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Preservation by Homomorphisms and Infinitary Languages

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Abstract In this paper we study when sentences of infinitary languages are preserved by homomorphisms. This is done by using generalized Henkin construction. By the same technique we can also study when a sentence has an equivalent sentence which is in normal form.

The so-called Hintikka game, which is a straightforward generalization of Henkin construction, is used in, e.g., Hyttinen [1], [2] and Oikkonen [6]. In this paper we refine the game and prove a preservation theorem by using it.

Throughout this paper we assume that κ is weakly compact. This is done because in the proofs we construct certain trees, which have no branches of length $\geq \kappa$ and no nodes that have $\geq \kappa$ immediate successors, and then under the assumption that κ is weakly compact we know that the trees are of cardinality $< \kappa$.

We begin this chapter by defining the language $M_{\kappa\kappa}$. This language was first defined and studied by M. Karttunen in [4]. By a λ κ-tree T we mean a tree such that each node has $< \lambda$ immediate successors, there are no branches of length $\geq \kappa$ and if x and y are limit nodes and $\{t \in T \mid t < x\} = \{t \in T \mid t < y\}$ then $x = y$.

1.1 Definition $\phi = (T, l)$ is a formula of $M_{\kappa\kappa}$ if

(1) T is a $\kappa\kappa$-tree without branches of limit length, i.e. every branch has a maximal element;
(2) l is a labeling function with the properties
  a: if $t \in T$ does not have any successors then $l(t)$ is either an atomic or negated atomic formula;
  b: if $t \in T$ has exactly one immediate successor then $l(t)$ is of the form $\exists x$ or $\forall x$, x variable;
  c: if $t \in T$ has more than one immediate successor then $l(t)$ is either $\vee$ or $\wedge$.

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If $\phi = (T, l)$ and $u \in T$ then we write ϕ^u for the subformula $(T', l | T')$ of ϕ where $T' = \{u' \in T | u' \geq u\}$, also a $\kappa\kappa$-tree.

Semantics for $M_{\kappa\kappa}$ is defined by a semantic game. Let $\mathfrak{A}$ be a model and let $\phi = (T, l)$ be a sentence of $M_{\kappa\kappa}$.

1.2 Definition The semantic game $S(\mathfrak{A}, \phi)$ is a game of two players, A and E. When the game begins, the players are in the root of T and during the game the players go up the tree T. At each move the players are in some node $t \in T$ and it depends on $l(t)$ how they continue the game:

1. If $l(t) = \vee (\wedge)$ then $E(A)$ chooses one immediate successor of t to be the node where the players go next.
2. If $l(t) = \exists x \ (\forall x)$ then $E(A)$ chooses an element $x^{\mathfrak{A}}$ from $\mathfrak{A}$ to be an interpretation of x. The players go then to the immediate successor of t.
3. If $l(t) = \phi(\bar{x})$ then the game is over and E has won if

$$A \models \phi(\bar{x}) [\overline{x^{\mathfrak{A}}}] \ .$$

The concept of winning strategy is defined as usual.

Let ϕ be a sentence of $M_{\kappa\kappa}$ and $\mathfrak{A}$ model.

1.3 Definition $\mathfrak{A} \models \phi$ if E has a winning strategy for $S(\mathfrak{A}, \phi)$.

A function $f: \mathfrak{A} \to \mathfrak{B}$ is a homomorphism if it is onto and for all positive atomic formulas $\phi(\bar{x})$ if $\mathfrak{A} \models \phi(\bar{a})$ then $\mathfrak{B} \models \phi(f(\bar{a}))$.

Let $(C,<)$ be a well-ordered set of power κ of new constant symbols. By $M_{\kappa\kappa}(C)$ we mean the original language $M_{\kappa\kappa}$ together with the new constants.

1.4 Definition Let $\mathfrak{A}$ and $\mathfrak{B}$ be models in the language of power $\leq \kappa$, and let $T_{\mathfrak{A}}$ and $T_{\mathfrak{B}}$ be theories of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then $H(\mathfrak{A},\mathfrak{B})$ is a game of length κ played by A and E. For every move $\alpha < \kappa$ first A asks a question and then E answers the question by choosing some $S_\alpha^{\mathfrak{A}}, S_\alpha^{\mathfrak{B}} \subseteq M_{\kappa\kappa}(C)$ of power $< \kappa$. There are eight different ways to form the question:

1. A chooses some $\phi \in T_{\mathfrak{A}}$ or $\in T_{\mathfrak{B}}$; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that $\phi \in S_\alpha^{\mathfrak{A}}$ or $\in S_\alpha^{\mathfrak{B}}$ respectively.
2. A chooses a closed term t; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that $t = t$, $t = c \in S_\alpha^{\mathfrak{A}}$ and $\in S_\alpha^{\mathfrak{B}}$ for some $c \in C$.
3. A chooses $t = t' \in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}}$ or $\in \bigcup_{\beta<\alpha} S_\alpha^{\mathfrak{B}}$, where t and t' are closed term; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\beta^{\mathfrak{B}}$ so that $t' = t \in S_\beta^{\mathfrak{A}}$ or $\in S_\alpha^{\mathfrak{B}}$ respectively.
4. A chooses $\exists x \phi(\bar{x}) \in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}}$ or $\in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{B}}$; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that $\phi(\bar{c}) \in S_\alpha^{\mathfrak{A}}$ or $\in S_\alpha^{\mathfrak{B}}$ respectively for some $\bar{c} \in C$.
5. A chooses $\forall \bar{x} \phi \in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}}$ or $\in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{B}}$ and some sequence $\bar{t}$ of closed terms; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that $\phi(\bar{t}) \in S_\alpha^{\mathfrak{A}}$ or $\in S_\alpha^{\mathfrak{B}}$ respectively.
6. A chooses $\vee \Phi \in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}}$ or $\in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{B}}$; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that for some $\phi \in \Phi$ $\phi \in S_\alpha^{\mathfrak{A}}$ or $S_\alpha^{\mathfrak{B}}$ respectively.
7. A chooses $\wedge \Phi \in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}}$ or $\in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{B}}$ and some $\phi \in \Phi$; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that $\phi \in S_\alpha^{\mathfrak{A}}$ or $\in S_\alpha^{\mathfrak{B}}$ respectively.

8. A chooses $t = t'$ and $\phi(t) \in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}}$ or $\in \bigcup_{\beta<\alpha} S_\beta^{\mathfrak{B}}$ where t and t' are closed terms; then E must choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that $\phi(t') \in S_\alpha^{\mathfrak{A}}$ or $\in S_\alpha^{\mathfrak{B}}$ respectively.

Furthermore E must obey the following rules:

9. $\bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}} \subseteq S_\alpha^{\mathfrak{A}}$ and $\bigcup_{\beta<\alpha} S_\beta^{\mathfrak{B}} \subseteq S_\alpha^{\mathfrak{B}}$ and if $(\phi^{u_i}(\bar{t}_i))_{i<\xi}$, $\xi < \kappa$ is a sequence of sentences from $\bigcup_{\beta<\alpha} S_\beta^{\mathfrak{A}}$ or from $\bigcup_{\beta<\alpha} S_\beta^{\mathfrak{B}}$ such that
   (a) if i is successor then $\phi^{u_i}(\bar{t}_i)$ has been an answer to a question of the form 4–7 in which $\phi^{u_{i-1}}(\bar{t}_{i-1})$ exists,
   (b) if i is limit then $\phi^{u_i}(\bar{t}_i)$ has been obtained from the previous sentences by this rule, then respectively $\phi^u(\bar{t}) \in S_\alpha^{\mathfrak{A}}$ or $\in S_\alpha^{\mathfrak{B}}$, where $u = \sup_{i<\gamma} u_i$ and $\bar{t} = \bigcup_{i<\gamma} \bar{t}_i$.
10. For all positive atomic formulas ϕ either $\phi \notin S_\alpha^{\mathfrak{A}}$ or $\neg\phi \notin S_\alpha^{\mathfrak{A}}$ and either $\phi \notin S_\alpha^{\mathfrak{B}}$ or $\neg\phi \notin S_\alpha^{\mathfrak{B}}$.
11. For all positive atomic formula ϕ if $\phi \in S_\alpha^{\mathfrak{A}}$ then $\phi \in S_\alpha^{\mathfrak{B}}$.
12. Whenever E needs constants from C which do not exist in $S_\alpha^{\mathfrak{A}} \cup S_\alpha^{\mathfrak{B}}$ she must choose them to be minimal (in the ordering of C) among those not in $S_\alpha^{\mathfrak{A}} \cup S_\alpha^{\mathfrak{B}}$.
13. E must always choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ so that they are minimal (in the ordering given by $\subseteq$) among those that satisfy 1–12.

E wins the game if she can in each move $\alpha < \kappa$ choose $S_\alpha^{\mathfrak{A}}$ and $S_\alpha^{\mathfrak{B}}$ according to the rules. Otherwise A wins.

1.5 Lemma *Let $\mathfrak{A}$ and $\mathfrak{B}$ be as above. A does not have a winning strategy for $H(\mathfrak{A},\mathfrak{B})$ iff there exist $\mathfrak{A}'$ and $\mathfrak{B}'$ such that for all ϕ in $M_{\kappa\kappa}$ if $\mathfrak{A} \models \phi$ then $\mathfrak{A}' \models \phi$ and if $\mathfrak{B} \models \phi$ then $\mathfrak{B}' \models \phi$ and there exists a homomorphism from $\mathfrak{A}'$ to $\mathfrak{B}'$.*

Proof: If such $\mathfrak{A}'$ and $\mathfrak{B}'$ do exist then E wins by playing according to them. If such $\mathfrak{A}'$ and $\mathfrak{B}'$ do not exist then A wins by asking all the possible questions. For details see the similar result in [3].

By $nM_{\kappa\kappa}$ we mean the language consisting of those formulas ϕ of $M_{\kappa\kappa}$ such that $\neg\phi$ is expressible in $M_{\kappa\kappa}$. Especially then $L_{\kappa\kappa}$ is a sublanguage of $nM_{\kappa\kappa}$.

By $PP_{\kappa\kappa}$ we mean the language consisting of formulas of the form

$$Q \bigvee_{i\in I} \bigwedge_{j\in J_i} \phi^{ij},$$

where Q is a quantifier prefix of length $< \kappa$, $|I|$, $|J_i| < \kappa$, and ϕ^{ij} are positive atomic formulas. This language can be considered as a sublanguage of $M_{\kappa\kappa}$ which also gives the semantics for these formulas.

1.6 Theorem *Let κ be weakly compact and T a theory in the language $nM_{\kappa\kappa}$ of cardinality $\leq \kappa$. Assume that T is preserved by homomorphisms. Then there is an equivalent theory T' such that all its sentences are of the form*

$$\bigvee_{i<\kappa} \phi_i,$$

where ϕ_i, $i < \kappa$, are sentences of $PP_{\kappa\kappa}$.

The next two lemmas together prove the theorem.

1.7 Lemma *Let κ and T be as in the theorem above. Then (a) implies (b):*
(a) *If $\mathfrak{A} \models T$ and every sentence of $PP_{\kappa\kappa}$ which is true in $\mathfrak{A}$ is true in $\mathfrak{B}$ then $\mathfrak{B} \models T$.*
(b) *There is a theory T' equivalent to T such that all its sentences are of the form*

$$\bigvee_{i<\kappa} \phi_i,$$

where ϕ_i, $i < \kappa$, are sentences of $PP_{\kappa\kappa}$.

Proof: This proof goes as the proof of the relating result in [CK]. Define T' to be the set of all φ of required form, such that $T \models \varphi$. Let $\mathfrak{B} \models T'$. Let

$$\Phi = \{\phi \in PP_{\kappa\kappa} | B \not\models \phi\}.$$

To prove the lemma it is enough to show that there exists $\mathfrak{A} \models T$ such that $\mathfrak{A} \models \phi$ for all $\phi \in \Phi$. But if such $\mathfrak{A}$ does not exist then $T \models \vee\Phi$ and so $\mathfrak{B} \models \vee\, \Phi$, contradiction.

1.8 Lemma *Let κ and T be as in the theorem above. Assume $\mathfrak{A} \models T$ and every sentence of $PP_{\kappa\kappa}$ which is true in $\mathfrak{A}$ is true in $\mathfrak{B}$. Then $\mathfrak{B} \models T$.*

Proof: We assume $|\mathfrak{A}| \geq \kappa$, if not then $|\mathfrak{B}| \leq |\mathfrak{A}|$ and there must exist a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (because if $|\mathfrak{A}| < \kappa$ then we can express this in $PP_{\kappa\kappa}$), which implies the claim in the lemma immediately. Because T is a theory in $nM_{\kappa\kappa}$, it is enough to show that A does not have a winning strategy for $H(\mathfrak{A},\mathfrak{B})$. For a contradiction assume that A has a winning strategy s. Let U be the tree of all sequences $(q_0, a_0, \ldots, q_\gamma)$, $\gamma < \kappa$, of moves in $H(\mathfrak{A},\mathfrak{B})$ such that A has followed his winning strategy s and E has played according to the rules of $H(\mathfrak{A},\mathfrak{B})$. Then U is a $\kappa\kappa$-tree, because κ has the tree property $|U| < \kappa$. Let Φ be the conjunction of all ϕ such that $\phi \in T_{\mathfrak{A}}$ and ϕ has been a question of A in some sequence in U. Similarly let Ψ be the conjunction of all φ such that $\varphi \in T_{\mathfrak{B}}$ and φ has been a question of A in some sequence in U. Then Φ, Ψ are sentences of $M_{\lambda\lambda}$ for some $\lambda < \kappa$. Let us now consider only the similarity type which consists of those symbols that exist in Φ or Ψ.

Define

$$\Phi' = (\forall x_i, \exists y_i)_{i<\lambda} \bigvee_{j \in J} \wedge \phi'_j,$$

where $\{\phi'_j | j \in J\}$ is the family of all sets ϕ' such that there exists an onto mapping f from $\{x_i, y_i | i < \lambda\}$ to $\mathfrak{A}'$, $\mathfrak{A}' \subseteq \mathfrak{A}$, $\mathfrak{A}' \models \Phi$, $|\mathfrak{A}'| = \lambda$, and $\xi(\bar{x}, \bar{y}) \in \phi'$ iff ξ is a positive atomic formula and $\mathfrak{A}' \models \xi(\overline{f(x)}, \overline{f(y)})$.

Then $\mathfrak{A} \models \Phi'$ and $\Phi' \in PP_{\kappa\kappa}$. So $\mathfrak{B} \models \Phi'$. Let $\mathfrak{B}' \subseteq \mathfrak{B}$, $|\mathfrak{B}'| \leq \lambda$, be such that $\mathfrak{B}' \models \Phi' \wedge \Psi$. Then there is a submodel $\mathfrak{A}'$ of $\mathfrak{A}$, $\mathfrak{A}' \models \Phi$ and a homomorphism from $\mathfrak{A}'$ to $\mathfrak{B}'$.

Let $q = (q_0, a_0, \ldots, q_\gamma)$ be the sequence of moves in $H(\mathfrak{A},\mathfrak{B})$ such that A has followed s and E has played according to the models $\mathfrak{A}'$ and $\mathfrak{B}'$ and each answer has been according to the rules but to the question q_γ she cannot answer anymore according to the rules. Then $q \notin U$ and we have derived a contradiction.

So Theorem 1.6 has been proved.

1.9 Corollary *Let κ be weakly compact such that below κ there are cofinally many weakly compact cardinals. If $\phi \in nM_{\kappa\kappa}$ and is preserved under homomorphisms then there is an equivalent sentence in $PP_{\kappa\kappa}$.*

Proof: Use Theorem 1.6 and push the disjunction and the conjunction behind the quantifier prefix and change the order of disjunctions and conjunctions.

Let $NM_{\kappa\kappa}$ be the language obtained from $M_{\kappa\kappa}$ by closing it under negation and under conjunctions and disjunctions of power $< \kappa$. The next theorem is essentially due to Magidor [5].

1.10 Theorem [Magidor] *Let κ be extendible. Then $NM_{\kappa\kappa}$ is κ-compact.*

Proof: As the related result in [5].

1.11 Corollary *Let κ be extensible and T a theory in the language $nM_{\kappa\kappa}$ of cardinality $\leq \kappa$. Assume T is preserved in homomorphisms. Then there is an equivalent theory T' in the language $PP_{\kappa\kappa}$.*

Proof: As the proof of Theorem 1.6 except that now in Lemma 1.7 we can avoid the big disjunction by compactness.

If we study when a theory is preserved under isomorphisms (!) instead of homomorphisms, we get a result that tells us when a sentence of $M_{\kappa\kappa}$ has an equivalent sentence which is in normal form.

By $PL_{\kappa\kappa}$ we mean the language consisting of formulas of the form $Q\phi$, where Q is a quantifier prefix of length $< \kappa$ and ϕ is a quantifier-free formula of $L_{\kappa\kappa}$. Again $PL_{\kappa\kappa}$ can be considered as a sublanguage of $M_{\kappa\kappa}$ which gives us the semantics.

1.12 Theorem *Let κ be weakly compact. If T is a theory in $nM_{\kappa\kappa}$ of cardinality $\leq \kappa$, then there is an equivalent theory T', which contains only sentences of the form $\bigvee_{i<\kappa}\phi_i$ where $\phi_i \in PL_{\kappa\kappa}$, $i < \kappa$.*

Proof: Just as the proof of Theorem 1.6 except study now when a theory is preserved under isomorphisms.

1.13 Corollary *Let κ be weakly compact such that under κ there are cofinally many weakly compact cardinals. If $\phi \in nM_{\kappa\kappa}$ then there is an equivalent sentence in $PL_{\kappa\kappa}$.*

Proof: Use Theorem 1.12 and push the disjunction and the conjunction behind the quantifier prefix.

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