar{\pi}_0 \dashv \bar{\Delta}$ *such that the diagram*

$$\begin{tikzcd}
\mathbb{S}(\Gamma(\Omega)) \arrow[r, "\bar{\Delta}" description] \arrow[d, "U" description] & \mathcal{E}(\Omega) \arrow[l, bend right, "\bar{\pi}_0"'] \arrow[l, bend left, "\bar{\Gamma}"] \arrow[d, "U" description] \\
\mathbb{S} \arrow[u, bend left, "L"] \arrow[u, bend right, "R"'] \arrow[r, "\Delta" description] & \mathcal{E} \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "\Gamma"] \arrow[u, bend left, "L"] \arrow[u, bend right, "R"']
\end{tikzcd}$$

is commutative.

Furthermore, if $\pi_0 : \mathcal{E} \to \mathbb{S}$ *satisfies the Frobenius's condition, then so does* $\bar{\pi}_0 : \mathcal{E}(\Omega) \to \mathbb{S}(\Gamma(\Omega))$.

Proof: We define $\bar{\pi}_0 : \mathcal{E}(\Omega) \to \mathbb{S}(\Gamma(\Omega))$ as the map which sends $(\mathsf{E}, \delta_{\mathsf{E}})$ into $(\pi_0(\mathsf{E}), \delta_{\pi_0(\mathsf{E})})$, where $\delta_{\pi_{0(\mathsf{E})}}$ will be defined presently, and a morphism $f: (\mathsf{E}, \delta_{\mathsf{E}}) \to (F, \delta_F)$ into $\pi_0(f) : \pi_0(\mathsf{E}) \to \pi_0(F)$. Later on, we shall check that $\bar{\pi}_0$ is a functor.

We first define the transpose of $\delta_{\pi_{0(\mathsf{E})}}$, namely $\delta_{\Delta\pi_0\mathsf{E}} : (\Delta\pi_0\mathsf{E})^2 \to \Omega$, as the final structure on $\Delta\pi_0\mathsf{E}$ for which $\eta_{\mathsf{E}} : \mathsf{E} \to \Delta\pi_0\mathsf{E}$ is a morphism of $\mathcal{E}(\Omega)$. In other words, $\delta_{\Delta\pi_0\mathsf{E}}$ is the smallest coincidence relation δ on $\Delta\pi_0\mathsf{E}$ such that $\delta_{\mathsf{E}} \leq \delta \circ \eta_{\mathsf{E}}^2$.

To make this definition precise, let $\mathbb{D} = \{\delta \in \Omega^{(\Delta\pi_0\mathsf{E})^2} \mid \delta$ is a coincidence relation and $\delta_{\mathsf{E}} \leq \delta \circ \eta_{\mathsf{E}}^2\} \hookrightarrow \Omega^{(\Delta\pi_0\mathsf{E})^2}$. Furthermore, let $e : \mathbb{D} \times (\Delta\pi_0\mathsf{E})^2 \rightarrowtail \Omega^{(\Delta\pi_0\mathsf{E})^2} \times (\Delta\pi_0\mathsf{E})^2 \xrightarrow{ev} \Omega$ be the restriction of the evaluation to functions in $\mathbb{D}$. If $\bar{e}$: $(\Delta\pi_0\mathsf{E})^2 \to \Omega^{\mathbb{D}}$ is its exponential transpose, we define $\delta_{\Delta\pi_0\mathsf{E}}$ to be the composite

$$\delta_{\Delta\pi_0\mathsf{E}} : (\Delta\pi_0\mathsf{E})^2 \xrightarrow{\bar{e}} \Omega^{\mathbb{D}} \xrightarrow{\wedge_{\mathbb{D}}} \Omega.$$

Using set-theoretical notation

$$\delta_{\Delta\pi_{0\mathsf{E}}} = \bigcap \{\delta \in \Omega^{(\Delta\pi_0\mathsf{E})^2} \mid \delta \in \mathbb{D}\}.$$

Claim

(a) $d_{\mathsf{E}} \leq \delta_{\Delta\pi_0\mathsf{E}} \circ \eta^2_{\mathsf{E}}$

(b) *If d and d' are coincidence relations on X, then $d \leq d'$ iff* $\mathrm{tr}(d) \leq \mathrm{tr}(d')$

(c) *If d is a coincidence relation on $\pi_0(\mathsf{E})$, then $\delta_{\pi_0\mathsf{E}} \leq d$ iff $\delta_{\mathsf{E}} \leq \mathrm{tr}(d) \circ \eta^2_{\mathsf{E}}$.*

Proof:

Ad(a): $\delta_{\mathsf{E}} \leq \delta \circ \eta^2_{\mathsf{E}}$ for all $\delta \in \mathbb{D}$ (by the very definition of $\mathbb{D}$). Therefore, $\delta_{\mathsf{E}} \leq \bigcap\{\delta \circ \eta^2_{\mathsf{E}} \mid \delta \in \mathbb{D}\} = \delta_{\Delta\pi_0\mathsf{E}} \circ \eta^2_{\mathsf{E}}$.

Ad(b): simple exercise in transposition as in Proposition 4.1.

Ad(c): $\Rightarrow$: from (a),

$\delta_{\mathsf{E}} \leq \mathrm{tr}(\delta_{\pi_0(\mathsf{E})}) \circ \eta^2_{\mathsf{E}} \leq \mathrm{tr}(d) \circ \eta^2_{\mathsf{E}}$

$\Leftarrow$: Since d is a coincidence relation, so is $\mathrm{tr}(d)$ (Proposition 4.1) and hence

$$\delta_{\Delta\pi_0\mathsf{E}} = \mathrm{tr}(\delta_{\pi_0\mathsf{E}}) \leq \mathrm{tr}(d).$$

This implies (by (b)), that $\delta_{\pi_0(\mathsf{E})} \leq d$.

Let us show that $\bar{\pi}_0$ is a functor; more specifically, let us show that $\pi_0(f)$: $\pi_0(\mathsf{E}) \to \pi_0(F)$ is a morphism in $\mathbb{S}(\Gamma(\Omega))$, provided that f is a morphism in $\mathcal{E}(\Omega)$.

In fact, consider the coincidence relation $d = \delta_{\pi_0(F)} \circ (\pi_0 f)^2$. Since $\mathrm{tr}(d) \circ \eta^2_{\mathsf{E}} = \mathrm{tr}(\delta_{\pi_0}(F) \circ \eta^2_F \circ f^2$, we have that

$$\delta_{\mathsf{E}} \leq \delta_F \circ f^2 \leq \mathrm{tr}(\delta_{\pi_0(F)}) \circ \eta^2_F \circ f^2 = \mathrm{tr}(d) \circ \eta^2_{\mathsf{E}}\,.$$

By applying (c), we conclude that $\delta_{\pi_0(\mathsf{E})} \leq d$. We now check that $\bar{\pi}_0 \dashv \bar{\Delta}$, i.e., that we have a natural bijection

$$\frac{\pi_0(\mathsf{E},\delta_{\mathsf{E}}) \xrightarrow{f} (X,\delta_X) \in \mathbb{S}(\Gamma(\Omega))}{(\mathsf{E},\delta_{\mathsf{E}}) \xrightarrow{\mathrm{tr}(f)} \Delta(X,\delta_X) \in \mathcal{E}(\Omega)\,.}$$

Since tr( ) gives such a bijection for maps, we need only show that f is a morphism iff $\mathrm{tr}(f)$ is a morphism. But we have the following equivalences:

$$\frac{\pi_0(f) \text{ is a morphism}}{\dfrac{\delta_{\pi_0(\mathsf{E})} \leq \delta_X \circ f^2}{\dfrac{\delta_{\mathsf{E}} \leq \mathrm{tr}(\delta_X \circ f^2) \circ \eta^2_{\mathsf{E}}}{\dfrac{\delta_{\mathsf{E}} \leq \delta_{\Delta X} \circ (\Delta f)^2 \circ \eta^2_{\mathsf{E}}}{\delta_{\mathsf{E}} \leq \delta_{\Delta X} \circ \mathrm{tr}(f)^2}}}}$$

Assume now that $\pi_0 : \mathcal{E} \to \mathbb{S}$ satisfies the Frobenius condition. We shall prove that its lifting satisfies the same condition, i.e., we shall prove that if $(F,\delta_F) = (\mathsf{E},\delta_{\mathsf{E}}) \times (\Delta Y,\delta_{\Delta Y})$, then $(\pi_0(F),\delta_{\pi_0(F)}) = (\pi_0(\mathsf{E}),\delta_{\pi_0(\mathsf{E})}) \times (Y,\delta_Y)$.

By assumption, $\pi_0(F) = \pi_0(\mathsf{E}) \times Y$ and we need only prove that

$$\delta_{\pi_0(F)}((a,y),(a',y')) = \delta_{\pi_0(\mathsf{E})}(a,a') \wedge \delta_Y(y,y').$$

We claim that this condition is equivalent to

(*) $\delta_{\Delta\pi_0(F)}((\alpha,\zeta),(\alpha',\zeta')) = \delta_{\Delta\pi_0\mathsf{E}}(\alpha,\alpha') \wedge \delta_{\Delta Y}(\zeta,\zeta')$.

In fact, this follows from the last claim (b) and the fact that the transpose of $[((a,a'),(y,y')) \mapsto \delta_{\pi_0(\mathsf{E})}(a,a') \wedge \delta_Y(y,y')]$ is $[((\alpha,\alpha'),(\zeta,\zeta')) \mapsto \delta_{\Delta\pi_0\mathsf{E}}(\alpha,\alpha') \wedge \delta_{\Delta Y}(\zeta,\zeta')]$ as follows from the equivalence

$$\frac{\pi_0(\mathsf{E})^2 \times Y^2 \xrightarrow{\langle \delta_{\pi_0(\mathsf{E})}\circ\pi_1,\, \delta_Y\circ\pi_2\rangle} \Gamma(\Omega)\times\Gamma(\Omega) \xrightarrow{\Gamma(\wedge)} \Gamma(\Omega)}{\begin{tikzcd}(\Delta\pi_0\mathsf{E})^2 \times \Delta Y^2 \arrow[r, "\langle \Delta\delta_{\pi_0(\mathsf{E})}\circ\pi_1,\, \Delta\delta_Y\circ\pi_2\rangle"] \arrow[dr, "\langle \delta_{\Delta\pi_0\mathsf{E}}\circ\pi_1,\, \delta_{\Delta Y}\circ\pi_2\rangle"'] & \Delta\Gamma\Omega\times\Delta\Gamma\Omega \arrow[r, "\Delta\Gamma\wedge"] \arrow[d, "\epsilon\times\epsilon"'] & \Delta\Gamma\Omega \arrow[d, "\epsilon"] \\ & \Omega\times\Omega \arrow[r, "\wedge"'] & \Omega\end{tikzcd}}$$

(Notice that the square is commutative by naturality of ϵ.)

$$\delta_{\Delta\pi_0 F}((\alpha,\zeta),(\alpha',\zeta')) = \delta_{\Delta\pi_0\mathsf{E}}(\alpha,\alpha') \wedge \delta_{\Delta Y}(\zeta,\zeta').$$

But $\le$ follows from functoriality. To show $\ge$, define

$$\delta(\alpha,\alpha') = \bigcap_{(\zeta,\zeta')\in\Delta Y^2} [\delta_{\Delta Y}(\zeta,\zeta') \to \delta_{\Delta\pi_0 F}(\alpha,\zeta),(\alpha',\zeta'))].$$

Claim $\delta_{\mathsf{E}} \le \delta\circ\eta_{\mathsf{E}}^2$.

Proof: $\delta(\eta_{\mathsf{E}}(e),\eta_{\mathsf{E}}(e') = \bigcap_{(\zeta,\zeta')\in\Delta Y^2} [\delta_{\Delta Y}(\zeta,\zeta') \to \delta_{\Delta\pi_0 F}((\eta_{\mathsf{E}}(e),\zeta),(\eta_{\mathsf{E}}(e'),\zeta'))]$. Since $\delta_F \le \delta_{\Delta\pi_0 F}\circ\eta_F^2 = \delta_{\Delta\pi_0 F}\circ\eta_E^2\times\Delta Y^2$,

$$\begin{aligned}\delta(\eta_{\mathsf{E}}(e),\eta_{\mathsf{E}}(e') &\ge \bigcap_{(\zeta,\zeta')\in\Delta Y^2} [\delta_{\Delta Y}(\zeta,\zeta') \to \delta_F((e,\zeta),(e',\zeta'))] \\ &\ge \delta_{\mathsf{E}}(e,e'),\end{aligned}$$

since $\delta_F((e,\zeta),(e',\zeta') = \delta_{\mathsf{E}}(e,e') \wedge \delta_{\Delta Y}(\zeta,\zeta')$.

By claim (c), $\delta_{\Delta\pi_0(\mathsf{E})} \le \delta$ and we conclude

$$\delta_{\Delta\pi_0\mathsf{E}}(\alpha,\alpha') \wedge \delta_{\Delta Y}(\zeta,\zeta') \le \delta(\alpha,\alpha') \wedge \delta_{\Delta Y}(\zeta,\zeta') \le \delta_{\Delta\pi_0 F}((\alpha,\zeta),(\alpha',\zeta')).$$

The rest of the proof is straightforward and left to the reader.

The obvious next question is whether $\bar{\pi}_0 : \mathcal{E}(\Omega) \to \mathbb{S}(\Gamma(\Omega))$ satisfies the generalized Frobenius condition whenever $\pi_0 : \mathcal{E} \to \mathbb{S}$ does.

To answer this question, we need some preliminaries.

Lemma 4.5 *Let* $(\mathsf{E},\delta_{\mathsf{E}}) \in \mathcal{E}(\Omega)$. *Then* $\delta_{\Delta\Pi_0\mathsf{E}} : (\Delta\pi_0\mathsf{E})^2 \to \Omega$ *is the smallest coincidence relation containing the map* $\delta_0^{\mathsf{E}} : (\Delta\pi_0\mathsf{E})^2 \to \Omega$ *defined by* $\delta_0^{\mathsf{E}}(\alpha,\alpha') = \bigvee_{\eta_{\mathsf{E}}(e)=\alpha,\,\eta_{\mathsf{E}}(e')=\alpha'} \delta_{\mathsf{E}}(e,e')$.

Proof: From $\delta_{\mathsf{E}} \le \delta_{\Delta\pi_0\mathsf{E}}\circ\eta_{\mathsf{E}}^2$ we conclude that

$$\delta_0^{\mathsf{E}}(\alpha,\alpha') = \bigvee_{\substack{\eta_{\mathsf{E}}(e)=\alpha \\ \eta_{\mathsf{E}}(e')=\alpha'}} \delta_{\mathsf{E}}(e,e') \le \delta_{\Delta\pi_0\mathsf{E}}(\alpha,\alpha').$$

Furthermore, it is obvious that $\delta_{\mathsf{E}} \le \delta_0^{\mathsf{E}}\circ\eta_{\mathsf{E}}^2$, and this implies that the smallest coincidence relation containing δ_0^{E}, namely $\overline{\delta_0^{\mathsf{E}}}$, satisfies the conditions

$$\overline{\delta_0^{\mathsf{E}}} \le \delta_{\Delta\pi_0\mathsf{E}}$$

$$\delta_{\mathsf{E}} \le \overline{\delta_0^{\mathsf{E}}}\circ\eta_{\mathsf{E}}^2.$$

Since $\delta_{\Delta\pi_0\mathsf{E}}$ is the smallest coincidence relation satisfying the second condition, $\overline{\delta_0^{\mathsf{E}}} = \delta_{\Delta\pi_0\mathsf{E}}$.

Recalling from the remark after Proposition 2.6.2 (§2.6) that $\mathcal{E}(\Omega)$ can be alternatively described as the category of partial equivalence relations on objects of $\mathcal{E}$, we may formulate the previous lemma as follows:

Lemma 4.6 *Let* $(\mathsf{E},\delta_{\mathsf{E}}) \in \mathcal{E}(\Omega)$. *Then* $\delta_{\Delta\pi_0\mathsf{E}}^{-1}(T)$ *is the smallest partial equivalence relation on* $\Delta\pi_0\mathsf{E}$ *containing* $S \rightarrowtail (\Delta\pi_0\mathsf{E})^2$, *where S is given as the image factorization*

$$\begin{tikzcd} \mathsf{E}^2 \arrow[rr, "\eta_{\mathsf{E}}^2"] \arrow[dr] & & (\Delta\pi_0\mathsf{E})^2 \\ & S \arrow[ur, hook] & \end{tikzcd}$$

of η_{E}^2.

Lemma 4.7 *Assume that* $\pi_0 : \mathcal{E} \to \mathbb{S}$ *satisfies the generalized Frobenius condition. If*

$$\begin{tikzcd} (F,\delta_F) \arrow[r] \arrow[d] & (\mathsf{E},\delta_{\mathsf{E}}) \arrow[d, "u"] \\ (\Delta Y,\delta_{\Delta Y}) \arrow[r, "\Delta f"'] & (\Delta X,\delta_{\Delta X}) \end{tikzcd}$$

is a pull-back, then for all $(\alpha,\zeta),(\alpha',\zeta') \in \Delta\pi_0 F$,

$$\delta_0^{\mathsf{E}}(\alpha,\alpha') \wedge \delta_{\Delta Y}(\zeta,\zeta') = \delta_0^F((\alpha,\zeta),(\alpha',\zeta')).$$

Proof: $\delta_0^{\mathsf{E}}(\alpha,\alpha') \wedge \delta_{\Delta Y}(\zeta,\zeta') = \bigvee_{\eta_{\mathsf{E}}(e)=\alpha,\, \eta_{\mathsf{E}}(e')=\alpha'} \delta_{\mathsf{E}}(e,e') \wedge \delta_{\Delta Y}(\zeta,\zeta')$. Since the diagram

$$\begin{tikzcd} \mathsf{E} \times \Delta Y \arrow[r, "\eta_{\mathsf{E}} \times \Delta Y"] & \Delta\pi_0\mathsf{E} \times \Delta Y \\ F \arrow[u, hook] \arrow[r, "\eta_F"'] & \Delta\pi_0 F \arrow[u, hook] \end{tikzcd}$$

is a pull-back, the right-hand side of the last equality is equal to

$$\bigvee_{\substack{\eta_F(e,\zeta)=(\alpha,\zeta) \\ \eta_F(e',\zeta)=(\alpha',\zeta')}} \delta_{\mathsf{E}}(e,e') \wedge \delta_{\Delta Y}(\zeta,\zeta') = \delta_0^F(\alpha,\zeta),(\alpha',\zeta')).$$

Corollary 4.8 *Let* $\mathcal{E} \to \mathbb{S}$ *be a geometric morphism* $\Delta \dashv \Gamma$ *such that* Δ *has a left adjoint* $\pi_0 \dashv \Delta$ *satisfying the generalized Frobenius condition. If for every* $\mathsf{E} \in \mathcal{E}$, *the image of a partial equivalence relation* $R \hookrightarrow \mathsf{E}^2$ *under the map* $\mathsf{E}^2 \xrightarrow{\eta_{\mathsf{E}}^2} (\Delta\pi_0\mathsf{E})^2$ *is again a partial equivalence relation, then the lifting* $\bar{\pi}_0 : \mathcal{E}(\Omega) \to \mathbb{S}(\Gamma(\Omega))$ *also satisfies the generalized Frobenius condition*

Proof: In this case, $\delta_{\Delta\pi_0\mathsf{E}} = \overline{\delta_0^{\mathsf{E}}} = \delta_0^{\mathsf{E}}$.

In particular, in Example 1 (§2.2), i.e.,

$$\mathrm{Sets} \underset{\Gamma}{\overset{\pi_0}{\leftleftarrows}} \mathrm{Sets}^I,$$

the lifting of π_0 satisfies the generalized Frobenius condition, since $\eta_{(Xi)_i}$: $(X_i)_i \to (\amalg_i X_i)_i$ is monic.

On the other hand, the following example (particular case of Example 2 (§2.2)) shows that the lifting of π_0 does not always satisfy this generalized Frobenius condition (even when π_0 does). Consider $\mathfrak{2} = \{0 \to 1\}$ and

$$\mathrm{Set} \underset{\Gamma}{\overset{\pi_0}{\leftrightarrows}}\!\!\xrightarrow{\Delta} \mathrm{Set}^{\mathfrak{2}op}.$$

In this case, $\Delta S = (S \xrightarrow{1_S} S)$, $\pi_0(\mathsf{E}_0 \xrightarrow{f} \mathsf{E}_1) = \mathsf{E}_1$, and $\Gamma(\mathsf{E}_0 \xrightarrow{f} \mathsf{E}_1) = \mathsf{E}_0$, with obvious actions on morphisms. The object of truth-values is $\Omega = (\Omega_0 \xrightarrow{\epsilon} \Omega_1)$, where $\Omega_0 = \{0,\frac{1}{2},1\}$, $\Omega_1 = \{0,1\}$, $\epsilon(0) = 0$, $\epsilon(\frac{1}{2}) = \epsilon(1) = 1$. Using our identification $\mathcal{E}(\Omega) = \mathrm{PER}(\mathcal{E})$, define (E, R) as follows:

$$\mathsf{E} = (\mathsf{E}_0 = \{0,1,2,3\} \xrightarrow{\alpha} \{0,1,2\} = \mathsf{E}_1)$$

with $\alpha(0) = 0$, $\alpha(1) = \alpha(2) = 1$, $\alpha(3) = 2$;

$$R = (R_0 \xrightarrow{\beta} R_1) \text{ where}$$
$$R_0 = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0),(2,3),(3,2)\}$$
$$R_1 = \mathsf{E}_1^2$$

and β is the restriction of α^2.

Define (F, S) by means of the pull-back diagram

$$\begin{tikzcd}
(F,S) \arrow[r] \arrow[d] & (\mathsf{E},R) \arrow[d, "u"] \\
(\Delta\{0\},(\Delta\{0\})^2) \arrow[r, tail, "\Delta i"'] & (\Delta\{0,1\},(\Delta\{0,1\})^2)
\end{tikzcd}$$

where $i = \{0\} \rightarrowtail \{0,1\}$ is the inclusion, $u_0(0) = u_0(3) = 0$, $u_0(1) = u_0(2) = 1$, $u_1(0) = u_1(2) = 0$ and $u_1(1) = 1$. Obviously, u is a morphism. From the definition of (F, S),

$$F = (\{0,3\} \xrightarrow{\alpha'} \{0,2\})$$
$$S = (S_0 \xrightarrow{\beta'} S_1),$$

where $S_0 = \{(0,0),(3,3)\}$, $S_1 = \{0,2\}^2$, α' is the restriction of α and β' is the restriction of β^2.

The map $\mathsf{E} \xrightarrow{\eta_\mathsf{E}} \Delta\pi_0\mathsf{E}$ is given by the diagram

$$\begin{tikzcd}
\mathsf{E}_0 \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \mathsf{E}_1 \arrow[d, "\mathrm{Id}_{\mathsf{E}_1}"] \\
\mathsf{E}_1 \arrow[r, "\mathrm{Id}_{\mathsf{E}_1}"'] & \mathsf{E}_1
\end{tikzcd}$$

We compute the image of R under η_E^2 to obtain $R' = (R_0' \xrightarrow{\gamma} R_1')$ where $R_0' = \{(0,0),(1,1),(2,2),(0,1),(1,0),(1,2),(2,1)\}$, $R_1' = \mathsf{E}_1^2$ and γ is the restriction of the identity.

In other words, the smallest partial equivalence relation containing R', $\delta^{-1}_{\Delta\pi_0\mathsf{E}}(T) = \mathsf{E} \times \mathsf{E}$. Thus, $\delta_{\pi_0\mathsf{E}} = \delta_T$.

On the other hand, $F \xrightarrow{\eta_F} \Delta\pi_0 F$ is given by

$$\begin{tikzcd} F_0 \arrow[r, "\alpha'"] \arrow[d, "\alpha'"'] & F_1 \arrow[d, "\mathrm{Id}_{F_1}"] \\ F_1 \arrow[r, "\mathrm{Id}_{F_1}"'] & F_1 \end{tikzcd}$$

The image of S under η_F^2 is $S' = (S'_0 \xrightarrow{\delta'} S'_1)$, where $S'_0 = \{(0,0),(2,2)\}$, $S'_1 = \{0,2\}^2$, and δ' is the restriction of the identity. But then S' is already the partial equivalence relation $\delta_{\Delta\pi_0 F}^{-1}(T)$.

Since in this case $\delta_{\pi_0(F)} = (\delta_{\Delta\pi_0 F})_0$, we obtain that $\delta_{\pi_0(F)}(0,2) \neq \top$ and hence that the diagram

$$\begin{tikzcd} (\pi_0(F),\delta_{\pi_0(F)}) \arrow[r] \arrow[d] & (\pi_0(\mathsf{E}),\delta_{\pi_0(\mathsf{E})}) \arrow[d, "\mathrm{tr}(u)"] \\ (\{0\},\delta_\top) \arrow[r, tail, "i"'] & (\{0,1\},\delta_\top) \end{tikzcd}$$

is *not* a pull-back.

We finish this section with a reformulation of Frobenius conditions for $\bar{\pi}_0$ in terms of properties of $\bar{\Delta}$.

Proposition 4.9 *Let $\mathcal{E} \to \mathbb{S}$ be a geometric morphism $\Delta \dashv \Gamma$. Assume that $\bar{\Delta}: S(\Gamma(\Omega)) \to \mathcal{E}(\Omega)$ has a left adjoint $\bar{\pi}_0 \dashv \bar{\Delta}$. Then $\bar{\Delta}$ preserves exponentials (respectively Π'_f operations) iff $\bar{\pi}_0: \mathcal{E}(\Omega) \to \mathbb{S}(\Gamma(\Omega))$ satisfies the Frobenius condition (respectively the generalized Frobenius condition).*

Proof: This is just a formal computation (cf. Theorem 5 of Barr–Paré [1]): for the case of exponentials, e.g.,

$$[\bar{\pi}_0(\mathsf{E}) \times X, Y]_{\mathbb{S}(\Gamma\Omega)} \simeq [\bar{\pi}_0(\mathsf{E}), Y^X]_{\mathbb{S}(\Gamma\Omega)} \simeq [\mathsf{E}, \bar{\Delta}(Y^X)]_{\mathcal{E}(\Omega)}$$

$$[\bar{\pi}_0(\mathsf{E} \times \bar{\Delta}X), Y]_{\mathbb{S}(\Gamma\Omega)} \simeq [\mathsf{E} \times \bar{\Delta}X, \bar{\Delta}Y]_{\mathcal{E}(\Omega)} \simeq [\mathsf{E}, \bar{\Delta}Y^{\bar{\Delta}X}]_{\mathcal{E}(\Omega)}.$$

Therefore, (by Yoneda) $\bar{\pi}_0(\mathsf{E} \times \bar{\Delta}X) \simeq \bar{\pi}_0(\mathsf{E}) \times X$ iff $\bar{\Delta}(Y^X) \simeq \bar{\Delta}Y^{\bar{\Delta}X}$. The case of Π'_f operations is similar.

Corollary 4.10 *Let $\mathcal{E} \to \mathbb{S}$ be a geometric morphism $\Delta \dashv \Gamma$. Assume that Δ has a left adjoint π_0 satisfying the Frobenius condition. Then the lifting $\bar{\Delta}: \mathbb{S}(\Gamma(\Omega) \to \mathcal{E}(\Omega)$ preserves exponentials.*

Remark The counterexample $\mathrm{Set}^{\mathfrak{2}^{op}} \to \mathrm{Set}$ together with Proposition 4.9 shows that $\bar{\Delta}: \mathrm{Set}(\Gamma(\Omega)) \to \mathrm{Set}^{\mathfrak{2}^{op}}(\Omega)$ does not preserve the Π_f operators although $\pi_0: \mathrm{Set}^{\mathfrak{2}^{op}} \to \mathrm{Set}$ satisfies the generalized Frobenius condition.

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Appendix In this Appendix, we derive the main result of Example 3 (§2.2), namely the relation between forcing relation for $\Box\varphi$ in terms of forcing relation for φ.

Recall that we had a topos $\mathcal{E} \to \mathbb{S}$ bounded. This means that $\mathcal{E}$ may be presented as the category of sheaves over a site $\mathbb{C}$ in $\mathbb{S}$. We have the diagram

$$\begin{tikzcd} & \mathrm{Sh}_{\mathbb{S}}(\mathbb{C}) = \mathcal{E} \arrow[dl, shift left, "\Gamma"] \arrow[d, shift left, "i"] \\ \mathbb{S} \arrow[ur, shift left, "\Delta"] \arrow[r, shift left, "\Delta_0"] & \mathbb{S}^{\mathbb{C}^{op}} \arrow[l, shift left, "\Gamma_0"] \arrow[u, shift left, "a"] \end{tikzcd}$$

where a is the “associated sheaf functor”, Δ_0 is the constant presheaf $\Delta_0(S)(C) = S \ \forall C \in \mathbb{C}$, $\Gamma_0(F) = \varprojlim_{\mathbb{C}^{op}} F, \Delta = a\Delta_0, \Gamma = \Gamma_0 i.$

To describe the maps $\Omega_{\mathbb{S}} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathcal{E}})$ we recall some definitions and a lemma (cf. Moerdijk [16]): a *sieve* K is a set (in the sense of $\mathbb{S}$) of objects of $\mathbb{C}$ such that $C \in K$ and $C' \to C \in \mathbb{C}$ implies $C' \in K$. The *closure* $\mathrm{cl}(K)$ of a sieve K is $\{C \in \mathbb{C} \mid \exists\{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C) \forall i \in I(C_i \in K)\}$. A sieve is *closed* if it coincides with its closure.

Lemma A.1 $\Gamma(\Omega_{\mathcal{E}}) =$ *set of closed sieves*
$\delta(p) = \mathrm{cl}\{C \in \mathbb{C} \mid p\}$
$\gamma(K) = \| \forall C \in \mathbb{C}(C \in K) \|$.

Proof: We first show that $\Gamma_0(\Omega) = \varprojlim_{\mathbb{C}^{op}} \Omega$, where Ω is the object of truth-values of $\mathbb{S}^{\mathbb{C}^{op}}$, is in bijective correspondence with the set of sieves on $\mathbb{C}$. Indeed, defining $\psi((\mu_C)_{C\in\mathbb{C}}) = \{C \in \mathbb{C} \mid \mathrm{id}_C \in \mu_C\}$ and $\Phi(K) = \{\{f: C' \to C \mid C' \in K\}\}_{C\in\mathbb{C}}$ one easily checks that ψ and Φ are inverse to each other and establish the desired bijection. Let us recall that $\Omega_{\mathcal{E}}$ is defined by the equalizer $i\Omega_{\mathcal{E}} \hookrightarrow \Omega \underset{\mathrm{id}_\Omega}{\overset{j}{\rightrightarrows}} \Omega$, where j is the topology defined by the site $\mathbb{C}$, i.e.,

$$j_C(K) = \{f : C' \to C \mid \exists \{C_i' \xrightarrow{\alpha_i} C'\}_{i\in I} \in \mathrm{Cov}(C') \forall i f \circ \alpha_i \in K\}.$$

This allows us to conclude that the couple ψ, Φ restricts to a bijection between elements in $\Gamma(\Omega_{\mathcal{E}}) = \Gamma_0(i\Omega_{\mathcal{E}}) = \varprojlim_{\mathbb{C}^{op}} i\Omega_{\mathcal{E}}$ and closed sieves. By identifying $\Gamma(\Omega_{\mathcal{E}})$ with closed sieves on $\mathbb{C}$, the counit $\Delta\Gamma\Omega_{\mathcal{E}} \xrightarrow{\mathcal{E}_\Omega} \Omega_E$ is now given by $(\mathcal{E}_\Omega)_C(K) = (\Phi(K))_C =$ the C^{th} component of $\Phi(K)$. (This argument was suggested by M. Zawadowski.)

To show the formula for $\delta(p)$, we first assume that p is a global section $p: \mathbb{1} \to \Omega_s$ and then go over to $\mathbb{S}/S$. Let $P \rightarrowtail \mathbb{1} \in \mathbb{S}$ be the subobject classified by p. Then $* \in \Delta P(C)$ iff $\exists \{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)\ \forall i \in I * \in P$. By identifying $\Delta P \hookrightarrow \mathbb{1}$ with the closed sieve $\{C \in \mathbb{C} \mid * \in \Delta P(C)\}$, we conclude that

$$\begin{aligned}\delta(p) &= \mathrm{cl}\{C \in \mathbb{C} \mid * \in P\}\\ &= \mathrm{cl}\{C \in \mathbb{C} \mid p\}.\end{aligned}$$

The formula for $\gamma(K)$ follows from the adjunction $\delta \dashv \gamma$.

We would now like to find the forcing relation $C \Vdash \Box\varphi[\xi]$, $\xi \in \Delta S(C)$ in terms of the forcing relation for φ. To solve this problem, we need some preliminaries.

Lemma A.2 *Consider the diagram*

$$A \underset{U_0}{\overset{F_0}{\rightleftarrows}} B \underset{U_1}{\overset{F_1}{\rightleftarrows}} C,$$

where $F_i \dashv U_i$ $(i = 0,1)$ *with units given by* $\mathrm{Id_A} \xrightarrow{\eta_0} U_0F_0$, $\mathrm{Id}_B \xrightarrow{\eta_1} U_1F_1$ *and counits given by* $F_0U_0 \xrightarrow{\epsilon_0} \mathrm{Id}_B$, $F_1U_1 \xrightarrow{\epsilon_1} \mathrm{Id}_C$. *Then* $F_1F_0 \dashv U_0U_1$ *with unit given by*

$$\eta : \mathrm{Id}_A \to U_0U_1F_1F_0$$
$$A \mapsto U_0(\eta_1)_{F_0(A)} \circ (\eta_0)_A$$

and counit given by

$$\epsilon : F_1F_0U_0U_1 \to \mathrm{Id}_C$$
$$C \mapsto (\epsilon_0)_C = (\epsilon_1)_C \circ F_1(\epsilon_{0U_1(C)}).$$

Proof: Simple computation.

Corollary A.3 *The following diagram is commutative*

$$\begin{tikzcd}
F_0U_0U_1(C) \arrow[rr, "(\eta_1)_{FUU_1'(C)}"] \arrow[dr, "\epsilon_{0U_1(C)}"'] & & U_1F_1F_0U_0U_1'(C) \arrow[dl, "U_1(\epsilon_C)"] \\
& U_1(C) &
\end{tikzcd}$$

Proof: Look at its transpose and use Lemma A.2.

We now apply these results to

$$\mathbb{S} \underset{\Gamma_0}{\overset{\Delta_0}{\rightleftarrows}} \mathbb{S}^{C^{op}} \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Sh}_{\mathbb{S}}(\mathbb{C}).$$

We let $\mathrm{Id}_{\mathbb{S}} \xrightarrow{\eta_0} \Gamma_0\Delta_0'$, $\mathrm{Id}_{\mathbb{S}^{\mathbb{C}^{op}}} \xrightarrow{\eta_1} ia$ be the units and $\Delta_0\Gamma_0 \xrightarrow{\epsilon_0} \mathrm{Id}_{\mathbb{S}^{\mathbb{C}^{op}}}$, $ai \xrightarrow{\epsilon_1} \mathrm{Id}_{\mathrm{Sh}_{\mathbb{S}}(\mathbb{C})}$ be the counits of the corresponding adjunctions. Furthermore, we define $\Delta = a\Delta_0$, $\Gamma = \Gamma_0 i$ and let η, ϵ be the unit and counit, respectively.

Let $\Delta S \xrightarrow{\varphi} \Omega_{\mathcal{E}} \in \mathcal{E}$ be given and consider the diagram (*)

(*)

$$\begin{tikzcd}
& \Delta_0\Gamma\Delta S \arrow[r, "\Delta_0\Gamma\varphi"] \arrow[d, "(\epsilon_0)_{i\Delta S}"] & \Delta_0\Gamma\Omega \arrow[d, "(\epsilon_0)_{i\Omega}"] \\
\Delta_0 S \arrow[ur, "\Delta_0\eta_S"] \arrow[r, "(\eta_1)_{\Delta_0 S}"'] & i\Delta S \arrow[r, "i\varphi"'] & i\Omega
\end{tikzcd}$$

The square commutes by naturality of ϵ_0 and the triangle also commutes:

$$\begin{tikzcd}
\Delta_0 S \arrow[r, "\Delta_0\eta S"] \arrow[d, "(\eta_1)_{\Delta_0 S}"'] & \Delta_0\Gamma\Delta S \arrow[d, "(\eta_1)_{\Delta_0\Gamma\Delta S}"] \\
ia\Delta_0 S \arrow[r, "ia\Delta_0\eta_S = i\Delta\eta_S"'] \arrow[dr, "\mathrm{id}"'] & ia\Delta_0\Gamma\Delta S \arrow[d, "i\epsilon_{\Delta S}"] \\
& i\Delta S
\end{tikzcd}$$

In fact, $i\epsilon_{\Delta S} \circ (\eta_1)_{\Delta_0\Gamma\Delta S} = (\epsilon_i)_{i\Delta S}$ by the corollary and $i\epsilon_{\Delta S} \circ i\Delta\eta_S = i(\epsilon_{\Delta S} \circ \Delta\eta_S) = i(\mathrm{Id}) = \mathrm{Id}$.

We compute (*) at $C \in \mathbb{C}$ to obtain (in $\mathbb{S}$)

$$\begin{tikzcd}
& \Gamma\Delta S \arrow[r, "\Gamma(\varphi)"] \arrow[d] & \Gamma(\Omega) = \Gamma_0(i\Omega) \arrow[d, "((\epsilon_0)_{i\Omega})_C"] \\
S \arrow[ur, "\eta_S"] \arrow[r, "((\eta_1)_{\Delta_0 S})_C"'] & i\Delta S(C) \arrow[r, "(i\varphi)_C"'] & (i\Omega)(C)
\end{tikzcd}$$

The map $(\epsilon_1)_{i\Omega}$ is just what we called $(\Phi(K)_C$ in Lemma A.1.
Let $l_S = (\eta_1)_{\Delta_0 S}$ to simplify the notation.

Proposition A.4 $C \Vdash \varphi[(l_S)_C(s)]$ *iff* $C \in \Gamma(\varphi)(\eta_S(s))$.

Proof: We have the following equivalences:

$$\begin{array}{ll}
C \Vdash \varphi[(l_S)(s)] & \\
\hline
 & \text{(def of } \Vdash) \\
(i\varphi)_C((l_S)_C(s) = T_C & \\
\hline
 & (*) \\
((\epsilon_1)_{i\Omega})_C\Gamma(\varphi)(\eta_S(s)) = T_C & \\
\hline
 & \text{(def of } (\epsilon_1)_{i\Omega}). \\
C \in \Gamma(\varphi)(\eta_S(s)) &
\end{array}$$

The following answers the question about $\Vdash$ for $\Box$:

Corollary A.5

(1) $C \Vdash \Box\varphi[(l_S)_C(s)]$ *iff* $\exists\{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)\ \forall i \in I\ \forall C' \in \mathbb{C}\ C' \Vdash \varphi[(l_S)_{C'}(s)]$.

(2) $C \Vdash \Box\varphi[\xi]$ *iff* $\exists\{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)\ \forall i \in I\ \exists s_i \in S\ \xi | C_i = (l_s)_{C_i}(s_i) \wedge \exists\{C_{ij} \to C_i\}_j \in \mathrm{Cov}(C_i) \forall j \forall C' \in \mathbb{C}\ C' \Vdash \varphi[(l_S)_{C'}(s_i)]$.

Proof:

(1) From Proposition A.4 and the definition of $\Box$, we obtain

$$\begin{aligned} C \Vdash \Box\varphi[(l_S)_C(s)] \text{ iff } & C \in \Box\Gamma(\varphi)(\eta_S(s)) = \delta\gamma(\Gamma(\varphi)(\eta_S(s))) \\ \text{iff } & C \in cl\{C \in \mathbb{C} \mid \| \forall C' \in \mathbb{C}(C' \in \Gamma(\varphi)(\eta_S(s)) \| \} \\ \text{iff } & \exists\{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)\ \forall i \in I\ \forall C' \in \mathbb{C} \\ & \quad C' \in \Gamma(\varphi)(\eta_s(s)) \\ \text{iff } & \exists\{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)\ \forall i \in I\ \forall C' \in \mathbb{C} \\ & \quad C' \Vdash \varphi[(l_S)_{C'}(s)] \end{aligned}$$

(2)
$$\begin{aligned} C \Vdash \Box\varphi[\xi] \quad \text{iff } & \exists\{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)\ \forall i \in I\ \exists s_i \in S \\ & \quad \xi | C_i = (l_S)_{C_i}(s_i) \wedge C_i \Vdash \Box\varphi[(l_S)_{c_i}(s_i)] \\ \text{iff } & \exists\{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)\ \forall i \in I\ \exists s_i \in S \\ & \quad \xi | C_i = (l_S)_{C_i}(s_i) \wedge \exists\{C_{ij} \to C_i\} \\ & \qquad \in \mathrm{Cov}(C_i)\ \forall_j \forall C' \in \mathbb{C} \\ & \quad C' \Vdash \varphi[(l_S)_{C'}(s_i)]. \end{aligned}$$