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Henkin's Completeness Proof: Forty Years Later

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1 In his 1949 paper, "The completeness of the first-order calculus", Henkin developed what is now called *the method of (individual) terms*[1] to establish that *every consistent set of statements of a first-order language L has a model of cardinality* α, α the number of statements of L. The idea is to start with such a set S, construct a so-called *term-extension* L^+ *of* L by adding α new terms to the vocabulary of L,[2] extend S to a *maximally consistent* and *term-complete* set S_∞ of statements of L^+,[3] and construct a model of S_∞ whose domain consists of the terms of L^+. When restricted to L, the model in question automatically constitutes one of S. Henkin's result has come to be known as *the Strong Completeness Theorem for First-Order L*. Another, and more familiar, version of the theorem has it that *if a statement A of L is true in every model of a set S of statements of L, then A is provable from S*. Henkin himself did not bother to prove this. He merely proved the special case of it, known as *the Weak Completeness Theorem for First-Order L*, where S is $\varnothing$.

A model like the one Henkin constructed for his set S_∞ is commonly known as a *Henkin model*. It is the kind of model in which each member of the domain "has a name". Henkin accomplished this by making each member of his domain a name of itself, a radical move at the time. In consequence, though, the restriction of his model to L does not constitute a Henkin model of S, a pity in the event that S does have such a model.

To commemorate the publication of Henkin's paper, we offer here two new completeness proofs for first-order L.[4] The language considered by Henkin had an unspecified number of terms to start with, but those played no special role in his proof. The one we construct in Section 2 has denumerably many, and these will play a crucial role in our proofs. In the first of them, begun in Section 3 and concluded in Section 5, no new terms will be added; in the second, presented in Section 5 and relating to truth-value semantics,[5] denumerably many will be. The proofs are sharpenings of proofs of Leblanc's in [10]. They have two cases each, Case One minding the consistent sets of statements of L that extend *without* the

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use of new terms to a maximally consistent and term-complete one, and Case Two minding the rest. Importantly, given our purpose, the former sets prove to be the sets of statements of L that have a Henkin model. And, importantly in its own right and for our two proofs, which hinge on this, the former sets also prove to be the sets of statements of L that are (what we call) *instantially consistent*, i.e. the sets of statements of L from which no contradiction is (what we call) *instantially provable.*

The notion of instantial provability is a straightforward generalization of the notion of *provability in omega-logic*; and its distinctive rule, according to which a universal quantification of L is provable from the set of its infinitely many *instances* in L, is a straightforward generalization of the *omega-rule* of omega-logic.[6] Henkin had already identified the consistent sets of statements of L that have a Henkin model in a 1954 paper, "A generalization of the concept of omega-consistency", but that identification, we feel, is less natural than the one offered here.

We touch in Section 6 on the history of these various notions, and then turn to more special, but rather intriguing, matters. As we will have shown in Section 4, any term-complete set of statements of L that is consistent in the standard sense is also instantially consistent. So there might be a temptation to conclude that consistency in the standard sense *plus* term-completeness amounts to instantial consistency. Not so: the set of all the substitution instances in L of a quantification $(\forall X)A$ of L, though instantially consistent, is generally not term-complete. There might also be a temptation to think that term-consistency and instantial consistency are the same. Not so again, though counterexamples are harder to come by. Henkin had constructed in his 1954 paper a term-consistent set of statements of L that in effect is not instantially consistent: it featured two monadic predicates *and* the identity sign '='. And we constructed another in Weaver [14], which featured two monadic predicates *but not* '='. We construct two more here, one featuring one dyadic predicate *but not* '=', the other featuring *one* monadic predicate *and* '='. The last of these counterexamples cannot be improved upon: as we go on to show, any term-consistent set of statements of L that features a single monadic predicate *but not* '=' is instantially consistent, and hence has a Henkin model.

2 The first-order language L whose strong completeness we prove has as its *primitive signs* countably (i.e., finitely or denumerably) many predicates, these denumerably many (individual) variables:

$$x, y, z, x', y', z', \ldots,$$

these denumerably many terms:

$$\mathrm{t}_1, \mathrm{t}_2, \ldots,$$

listed here in alphabetical order, the three logical operators '$\sim$', '$\supset$', and '$\forall$', the two parentheses '(' and ')', and the comma ','. And its denumerably many *statements*, presumed to be arranged in some alphabetical order, are of the following four forms: (i) $P(T_1, T_2, \ldots, T_n)$, where P is an n-adic ($n = 1,2,\ldots$) predicate of L and $T_1, T_2, \ldots, T_n$ are n not necessarily distinct terms of L;

(ii) $\sim A$, where A is a statement of L; (iii) $(A \supset B)$, where A and B are statements of L; and (iv) $(\forall X)A$, where X is a variable of L and the result $A(T/X)$ of replacing X everywhere in A by a term T of L is a statement of L.[7] We presume the four logical operators '&', 'v', '≡', and '∃', used only in Section 6, to be defined in the customary manner. And, for brevity's sake, we omit outer parentheses whenever clarity allows.

Usually, one would acknowledge as *the axioms of L* the statements of L of such forms as these six:

A1 $A \supset (B \supset A)$
A2 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3 $(\sim A \supset \sim B) \supset (B \supset A)$
A4 $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$
A5 $A \supset (\forall X)A$
A6 $(\forall X)A \supset A(T/X)$,

plus the statements of L of the form $(\forall X)(A(X/T))$, where A is an axiom of L and T is a term of L. Given a statement A of L and a set S of statements of L, one would next acknowledge as a *proof of A from S in the standard sense* any finite sequence of statements of L whose last entry is A, and every one of whose entries is: (i) a member of S, (ii) an axiom of L, or (iii) the consequent C of a conditional $B \supset C$ of L and both B and $B \supset C$ occur earlier in the sequence. And one would then say that A is *provable from S in the standard sense*, for short,

$$S \vdash A,$$

if there exists a proof of A from S in the standard sense.

A *proof of A from S in the instantial sense*, on the other hand, would be any *countable* sequence of statements of L whose last entry is A, and every one of whose entries is as in (i)–(iii) above or is (iv) a quantification $(\forall X)B$ of L and all the substitution instances $B(t_1/X), B(t_2/X), \ldots$ of $(\forall X)B$ in L occur earlier in the sequence. And one would say that A is *instantially provable from S*, for short,

$$S \vdash_i A,$$

if there exists a proof of A from S in the instantial sense.

In each account, (iii) is of course modus ponens, the *Elimination Rule for* '⊃' of *natural deduction*. In the second, (iv) is the rule of infinite instantial induction. As it is a (nonstandard) *Introduction Rule for* '∀', we call it $\forall I_i$.[8]

As the reader well knows, a statement of L is provable in the standard sense from an infinite set of statements of L if, and only if, provable in that sense from a finite subset of it (*Point One*). However, a statement of L may be provable in the instantial sense from an infinite set of statements of L and yet not be provable in that sense from any finite subset of it (*Point Two*). $(\forall X)A$, for example, is provable in the instantial sense from the set of its substitution instances in L, but it is not generally provable in that sense from any finite subset of that set.

Used here for convenience and novelty are accounts of provability equivalent to, but quite different from, the foregoing. They dispense, in particular, with axioms and proofs. The account of provability in the standard sense is in two parts. The first one attends to provability from a finite set of statements of L. Its eight clauses are counterparts of rules of natural deduction, and they bear the names of those rules. The other part, attending to provability from an infinite set of statements of L, is *Point One*. Because of it the sets S and S' in the first part may of course be infinite as well as finite. Expectedly, given *Point Two*, the account of instantial provability attends at a stroke to provability from a finite set of statements of L and provability from an infinite one.

Provability in the standard sense

Part One *Suppose S a finite set of statements of L.*

1. *Structural Clauses:*

$$S \vdash A \textit{ for any member } A \textit{ of } S \quad (\textit{Reiteration})$$

$$\textit{If } S \vdash A, \textit{ then } S \cup S' \vdash A$$
$$\textit{for any finite set } S' \textit{ of statements of } L \quad (\textit{Thinning})^9$$

2. *Intelim Clauses:*

If $S \cup \{A\} \vdash B$ *and* $S \cup \{A\} \vdash {\sim}B$ *for any statement* B *of* L, *then* $S \vdash {\sim}A$	$({\sim}\mathrm{I})$	*If* $S \vdash {\sim}{\sim}A$, *then* $S \vdash A$	$({\sim}\mathrm{E})$
If $S \cup \{A\} \vdash B$, *then* $S \vdash A \supset B$	$(\supset\mathrm{I})$	*If* $S \vdash A$ *and* $S \vdash A \supset B$, *then* $S \vdash B$	$(\supset\mathrm{E})$
If $S \vdash A(T/X)$ *for any term* T *of* L *foreign to* S *and to* $(\forall X)A$, *then* $S \vdash (\forall X)A$	$(\forall\mathrm{I})$	*If* $S \vdash (\forall X)A$, *then* $S \vdash A(T/X)$ *for any term* T *of* L	$(\forall\mathrm{E})$

Part Two *Suppose S an infinite set of statements of L. Then $S \vdash A$ if, and only if, $S' \vdash A$ for at least one finite subset S' of S.*

Provability in the instantial sense

Part One *Same eight rules as in Part One of the preceding table, but with:* (i) *S and S' allowed to be infinite and* (ii) 'i' *subscripted everywhere to* '$\vdash$'.

Part Two *Rule distinctive of provability in the instantial sense, with S allowed again to be infinite*:

$$\textit{If } S \vdash_i A(T/X) \textit{ for every term } T \textit{ of } L,$$
$$\textit{then } S \vdash_i (\forall X)A \quad (\forall \mathrm{I}_i)$$

Lemma 1 holds by definition. Its converse fails, of course: though instantially provable from the set $\{P(T): \textit{for any term } T \textit{ of } L\}$, '$(\forall x)P(x)$' is not provable in the standard sense from any finite subset of the set nor – as a result – from the set itself.

Lemma 1 *If $S \vdash A$, then $S \vdash_i A$.*

A number of syntactical notions, some of them already mentioned in the previous pages, require definition. Suppose S a set of statements of L. We shall say that S is

(i) *consistent in the standard sense* if there exists no statement A of L such that both $S \vdash A$ and $S \vdash \sim A$, *inconsistent in the standard sense* otherwise,

(ii) *instantially consistent* if there exists no statement A of L such that both $S \vdash_i A$ and $S \vdash_i \sim A$, *instantially inconsistent* otherwise,

(iii) *maximally consistent* if, and only if, S is consistent in the standard sense and, for any statement A of L that does not belong to $S, S \cup \{A\}$ is inconsistent in the standard sense,

(iv) *term-consistent* if there exists no quantification $(\forall X)A$ of L such that $S \vdash A(T/X)$ for every term T of L and yet $S \vdash \sim(\forall X)A$, *term-inconsistent* otherwise,[10] and

(v) *term-complete* if, and only if, for every quantification $(\forall X)A$ of L, $S \vdash (\forall X)A$ if $S \vdash A(T/X)$ for every term T of L.[11]

And for brevity's sake we shall say that S constitutes a *Henkin set* if, and only if, S is both maximally consistent and term-complete.

A familiar set of statements of L that is consistent in the standard sense but not term-consistent is this:

$$\{\mathrm{P}(T)\colon \textit{for any term } T \textit{ of } L\} \cup \{\sim(\forall x)\mathrm{P}(x)\},$$

and one of the sets shown in Section 4 to be term-consistent but not instantially consistent is this binary counterpart of it:

$$\{\mathrm{P}(T, T')\colon \textit{for any term } T \textit{ and any term } T' \textit{ of } L\} \cup \{\sim(\forall x)(\forall y)\mathrm{P}(x, y)\}.$$

The notion of term-consistency is used only in Section 6.

Lemma 2

(a) *If S is instantially consistent, then S is consistent in the standard sense*;

(b) *If S is inconsistent in the standard sense, then so is at least one finite subset of S*;

(c) *If S is consistent in the standard sense, then so is either $S \cup \{A\}$ or $S \cup \{\sim A\}$ for any statement A of L*;

(d) *If S is instantially consistent, then so is either $S \cup \{A\}$ or $S \cup \{\sim A\}$ for any statement A of L*;

(e) *If $S \cup \{\sim A\} \vdash A$, then $S \cup \{\sim A\}$ is inconsistent in the standard sense*;

(f) *If $S \cup \{\sim A\} \vdash_i A$, then $S \cup \{\sim A\}$ is instantially inconsistent*;

(g) *If S is consistent in the standard sense and—for any statement A of L—either A belongs to S or $\sim A$ does, then A is maximally consistent*;

(h) *If S is maximally consistent and $S \nvdash A$, then $\sim A$ belongs to S.*

Proof: (a) By Lemma 1. (b) By Part Two of the account of provability in the standard sense. (c) Suppose that both $S \cup \{A\}$ and $S \cup \{\sim A\}$ are inconsistent in the standard sense. Then $S \vdash \sim A$ and $S \vdash \sim\sim A$ by $\sim$I, and hence S is likewise inconsistent in the standard sense. (d) Proof like that of (c). (e)–(f) $S \cup \{\sim A\} \vdash \sim A$ by *Reiteration*. So (e) and (f). (g) Suppose S such that—for any statement A of L—either A belongs to S or $\sim A$ does, and let B be an arbitrary statement of L that does not belong to S. Then by *Reiteration* both $S \cup \{B\} \vdash B$ and $S \cup \{B\} \vdash \sim B$, and hence $S \cup \{B\}$ is inconsistent in the standard sense. So, if S is consistent in that sense, then S is maximally consistent. (h) Suppose S is maximally consistent, and suppose $\sim A$ does not belong to S. Then $S \cup \{\sim A\}$ is inconsistent in the standard sense, and hence $S \vdash A$ by $\sim$I and $\sim$E. So, if $S \nvdash A$, then $\sim A$ has to belong to S.

Another version of Lemma 2(c) will turn up later.

Turning to model-theoretic matters, we understand of course by a *domain* any nonempty set. D being a domain, we understand by a *D-interpretation of (the terms and predicates of) L* any function that pairs each term of L with a member of D and each n-adic ($n = 1, 2, \ldots$) predicate of L with a subset of the n-th power D^n of D. And, D being a domain, $\mathcal{I}_D$ a D-interpretation of L, and T a term of L, we understand by a *T-variant of $\mathcal{I}_D$* any D-interpretation of L that is like $\mathcal{I}_D$ except for possibly pairing with T a member of D other than $\mathcal{I}_D(T)$.

Suppose next that D is a domain, A is a statement of L, and $\mathcal{I}_D$ is a D-interpretation of the terms of L. We say that *A is true on $\mathcal{I}_D$* if, and only if: (i) in the case that A is an atomic statement $P(T_1, T_2, \ldots, T_n)$, the n-tuple $\langle \mathcal{I}_D(T_1), \mathcal{I}_D(T_2), \ldots, \mathcal{I}_D(T_n)\rangle$ belongs to $\mathcal{I}_D(P)$, (ii) in the case that A is a negation $\sim B$, B is not true on $\mathcal{I}_D$, (iii) in the case that A is a conditional $B \supset C$, B is not true on $\mathcal{I}_D$ or C is, and (iv) in the case that A is a quantification $(\forall X)B$ and T is any term of L foreign to A, $B(T/X)$ is true on every T-variant of $\mathcal{I}_D$.[12]

Suppose then that D is a domain, $\mathcal{I}_D$ is a D-interpretation of L, and S is a set of statements of L. We say that *$\mathcal{I}_D$ constitutes a model of S*—equivalently, that *S has $\mathcal{I}_D$ as a model*—if, and only if, all the members of S are true on $\mathcal{I}_D$. And, to accommodate a familiar and handy phrase, we say of a statement A of L that it is *true in* a model $\mathcal{I}_D$ of a given set of statements of L if, and only if, A is true on $\mathcal{I}_D$.

Lastly, we call a D-interpretation $\mathcal{I}_D$ a *Henkin D-interpretation of L*—and, when $\mathcal{I}_D$ constitutes a model of a set S of statements of L, we say that it *constitutes a Henkin model* of S—if, and only if, $\mathcal{I}_D$ pairs *each member of D with a term of L* as well as each term of L with a member of D.[13] Plainly, a quantification $(\forall X)A$ of L is true on a Henkin D-interpretation $\mathcal{I}_D$ of L if, and only if, $A(T/X)$ is true on $\mathcal{I}_D$ for every term T of L.[14] Proof of the fact is in [10], but as it is lengthy we do not reproduce it here.

When '=' is added (as in Section 6) to the vocabulary of L, these changes are in order:

on p. 213, $T = T'$ counts as a *statement* of L when T and T' are terms of L,
on p. 215, Part One of the account of provability in the standard sense must feature these additional rules, in the second of which $A(T'//T)$ is the result of replacing T *somewhere* in A by T',

$$\vdash T = T \quad (=\text{I}) \qquad\qquad \textit{If } S \vdash A \textit{ and } S \vdash T = T', \textit{ then } S \vdash A(T'//T) \quad (=\text{E})^{15}$$

and

on this page, paragraph 1, clause (i) in the account of *A is true on $\mathcal{I}_D$* must read

(i) *in the case that A is an atomic statement of the form $P(T_1, T_2, \ldots, T_n)$, $\langle \mathcal{I}_D(T_1), \mathcal{I}_D(T_2), \ldots, \mathcal{I}_D(T_n)\rangle$ belongs to $\mathcal{I}_D(P)$, and in the case that A is an atomic statement of the form $T = T'$, $\mathcal{I}_D(T) = \mathcal{I}_D(T')$.*

The changes that in the presence of '=' must be brought to the completeness proofs in Sections 3–5 are like those Henkin brings to his own proof. We shall not rehearse them here.

3 Sharpening the first result obtained by Henkin in his 1949 paper, we establish that any instantially consistent set of statements of L extends to a Henkin set (=Theorem 1). Proof of this calls for Lemma 3, where the term T constitutes, if you will, a *witness of* $\sim(\forall X)A$. In Henkin that term would be new to L; here, by contrast, it is one of the terms of L. Two other versions of Lemma 3 will eventually turn up.

Lemma 3 *If* $S \cup \{\sim(\forall X)A\}$ *is instantially consistent, then so is* $S \cup \{\sim(\forall X)A, \sim A(T/X)\}$ *for at least one term* T *of* L.

Proof: Suppose $S \cup \{\sim(\forall X)A, \sim A(T/X)\}$ instantially inconsistent for every term T of L. Then, by $\sim$I and $\sim$E, $S \cup \{\sim(\forall X)A\} \vdash_i A(T/X)$ for every such T, hence by $\forall$I$_i$ $S \cup \{\sim(\forall X)A\} \vdash_i (\forall X)A$, and hence by Lemma 2(f) $S \cup \{\sim(\forall X)A\}$ is instantially inconsistent.

This attended to, let S (also called S_0 for convenience's sake) be a set of statements of L that is instantially consistent; for each n from 1 on, let S_n be defined as follows, with A_n there the alphabetically nth statement of L and in *Case 3* T the alphabetically earliest term of L such that, owing to Lemma 3, $S_{n-1} \cup \{A_n, \sim B(T/X)\}$ is instantially consistent:

$$S_n = \begin{cases} S_{n-1} \cup \{\sim A_n\} \textit{ if } S_{n-1} \cup \{A_n\} \textit{ is instantially inconsistent } (\textit{Case } 1) \\ S_{n-1} \cup \{A_n\} \textit{ if } S_{n-1} \cup \{A_n\} \textit{ is instantially consistent and } A_n \textit{ is not a} \\ \qquad\qquad \textit{negated quantification } (\textit{Case } 2) \\ S_{n-1} \cup \{A_n, \sim B(T/X)\} \textit{ if } S_{n-1} \cup \{A_n\} \textit{ is instantially consistent and} \\ \qquad\qquad A_n \textit{ is a negated quantification } \sim(\forall X)B \\ \qquad\qquad (\textit{Case } 3); \end{cases}$$

and let S_∞ be defined as follows:

$$S_\infty = \bigcup_{n=0}^{\infty} S_n.$$

By the construction of S_n, Lemma 2(d), and Lemma 3, $S_n (n = 1,2\ldots)$ is instantially consistent if S_{n-1} is. But, by supposition, S_0 is instantially consistent. So, by mathematical induction on $n, S_n (n = 0,1,\ldots)$ is instantially consistent, and hence by Lemma 2(a) is consistent in the standard sense. But, if so, then S_∞ too is consistent in that sense. For suppose S_∞ were inconsistent in the standard sense. Then by Lemma 2(b) so would be some finite subset of S_∞,[16] and hence by *Thinning* so would be S_n for some n or other. But, by the construction of S_n and that of S_∞, either A belongs to S_∞ or $\sim A$ does, this for any statement A of L. So by Lemma 2(g) S_∞ is maximally consistent. Suppose next that, for some quantification $(\forall X)B$ of $L, S \vdash B(T/X)$ for every term T of L, but $S_\infty \nvdash (\forall X)B$. Then by Lemma 2(h) $\sim(\forall X)B$ belongs to S_∞. But, if so, then by the construction of S_∞ there exists an n such that $\sim(\forall X)B$ belongs to S_n, and hence by the construction of S_n there exists a term of T of L such that $\sim B(T/X)$ belongs to S_n and hence to S_∞. So by *Reiteration* $S_\infty \vdash \sim B(T/X)$ for at least one term T of L, and hence S_∞ is inconsistent in the standard sense, contrary to the result just obtained. So S_∞ is term-complete. So:

Theorem 1 *If S is instantially consistent, then S extends to (a set of statements of L that constitutes) a Henkin set.*

The parenthesis is of course for emphasis. The Henkin set to which Henkin's own S in [5] extended was made up of statements of a certain term extension L^+ of L. As announced in Section 1, the one to which our S extends is made up exclusively of statements of L.

Suppose now that a set S constitutes a Henkin set, and—adapting Henkin's construction of a model to suit our own needs—let D be {'t_1','t_2', . . . }, and let $\mathcal{I}_D$ be this D-interpretation of L:

$$\mathcal{I}_D(T) = T \textit{ for each term } T \textit{ of } L$$

and

$$\mathcal{I}_D(P) = \{\langle T_1, T_2, \ldots, T_n\rangle : S \vdash P(T_1, T_2, \ldots, T_n)\}$$
$$\textit{for each n-adic } (n = 1,2, \ldots) \textit{ predicate } P \textit{ of } L.$$

Owing to the maximal consistency of S,

$$S \vdash \sim A \textit{ if, and only if, } S \nvdash A$$

and

$$S \vdash A \supset B \textit{ if, and only if, } S \nvdash A \textit{ or } S \vdash B;$$

and, owing to the maximal consistency and the term-completeness of S,

$$S \vdash (\forall X)A \textit{ if, and only if, } S \vdash A(T/X) \textit{ for every term } T \textit{ of } L.$$

So each member A of S is true on $\mathcal{I}_D$, as an induction on the number of logical operators in A will show. But $\mathcal{I}_D$ is a Henkin D-interpretation. So:

Lemma 4
(a) *If S constitutes a Henkin set, then S has a Henkin model;*
(b) *If S is a subset of a Henkin set, then S has a Henkin model.*

So *Case One* of our first *Completeness Proof for First-Order L*:

Theorem 2 *If S is instantially consistent, then S has a Henkin model.*

Case Two of the proof is to the effect that if S is consistent in the standard sense *but not* instantially consistent, then S again has a model, *but not* a Henkin one. We postpone consideration of it until Section 5, however, and devote the balance of this section and the next to various results about Henkin sets, Henkin models, etc.

Suppose first that $S \vdash_i A$, and look back at our account of instantial provability. A transfinite induction on the nine rules there will establish that $S \cup \{\sim A\}$ cannot have a Henkin model. So:

Lemma 5 *If S has a Henkin model, then S is instantially consistent.*

So by Lemma 4(b):

Theorem 3 *If S extends to a Henkin set, then S is instantially consistent.*

Theorem 1 can therefore not be improved upon: being instantially consistent is a *necessary*, as well as *sufficient*, condition for extending to a Henkin set. Nor as a result can Theorem 2. So:

Theorem 4
(a) *S has a Henkin model if, and only if, S is instantially consistent*;
(b) *S has a Henkin model if, and only if, S is a subset of a Henkin set.*

Clause (a) is a generalization of *the Completeness and Soundness Theorem for Omega-Logic.*[17]

4 Theorem 1 also holds for S consistent in the standard sense *and* term-complete rather than instantially consistent. The resulting theorem, the converse of which we already know to fail, has two important consequences regarding instantial consistency and instantial provability.

Lemma 6 *Let X be foreign to A. Then*:
(a) $\vdash (\forall X)(A \supset B) \supset (A \supset (\forall X)B)$;
(b) *If $S \vdash (\forall X)(A \supset B)$, then $S \cup \{A\} \vdash (\forall X)B$.*

Proof: (a) Let T be an arbitrary term of L foreign to $\{(\forall X)(A \supset B), A\}$ and to $(\forall X)B$. Then $\{(\forall X)(A \supset B), A\} \vdash (\forall X)(A \supset B)$ by *Reiteration*, hence $\{(\forall X)(A \supset B), A\} \vdash A \supset B(T/X)$ by $\forall$E,[18] hence $\{(\forall X)(A \supset B), A\} \vdash B(T/X)$ by *Reiteration* and $\supset$E, hence $\{(\forall X)(A \supset B), A\} \vdash (\forall X)B$ by $\forall$I and the hypothesis on T, and hence (a) by $\supset$I. (b) Suppose $S \vdash (\forall X)(A \supset B)$. Then $S \cup \{A\} \vdash (\forall X)(A \supset B)$ by *Thinning*. But $S \cup \{A\} \vdash (\forall X)(A \supset B) \supset (A \supset (\forall X)B)$ by (a) and *Thinning*. Hence $S \cup \{A\} \vdash A \supset (\forall X)B$ by $\supset$E, and hence $S \cup \{A\} \vdash (\forall X)B$ by *Reiteration* and $\supset$E.

Lemma 7 *If S is term-complete, then so is $S \cup \{A\}$ for any statement A of L.*

Proof: Suppose S is term-complete; and, A being an arbitrary statement of L and $(\forall X)B$ an arbitrary quantification of L, suppose $S \cup \{A\} \vdash B(T/X)$ for every term T of L. Then $S \vdash A \supset B(T/X)$ for every such T by $\supset$I, hence $S \vdash (\forall X)(A \supset B)$ by the term-completeness of S, and hence $S \cup \{A\} \vdash (\forall X)B$ by Lemma 6(b), this when X is foreign to A. Otherwise, suppose Y a variable foreign to A. Then, clearly, $S \cup \{A(Y/X)\} \vdash B(T/X)$ for every T, hence $S \cup \{A(Y/X)\} \vdash (\forall X)B$, and hence $S \cup \{A\} \vdash (\forall X)B$. So $S \cup \{A\}$ is term-complete.

Lemma 8
(a) *If S is consistent in the standard sense and term-complete, then so is either $S \cup \{A\}$ or $S \cup \{\sim A\}$ for any statement A of L;*
(b) *If $S \cup \{\sim (\forall X)A\}$ is consistent in the standard sense and term-complete, then so is $S \cup \{\sim (\forall X)A, \sim A(T/X)\}$ for at least one term of T of L.*

Proof: (a) By Lemma 2(c) and Lemma 7. (b) Suppose $S \cup \{\sim (\forall X)A\}$ is term-complete. Then by Lemma 7 so is $S \cup \{\sim (\forall X)A, \sim A(T/X)\}$ for every term T of L. Suppose further that $S \cup \{\sim (\forall X)A, \sim A(T/X)\}$ is inconsistent in the standard sense for every term T of L. Then, by $\sim$I and $\sim$E, $S \cup \{\sim (\forall X)A\} \vdash$

$A(T/X)$ for every such T, hence $S \cup \{\sim(\forall X)A\} \vdash (\forall X)A$ by the term-completeness of $S \cup \{\sim(\forall X)A\}$, and hence $S \cup \{\sim(\forall X)A\}$ is inconsistent in the standard sense by Lemma 2(e).

Given Lemma 8, proof that S extends to a Henkin set when consistent in the standard sense and term-complete mimicks the proof of Theorem 1. You define $S_n (n = 0,1,\ldots)$ and S_∞ as before; presuming that $S_{n-1} (n = 1,2,\ldots)$ is consistent in the standards sense and term-complete, you next show—using Lemma 8 in lieu of Lemma 2(d) and Lemma 3—that S_n also is; and given that $S_n (n = 0,1,\ldots)$ is consistent in the standard sense, you then show in the same way as before that S_∞ constitutes a Henkin set. So:

Theorem 5 *If S is consistent in the standard sense and term-complete, then S extends to a Henkin set.*

The result is a generalization of another version of *the Strong Completeness Theorem for Omega-Logic.*

Hence these two results regarding instantial consistency and instantial provability:

Corollary 1 *Let S be term-complete. Then*:
(a) *S is consistent in the standard sense if, and only if, S is instantially consistent*;
(b) $S \vdash A$ *if, and only if,* $S \vdash_i A$.

Proof: (a) Suppose S is consistent in the standard sense. Then by Theorem 5 and Theorem 4(b) S has a Henkin model, and hence by Lemma 5 S is instantially consistent. Hence (a) by Lemma 2(a). (b) Suppose $S \vdash_i A$, in which case $S \cup \{\sim A\} \vdash_i A$ by *Thinning*. Then by Lemma 2(f) $S \cup \{\sim A\}$ is instantially inconsistent, hence by (a) $S \cup \{\sim A\}$ is inconsistent in the standard sense, and hence $S \vdash A$ by $\sim$I and $\sim$E. So (b) by Lemma 1.

Corollary 1 permits this alternative proof of Theorem 3. Suppose S were instantially inconsistent and yet did extend to a Henkin set S_∞. By *Thinning* S_∞ would be instantially inconsistent, hence by clause (a) of the corollary S_∞ would be inconsistent in the standard sense, and hence by Lemma 4(a) S_∞ would not constitute a Henkin set.

5 We exploit results in Leblanc [8] and [10] to conclude our *First Completeness Proof for First-Order L*. Throughout, we take a set S of statements of L to be *infinitely extendible* if, and only if, denumerably many terms of L are foreign to S.[19]

Lemma 9 *If* $S \cup \{\sim(\forall X)A\}$ *is consistent in the standard sense, then so is* $S \cup \{\sim(\forall X)A, \sim A(T/X)\}$ *for any term T of L foreign to* $S \cup \{\sim(\forall X)A\}$.

Proof: Like that of Lemma 3, but using $\forall$I in place of $\forall \mathrm{I}_i$.

Now suppose that in the preamble to the construction of the set S_n in Section 3, T is taken to be the alphabetically earliest term of L foreign to $S_{n-1} \cup \{A_n\}$.[20] Together with Lemma 2(c), Lemma 9 then delivers this second variant of Theorem 1:

Theorem 6 *If S is consistent in the standard sense and infinitely extendible, then S extends to a Henkin set.*[21]

And, together with Lemma 4(b), Theorem 6 delivers this variant of Theorem 2:

Theorem 7 *If S is consistent in the standard sense and infinitely extendible, then S has a Henkin model.*[22]

The model in question is of course that on p. 219, with {'t_1','t_2', . . . } serving as D and $\mathfrak{I}_D(T)$ taken to be T itself, this for each term T of L.[23]

Next, understand by

(i) *the double rewrite of a statement A of L* the result of simultaneously substituting 't_2' for 't_1', 't_4' for 't_2', . . . everywhere in A,

(ii) *the double rewrite of a set S of statements of L* the set S itself when S is empty, otherwise the set consisting of the double rewrites of the various members of S, and

(iii) *the double rewrite of a D-interpretation* $\mathfrak{I}_D$ the D-interpretation $\mathfrak{I}^2_D$, where

$$\mathfrak{I}^2_D(P) = \mathfrak{I}_D(P) \textit{ for each predicate } P \textit{ of } L$$

as before but

$$\mathfrak{I}^2_D(\text{'}t_i\text{'}) = \mathfrak{I}_D(\text{'}t_{2i}\text{'}) \textit{ for each } i \textit{ from } 1 \textit{ on.}$$

It is easily verified that

Lemma 10

(a) *If S is consistent in the standard sense, then so is the double rewrite of S*;

(b) *If the double rewrite of S is true on* $\mathfrak{I}_D$, *then S is true on the double rewrite of* $\mathfrak{I}_D$.

This done, suppose S consistent in the standard sense but not instantially consistent. S cannot be infinitely extendible, for if it were, then by Theorem 6 it would have a Henkin model, which by Theorem 4(a) it cannot have. However, the double rewrite of S is infinitely extendible, and by Lemma 10(a) is consistent in the standard sense. So by Theorem 7 that double rewrite has a Henkin model, the one on p. 218, and hence by Lemma 10(b) S itself has as a model the double rewrite of that model. Expectedly, the second model is not a Henkin one, there existing no term T of L such that $\mathfrak{I}^2_D(T) =$ 't_1', none such that $\mathfrak{I}^2_D(T) =$ 't_3', ...

So *Case Two* of our first *Completeness Proof for First-Order L*:

Theorem 8 *If S is consistent in the standard sense but is not instantially consistent, then S has a model—though not a Henkin one.*

So $S \cup \{\sim A\}$ is inconsistent in the standard sense if $S \cup \{\sim A\}$ has no model. So, by ~I and ~E, $S \vdash A$ if A is true in every model of S.

Importantly, {'t_1','t_2', . . . }—the domain in both cases of our first *Completeness Proof*—is denumerable. But, any set of statements of L that is consistent (in the standard sense) has a denumerable model. So, owing to *the Strong Soundness Theorem for First-Order L*, any set of statements of L that has a model is consistent (in the standard sense). So any set of statements of L that has a model has a denumerable one (= *Skolem's Generalization of Löwenheim's Theorem*). These are of course familiar results, but Henkin may have been the first to obtain them as corollaries of a *Strong Completeness Theorem*.

Turning to our second proof, take *the substatements in L of a statement A of L* to be: (i) A itself, (ii) B when A is a negation $\sim B$, (iii) B and C when A is a conditional $B \supset C$, (iv) $B(T/X)$ for any term T of L when A is a quantification $(\forall X)B$, *plus* (v) any substatement in L of any substatement of A in L; and take *the substatements in L of a set S of statements of L* to be the substatements in L of the various members of S.[24] Also, given a statement A of L, a set Σ consisting of all the atomic substatements of S in L and possibly other atomic statements of L as well, and a function $\mathfrak{A}$ from Σ to $\{\mathrm{T},\mathrm{F}\}$, said function called as usual a *truth-value assignment to the members of* Σ, count A *true on* $\mathfrak{A}$ if, and only if: (i) $\mathfrak{A}(A) = \mathrm{T}$ when A is atomic, (ii) B is not true on $\mathfrak{A}$ when A is a negation $\sim B$, (iii) B is not true on $\mathfrak{A}$ or C is when A is a conditional $B \supset C$, and (iv) $B(T/X)$ is true on $\mathfrak{A}$ for every term T of L when A is a quantification $(\forall X)B$.[25] Last, given a nonempty set S of statements of L and a truth-value assignment $\mathfrak{A}$ to the atomic substatements in L of the various members of S, count S *true on* $\mathfrak{A}$ if, and only if, every member of S is true on $\mathfrak{A}$.

As the reader may wish to verify, a Henkin set S of statements of L is true on this truth-value assignment $\mathfrak{A}$ to the atomic substatements of S in L (and on none other):

$$\mathfrak{A}(A) = \begin{cases} \mathrm{T} \text{ } \textit{if A belongs to S} \\ \mathrm{F} \text{ } \textit{otherwise.} \end{cases}$$

So *Case One* of our second *Completeness Proof for First-Order L*:

Theorem 9 *If S is nonempty and instantially consistent, then S is true on at least one truth-value assignment to the atomic substatements of S in L.*

But, clearly, no set of statements of L that is instantially inconsistent can be true on a truth-value assignment to its atomic substatements in L. So this counterpart of Theorem 4:

Theorem 10 *Let S be nonempty. Then*:
(a) *S is true on at least one truth-value assignment to the atomic substatements of S in L if, and only if, S is instantially consistent*;
(b) *S is true on at least one truth-value assignment to the atomic substatements of S in L if, and only if, S is a subset of a Henkin set.*

Owing to Theorem 10(a), the truth-value of a nonempty set S of statements of L—a function of the truth-values of the atomic substatements of S in L—is F always when S is instantially *inconsistent*; otherwise it is either T or F.[26]

Case Two of our second proof is just a bit more work. Let L^+ be the result of adding denumerably many new terms to the vocabulary of L, and extend to L^+ every definition in Section 2, that in this section of an infinitely extendible set of statements of L, and that in this section again of the substatements in L of a statement of L and of a set of statements of L. Clearly, all sets of statements of L are infinitely extendible *qua* sets of statements of L^+. So, owing to Theorem 6:

Theorem 11 *If a nonempty set S of statements of L is consistent in the standard sense, then S extends to a Henkin set of statements of L^+.*

So *Case Two* of our second *Completeness Proof for First-Order L*:

Theorem 12 *If a nonempty set S of statements of L is consistent in the standard but not the instantial sense, then there exists a truth-value assignment to the atomic substatements of S in L^+—though not to those in L only—on which S is true.*

In view of Theorem 12, the truth-value of a nonempty set S of statements of L is a function of the truth-values of the atomic substatements of S *in* L^+, that truth-value F when the set is *inconsistent in the standard sense*, otherwise either T or F.

One brand of truth-value semantics, that of Dunn and Belnap in [2], understands by a *term-extension* L^+ *of* L any first-order language that is exactly like L except for having countably many new terms; and it takes a set S of statements of L to be *truth-value verifiable* by *fiat* when S is empty, otherwise if S is true, for some term-extension L^+ of L, on at least one truth-value assignment to the atomic substatements of S in that L^+. The preceding results guarantee that *S is truth-value verifiable if, and only if, S is consistent in the standard sense.* With a set S of statements of L said to *logically imply* a statement A of L if, and only if, $S \cup \{\sim A\}$ is not truth-value verifiable, one may then conclude that *S logically implies A if, and only if, $S \vdash A$.* And, with a statement A of L said to be *logically true* if, and only if, $\varnothing$ logically implies A, one may further conclude that *A is logically true if, and only if, $\varnothing \vdash A$.* Another brand of truth-value semantics, that of Leblanc in [9], dispenses with term-extensions, rather taking S to be truth-value verifiable—when nonempty—if, and only if, the double rewrite S^2 of S is true on some truth-value assignment to the atomic substatements of S^2 in L. There too S logically implies A if, and only if, $S \vdash A$, and A is logically true if, and only if, $\varnothing \vdash A$.[27]

6 The rule $\forall I_i$ is our extension to terms *in general* of a rule in certain systems of arithmetic according to which a quantification $(\forall X)A$ is provable from the denumerably many results of replacing X everywhere in A by a *numeral.* The rule, proposed by Tarski in a lecture of 1927 and published seven years later in Tarski [13], was called by him—we noted in Section 1—*the rule of infinite induction.* Also known for a while as *Carnap's rule*, it is now commonly called *the omega-rule of omega-logic.*[28] The strong completeness of omega-logic follows from Theorem 3 above; and Theorem 3, conversely, can be had from *the Omitting Types Theorem* in [1] by a straightforward generalization of *the Omega-Completeness Theorem* there (Theorem 2.2.9 and Proposition 2.2.13, respectively). And it follows from results in Section 3 and results of Henkin's in [6] (Theorem 7, p. 194) that the notion of instantial consistency is equivalent to Henkin's notion of *strong* Γ*-consistency* for the case where Γ consists of all the terms of L. "Strongly Γ-consistent" is of course the other way we mentioned in Section 1 of identifying the sets of statements of L that have a Henkin model.

The notion of term-consistency is an extension to terms *in general* of the notion of omega-consistency introduced by Gödel in [4]; and the notion of term-completeness is a similar extension of the notion of omega-completeness introduced by Tarski in [13]. As for the notion of instantial consistency, it is our extension to terms *in general* of the notion of consistency in omega-logic. Sets of statements of L that are instantially consistent are term-consistent, of course, the way sets of statements of omega-logic that are consistent in omega-logic are omega-consistent. But, as reported in Section 1, sets of statements of L that are term-consistent are not always instantially consistent, and hence sets of statements of omega-logic that are omega-consistent are not always consistent in omega-logic. Henkin's own set in Henkin [6] is a case in point, as are these three sets which we mentioned in Section 1:

$$S_1 = \{(\exists x)\mathrm{P}_1(x)\} \cup \{\mathrm{P}_1(T) \supset {\sim}\mathrm{P}_2(T')\text{: } \textit{for any term } T \textit{ and any term } T' \textit{ of } L\}$$
$$\cup \{\mathrm{P}_1(T) \supset (\exists y)\mathrm{P}_2(y)\text{: } \textit{for any term } T \textit{ of } L\}$$

$$S_2 = \{{\sim}(\forall x)(\forall y)\mathrm{P}(x,y)\} \cup \{\mathrm{P}(T, T')\text{: } \textit{for any term } T \textit{ and any term } T' \textit{ of } L\}$$

and

$$S_3 = \{(\exists x)\mathrm{P}(x)\} \cup \{\mathrm{P}(T) \supset (\exists y)(y \neq \mathrm{t}_1 \;\&\; {\sim}\mathrm{P}(y))\text{: } \textit{for any term } T \textit{ of } L\}$$
$$\cup \{\mathrm{P}(T) \supset (T' \neq \mathrm{t}_1 \supset \mathrm{P}(T'))\text{: } \textit{for any term } T \textit{ and any term } T' \textit{ of } L\}.$$[29]

That S_2 is instantially inconsistent is immediate, given *Reiteration* and $\forall\mathrm{I}_\mathrm{i}$. Proof that the other two sets are instantially inconsistent calls for this lemma, proof of which is familiar:

Lemma 11 *Let X be foreign to B. Then $S \vdash_\mathrm{i} (\exists X)A \supset B$ if $S \vdash_\mathrm{i} (\forall X)(A \supset B)$.*

Since obvious steps deliver

$$S_1 \vdash_\mathrm{i} (\forall x)(\mathrm{P}_1(x) \supset {\sim}\mathrm{P}_2(T'))$$

for every term T' of L, we readily have

$$S_1 \vdash_\mathrm{i} (\forall y) \sim \mathrm{P}_2(y);$$

and, since similar steps deliver

$$S_1 \vdash_\mathrm{i} (\exists y)\mathrm{P}_2(y),$$

we have by the definition of '$\exists$'

$$S_1 \vdash_\mathrm{i} {\sim}(\forall y) \sim \mathrm{P}_2(y).$$

Lastly, similar steps and the definition of '&' will deliver both

$$S_3 \vdash_\mathrm{i} (\exists y)(y \neq \mathrm{t}_1 \;\&\; {\sim}\mathrm{P}(y))$$

and

$$S_3 \vdash_\mathrm{i} {\sim}(\exists y)(y \neq \mathrm{t}_1 \;\&\; {\sim}\mathrm{P}(y)).$$

Our proof that each of S_1, S_2, and S_3 is term-consistent uses the following lemma, which provides a sufficient (though *not* necessary) condition for a set of statements of L to be term-consistent.

Lemma 12 *If there exists a denumerable set Σ of terms of L such that, no matter the member T of Σ, every T-variant of a given model of S also constitutes a model of S, then S is term-consistent.*

Proof: Suppose a certain D-interpretation $\mathcal{I}_D$ of L constitutes a model of S; suppose there exists a denumerable set Σ of terms of L such that, no matter the term T of Σ, every T-variant $\mathcal{I}_D^T$ of $\mathcal{I}_D$ also constitutes a model of S; suppose there exists a quantification $(\forall X)A$ of L such that $S \vdash \sim(\forall X)A$; and let T' be the alphabetically first member of Σ that is foreign to $(\forall X)A$. Then, by *the Strong Soundness Theorem for L*, $(\forall X)A$ is not true on $\mathcal{I}_D$, and hence there exists at least one T'-variant $\mathcal{I}_D^{T'}$ of $\mathcal{I}_D$ on which $A(T/X)$ is not true. But, by the second supposition on S, S is true on $\mathcal{I}_D^{T'}$. So there exists a model of S in which $A(T'/X)$ is not true. So, by *the Strong Soundness Theorem for L*, $S \nvdash A(T/X)$ for at least one term T of L. So S is term-consistent.

So, tackling S_1, let D be $\{1,2,3\}$, and let $\mathcal{I}_D$ be any D-interpretation of L such that $\mathcal{I}_D(T) = 3$ for every term T of L, $\mathcal{I}_D(\text{'P}_1\text{'}) = \{1\}$, and $\mathcal{I}_D(\text{'P}_2\text{'}) = \{2\}$. '$(\exists x)\mathrm{P}_1(x)$' and '$(\exists y)\mathrm{P}_2(y)$' are clearly true on $\mathcal{I}_D$; and, since $\mathrm{P}_1(T)$ is not true on $\mathcal{I}_D$ for any term T of L, $\mathrm{P}_1(T) \supset \sim\mathrm{P}_2(T')$ is true on $\mathcal{I}_D$ for every term T and every term T' of L, and $\mathrm{P}_1(T) \supset (\exists y)\mathrm{P}_2(y)$ is true on $\mathcal{I}_D$ for every term T of L. So $\mathcal{I}_D$ constitutes a model of S_1. Consider now any term-variant $\mathcal{I}_D^{T^*}$ of $\mathcal{I}_D$ that assigns 2 to a certain term T^* of L. Since neither of 2 and 3 belongs to $\mathcal{I}_D^{T^*}(\text{'P}_1\text{'})$, $\mathrm{P}_1(T^*) \supset \sim\mathrm{P}_2(T')$, $\mathrm{P}_1(T) \supset \sim\mathrm{P}_2(T^*)$, and $\mathrm{P}_1(T^*) \supset (\exists y)\mathrm{P}_2(y)$ are all true on $\mathcal{I}_D^{T^*}$. Consider then any term-variant $\mathcal{I}_D^{T^*}$ of $\mathcal{I}_D$ that assigns 1 to a certain term T^* of L. Since neither of 1 and 3 belongs to $\mathcal{I}_D^{T^*}(\text{'P}_2\text{'})$, $\mathrm{P}_1(T^*) \supset \sim\mathrm{P}_2(T')$ and $\mathrm{P}_1(T) \supset \sim\mathrm{P}_2(T^*)$ are true on $\mathcal{I}_D^{T^*}$; and, since '$(\exists y)\mathrm{P}_2(y)$' is true on $\mathcal{I}_D^{T^*}$, $\mathrm{P}_1(T^*) \supset (\exists y)\mathrm{P}_2(y)$ is true on $\mathcal{I}_D^{T^*}$. So all the term-variants of $\mathcal{I}_D$ constitute models of S_1. So by Lemma 12 S_1 is term-consistent.[30]

Tackling S_2, let D be $\{1,2,3\}$ again, and let $\mathcal{I}_D$ be any D-interpretation of L such that $\mathcal{I}_D(T) = 3$ and $\mathcal{I}_D(\text{'P'}) = \{\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 3,1\rangle,\langle 3,2\rangle,\langle 1,3\rangle,\langle 2,3\rangle\}$. $\mathcal{I}_D$ clearly constitutes a model of S_2. Consider now any term-variant $\mathcal{I}_D^{T^*}$ of $\mathcal{I}_D$ that assigns either 2 or 1 to a certain term T^* of L. Since $\langle 2,2\rangle$ and $\langle 1,1\rangle$ belong to $\mathcal{I}_D^{T^*}(\text{'P'})$, $\mathrm{P}(T^*, T^*)$ is true on $\mathcal{I}_D^{T^*}$; since $\langle 3,2\rangle$ and $\langle 3,1\rangle$ also belong to $\mathcal{I}_D^{T^*}$, $\mathrm{P}(T, T^*)$ is true on $\mathcal{I}_D^{T^*}$; and, since $\langle 2,3\rangle$ and $\langle 1,3\rangle$ also belong to $\mathcal{I}_D^{T^*}$, $\mathrm{P}(T^*, T')$ is true on $\mathcal{I}_D^{T^*}$. So all the term-variants of $\mathcal{I}_D$ constitute models of S_2. So by Lemma 12 S_2 is term-consistent.

And, tackling S_3, let D be $\{1,2\}$ this time, and let $\mathcal{I}_D$ be any D-interpretation of L such that $\mathcal{I}_D(T) = 2$ for every term T of L and $\mathcal{I}_D(\text{'P'}) = \{1\}$. $\mathcal{I}_D$ clearly constitutes a model of S_3. Consider now any term-variant $\mathcal{I}_D^{T^*}$ of $\mathcal{I}_D$ that assigns 1 to a certain term T^* of L other than 't_1', a D-interpretation of L on which $\mathrm{P}(T^*)$ is true. Then '$(\exists y)(y \neq \mathrm{t}_1 \ \& \sim\mathrm{P}(y))$' is true on $\mathcal{I}_D^{T^*}$, since true on $\mathcal{I}_D$; and $T' \neq \mathrm{t}_1 \supset \mathrm{P}(T')$ is true on $\mathcal{I}_D^{T^*}$, since $\mathcal{I}_D^{T^*}(T')$ belongs to $\mathcal{I}_D^{T^*}(\text{'}P\text{'})$ when T' is the same as T^*, otherwise $T' \neq \mathrm{t}_1$ is false on $\mathcal{I}_D^{T^*}$.[31] So there exists a denumerable set Σ of terms of L, namely the set of all the terms of L other than 't_1', such that, no matter the member T^* of Σ, every T^*-variant of $\mathcal{I}_D$ constitutes a model of S_3. So by Lemma 12 S_3 is term-consistent. So all three of S_1, S_2, and S_3 are instantially inconsistent but nonetheless term-consistent.

Though sufficient for a set of statements to be term-consistent, the condition in Lemma 12—we parenthetically remarked—is *not* necessary. For proof, let L have as its predicates the denumerably many monadic predicates 'P_1', 'P_2', . . . ; let D be $\{1,2,\ldots\}$; let $\mathcal{I}_D$ be the D-interpretation of L such that $\mathcal{I}_D(\text{'t}_n\text{'}) = n$ and $\mathcal{I}_D(\text{'P}_n\text{'}) = \{n\}$ for each n from 1 on; and let S consist of the statements of L that are true on $\mathfrak{I}_D$. S is easily verified to be maximally consistent and term-complete, and hence term-consistent. Now let D be this time *any* set of cardinality no less than 2, and let $\mathfrak{I}_D$ be this time *any* D-interpretation of L that constitutes a model of S. It suffices to show that, for each term T of L, there exists a T-variant of $\mathfrak{I}_D$ that does not constitute a model of S. With an eye to that, let $\mathfrak{I}_D^{t_n}$ be, for each n from 1 on, the 't_n'-variant of $\mathfrak{I}_D$ such that $\mathfrak{I}_D^{t_n}$('t_n') $=$ $\mathfrak{I}_D$('t_m') for some m other than n.[32] By the construction of S, '$P_m(t_m)$' belongs to S; hence, $\mathfrak{I}_D$('t_m') belongs to $\mathfrak{I}_D$('P_m'); hence, $\mathfrak{I}_D^{t_n}$('t_n') belongs to $\mathfrak{I}_D$('P_m'); hence, $\mathfrak{I}_D^{t_n}$('t_n') belongs to $\mathfrak{I}_D^{t_n}$('P_m'); hence, '$P_m(t_n)$' is true on $\mathfrak{I}_D^{t_n}$; and, hence '$\sim P_m(t_n)$' is not true on $\mathfrak{I}_D^{t_n}$. But, by the construction of S again, '$\sim P_m(t_n)$' belongs to S. So $\mathfrak{I}_D^{t_n}$ does not constitute a model of S.

Proof of our closing result uses this lemma, proof of which we leave to the reader:

Lemma 13

(a) *Let X be foreign to B. If $S \vdash (\exists X)A$ and $S \vdash (\exists X) \sim A$, then $S \vdash \sim(\forall X) \sim ((A \;\&\; \sim B) \vee (\sim A \;\&\; B))$;*

(b) *If $S \vdash A \equiv B$, then $S \vdash \sim((A \;\&\; \sim B) \vee (\sim A \;\&\; B))$.*

This attended to, consider a first-order language L^1 that has a *single* predicate, say, 'P', that predicate a *monadic* one; and, given a domain D and a D-interpretation $\mathfrak{I}_D^1$ of L^1, let R be this binary relation on D:

$$\{\langle d, d'\rangle : d \in \mathfrak{I}_D^1(\text{'P'}) \textit{ if, and only if, } d' \in \mathfrak{I}_D^1(\text{'P'})\}.$$

R is easily seen to constitute an equivalence relation on D. So let $[d]$ be for each member d of D the equivalence set of d with respect to R; let $[D]$ be the set consisting of the resulting equivalence sets; and let $\mathfrak{I}_{[D]}^1$ be the $[D]$-interpretation of L^1 such that

$$\mathfrak{I}_{[D]}^1(T) = [\mathfrak{I}_D^1(T)] \textit{ for each term } T \textit{ of } L^1$$

and

$$\mathfrak{I}_{[D]}^1(\text{'P'}) = \{\mathfrak{I}_D^1(\text{'P'})\}.$$

It is easily verified that $[D]$ consists of just D when $\mathfrak{I}_D^1$('P') is either D or $\varnothing$, otherwise of both $\mathfrak{I}_D^1$('P') and its complement, and hence that $[D]$ is either of cardinality 1 or of cardinality 2. And it is easily verified that a statement of L^1, and, hence, a set of statements of L^1, is true on $\mathfrak{I}_D$ if, and only if, true on $\mathfrak{I}_{[D]}^1$.

Consider now a set S of statements of L^1 that has no Henkin model, and suppose *first* that either $S \nvdash (\exists x)P(x)$ or $S \nvdash (\exists x) \sim P(x)$. Then either $S \cup \{(\forall x) \sim P(x)\}$ or $S \cup \{(\forall x)P(x)\}$ is consistent in the standard sense. So, by the counterpart for L^1 of *the Strong Completeness Theorem for L*, there exists, for some domain D or other, a D-interpretation $\mathfrak{I}_D^1$ of L^1, and hence a $[D]$-interpretation $\mathfrak{I}_{[D]}^1$ of L^1, on which S is true. But under the present circumstances $[D]$ has to be of cardinality 1, and hence $\mathfrak{I}_{[D]}^1$ has to constitute a Henkin $[D]$-interpretation of L^1. So, S has a Henkin model, something ruled out here. So, $S \vdash (\exists x)P(x)$ and $S \vdash (\exists x) \sim P(x)$. So, by Lemma 13(a),

$$S \vdash \sim(\forall x) \sim ((P(x) \;\&\; \sim P(T')) \vee (\sim P(x) \;\&\; P(T')))$$

for any term T' of L^1. Suppose *next* that there exist a term T of L^1 and a term T' of L^1 other than T such that $S \nvdash \mathrm{P}(T) \equiv \mathrm{P}(T')$. Then there exist a term T of L^1 and a term T' of L^1 other than T such that $S^1 \cup \{\sim(\mathrm{P}(T) \equiv \mathrm{P}(T'))\}$ is consistent in the standard sense. So, by the counterpart for L^1 of *the Strong Completeness Theorem for L*, there exists, for some domain D or other, a D-interpretation $\mathcal{I}_D^1$ of L^1, and hence a $[D]$-interpretation $\mathcal{I}_{[D]}^1$ of L^1, on which S is true. But under the present circumstances $[D]$ has to be of cardinality 2 and $\mathcal{I}_{[D]}^1$ has to pair each one of the two members of $[D]$ with a term of L^1. Under the present circumstances, therefore, $\mathcal{I}_{[D]}^1$ has to constitute a Henkin $[D]$-interpretation of L^1. So S has a Henkin model, something ruled out here. So $S \vdash \mathrm{P}(T) \equiv \mathrm{P}(T')$ for any term T of L^1 and any term T' of L^1 other than T. So, by Lemma 13(b),

$$S \vdash \sim((\mathrm{P}(T) \ \&\ \sim\mathrm{P}(T')) \vee (\sim\mathrm{P}(T) \ \&\ \mathrm{P}(T')))$$

for any term T of L^1 and any term T' of L^1 other than T. So S is term-inconsistent.

So:

Theorem 13. *Let L^1 be a first-order language that has a single predicate, that predicate a monadic one, and let S be a set of statements of L^1. If S is term-consistent, then S has a Henkin model.*

Consider then a set S of statements of L that features a single predicate, that predicate the monadic predicate 'P' of L^1; and suppose S is term-consistent. Owing to Theorem 13, there exists for some domain D or other a Henkin D-interpretation $\mathcal{I}_D^1$ of L^1 on which—*in its capacity as a set of statements of L^1*—S is true. But $\mathcal{I}_D^1$ readily extends to a Henkin D-interpretation of L *as a whole* on which S is true as well: pair each predicate of L other than 'P' with $\varnothing$. So by Lemma 5 S is instantially consistent. So, given Lemma 2(i):

Corollary 2. *Let S be a set of statements of L that features a single predicate, that predicate a monadic one. Then S is term-consistent if, and only if, S is instantially consistent.*

So, whereas the set

$$\{\sim(\forall x)(\forall y)\mathrm{P}(x,y)\} \cup \{\mathrm{P}(T,T') : \textit{for any term } T \textit{ and any term } T' \textit{ of } L\}$$

is term-consistent *though* instantially inconsistent, its monadic counterpart

$$\{\sim(\forall x)\mathrm{P}(x)\} \cup \{\mathrm{P}(T) : \textit{for any term } T \textit{ of } L\}$$

is term-inconsistent *because* instantially inconsistent.

In summary, then, for a set S of first-order statements to have a *Henkin* model, it is (i) sufficient *and* necessary that S be instantially consistent (Theorem 2), (ii) sufficient, *but not* necessary, that S be consistent in the standard sense and *either* term-complete (Theorem 5) *or* infinitely extendible (Theorem 6), and (iii) necessary, *but not* sufficient, that S be term-consistent (Lemma 2(i)), unless of course S features a single predicate and that predicate is a monadic one (Corollary 2).[33]

NOTES

1. More exactly, *the method of (individual) constants.* But we prefer here to talk of terms.

2. Because of the role they eventually play, these new terms have been called *witness terms.*

3. The sets we call here *term-complete* and those we call below *term-consistent* are called in [14] and some other places *omega-complete* and *omega-consistent*, respectively. The present appellations seem more appropriate, as the omega terminology (due to Gödel in [4] and Tarski in [13]) has generally been used in connection with a special kind of terms, *numerals.*

4. The notion of a *double rewrite* used in the first of them was suggested by Hintikka to Leblanc at the time the latter was writing [7].

5. Truth-value semantics, it so happens, was anticipated by Henkin in his 1949 paper, a fact acknowledged by Dunn and Belnap in [2].

6. As the omega-rule of omega-logic has been called *the rule of infinite induction*, our generalization of it could be called *the rule of infinite instantial induction.* However, for a reason given in Section 2 we shall name it $\forall I_i$.

7. The statements of L are of course *closed* ones: indeed, *open statements* will play no special role here, nor will the distinction between *bound variables* and *free* ones. And, due to (iv), identical quantifiers cannot overlap in a statement of L. So, when both $(\forall X)A$ and A are statements, as happens in *A5* further down in the text, X is sure to be foreign to A, $(\forall X)A$ is sure to be a so-called *vacuous quantification*, and—for any term T of L—$A(T/X)$ is sure to be A.

8. $\forall I_i$ ensures, by the way, that if A is an axiom of L, then $(\forall X)(A(X/T))$ is instantially provable from any set of statements of L. So clause (ii) in the account of instantial provability could be weakened to read: "(ii) an axiom of L of one of the six forms A1–A6".

9. The two accounts could be refined some. In the first,

   $$\textit{If } S \vdash A, \textit{ then } S \cup \{B\} \vdash A \textit{ for any statement } B \textit{ of } L$$

   would obviously do as *Thinning*; and, given either version of *Thinning*,

   $$\{A\} \vdash A$$

   would obviously do as *Reiteration*. And, given $\forall I_i$, $\forall I$ could be dropped in the second account. However, in the absence of $\forall I$, Lemma 1 would no longer hold *by definition alone*, and a substitute proof that it does hold—though relatively straightforward—would nonetheless be too lengthy for inclusion here. So we forgo these refinements.

10. The notion defined here is that of *term-consistency as regards* '$\forall$'. S would be said to be *term-consistent as regards* '$\exists$' if there exists no quantification $(\exists X)A$ of L such that $S \vdash \sim A(T/X)$ for every term T of L and yet $S \vdash (\exists X)A$, *term-inconsistent as regards* '$\exists$' otherwise. The two notions are equivalent.

11. The notion defined here is that of *term-completeness as regards* '$\forall$'. S would be said to be *term-complete as regards* '$\exists$' if, and only if, for every quantification $(\exists X)A$ of L, $S \vdash A(T/X)$ for at least one term T of L if $S \vdash (\exists X)A$. The two notions are

equivalent when S is maximally consistent, but not necessarily so otherwise. For example, the set $\{A : \vdash A\}$ is term-complete as regards '$\forall$', but not as regards '$\exists$': '$(\exists y)((\exists x)P(x) \supset P(y))$' is provable in the standard sense from $\{A : \vdash A\}$, but $(\exists x)P(x) \supset P(T)$ is not for any term T of L. What the rule $\forall I_i$ does is to declare the set $\{A(T/X)$: *for any term* T *of* $L\}$ term-complete as regards '$\forall$'.

12. Clause (iv) would be course not do if L had only finitely many terms. We could require of T that it be *the alphabetically earliest term of* L *foreign to* A, but the present course often proves more convenient.

13. For this to be possible, D must of course be countable.

14. The resulting interpretation of $(\forall X)A$ is of course the *substitutional* one, that in clause (iv) above being by contrast the *objectual* one.

15. =I is obviously more of an axiom than a rule, but the labels '=' and '=E' do prove handy.

16. Only in this step and like ones taken in variants of this proof in Sections 4 and 5 is Lemma 2(b) appealed to.

17. Disconcertingly, the result is referred to on p. 81 of [1] as just *the Completeness Theorem for Omega-Logic.*

18. In view of Note 7 the condition on X here is equivalent to A being a statement of L and $A(T/X)$ is A itself. A like remark, but with B in place of A, will apply at two further points in the text.

19. The appellation is Robert K. Meyer's.

20. The construction of $S_1, S_2, \ldots$ may entail adding to S denumerably many statements of L of the form $\sim(\forall X)B$; and, when S is not instantially consistent, Lemma 9 requires that in each case the witness term T be foreign in effect to S. So, denumerably many terms of L—or on p. 224 of the term-extension L^+ of L—that are foreign to S must be on hand. Henkin requires instead that there be on hand as many new terms as there are primitive signs of L. But there are of course as many of them as there are statements of L. Hence the phrasing of the matter on p. 1.

21. The converse of Theorem 6 fails, of course: though not infinitely extendible, the set $\{P(T)$: *for any term* T *of* $L\}$ nonetheless extends to a Henkin set.

22. Hence any *finite* set of statements of L that is consistent in the standard sense has a Henkin model, and so does *any* set of statements of L that is consistent in the standard sense and features no term. Carnap would have described the latter set as one of *purely general statements* of L.

23. Lemma 9, by the way, has this other corollary, which, thanks to Theorem 5, makes for an alternative proof of Theorem 6. Suppose that S is consistent in the standard sense and infinitely extendible; suppose that, for some quantification $(\forall X)A$ of L, $S \vdash A(T/X)$ for every term T of L; suppose that $S \nvdash (\forall X)A$ nonetheless; and let T^* be the alphabetically earliest term of L that is foreign to $S \cup \{\sim(\forall X)A\}$. Then by *Thinning* $S \cup \{\sim(\forall X)A, \sim A(T^*/X)\} \vdash A(T^*/X)$, and hence by Lemma 2(e) $S \cup \{\sim(\forall X)A, \sim A(T^*/X)\}$ is inconsistent in the standard sense. But, since $S \nvdash (\forall X)A$, $S \cup \{\sim(\forall X)A\}$ is consistent in the standard sense, for otherwise $S \vdash (\forall X)A$ by ~I and ~E. Hence by Lemma 9 $S \cup \{\sim(\forall X)A, \sim A(T^*/X)\}$ is consistent in the standard sense. Hence the third supposition is untenable. Hence S, *if consistent in the standard sense and infinitely extendible, is term-complete*, and, hence, thanks to Theorem 5, extends to a Henkin set.

24. The point of the qualification 'in L' will appear shortly. The appellation 'substatement' is an adaptation to this context of the more customary 'subformula'.

25. The interpretation of $(\forall X)B$ here is of course the *substitutional* one.

26. And because of Theorem 10(b) Henkin sets have been called *truth sets*, an appellation which may have originated with Quine in [12]. See [10] for more on truth sets.

27. The truth-value semantics in [2], used in [10]–[11] also, is thus in line with our first *Completeness Proof* here, that in [9] in line with the second.

28. See pp. 212–214 of Feferman [3] for more on the history of Tarski's rule, and pp. 76–93 of [1] for more on omega-logic and related matters.

29. As the reader will have noticed, 'P' throughout the paper is a syntactic variable ranging over the predicates of L. 'P', on the other hand, is to be thought of as an actual predicate of L (and of L^1 on pp. 227–228): a dyadic predicate when appearing in the set S_2, a monadic one on all other occasions. 'P_1' and 'P_2' are likewise to be thought of as actual monadic predicates of L, as are 'P_1','P_2',... on pp. 226–227.

30. See [14] for another proof that S_1 is term-consistent. Our original version of the set featured denumerably many predicates. Elliott Mendelson pointed out to us that two suffice.

31. In the case that T' is distinct from 't_1', $\mathrm{P}(\mathrm{t}_1) \supset (T' \neq \mathrm{t}_1 \supset \mathrm{P}(T'))$ is not true on the 't_1'-variant $\mathfrak{I}_D^{\mathrm{t}_1}$ of $\mathfrak{I}_D$ that assigns 1 to 't_1': since $\mathfrak{I}_D^{\mathrm{t}_1}(T')$ has to be 2, $\mathrm{P}(\mathrm{t}_1)$ and $T' \neq \mathrm{t}_1$ are both true on $\mathfrak{I}_D^{\mathrm{t}_1}$ but $\mathrm{P}(T')$ is not. Hence our requiring of T^* that it be distinct from 't_1'.

32. It is to allow for this that we require of D *that it be of cardinality no less than* 2. When D is of cardinality 1, the only T-variant of $\mathfrak{I}_D$ is $\mathfrak{I}_D$ itself. In the absence of '=', any set of statements of L that has a model of cardinality 1 does of course have one of any cardinality greater than 1.

33. The main results of this paper were announced at the Annual Meeting of the Association for Symbolic Logic, University of California, Berkeley, January 13, 1990.

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