Notre Dame Journal of Formal Logic
Volume 32, Number 1, Winter 1991

Book Review

Stephen Pollard, *Philosophical Introduction to Set Theory*. University of Notre Dame Press, 1990. 180 pages.

What is a set? If Pollard's arguments are accepted, there should be no further need for philosophers of mathematics to pursue this question. This is because he argues that the two viable contenders for the title "mathematically adequate philosophy of set theory" are formalism and structuralism and for neither of these is this a question which it is meaningful to pursue. It would seem that Pollard favors structuralism, but he recognizes that he has not conclusively refuted his own arguments in favor of taking formalism seriously. Pollard's approach to the philosophical problems posed by set theory, whether one agrees with his conclusions or not, deserves careful consideration; he places many issues in a fresh light.

The book begins by asking why the attention of the philosopher of mathematics should be focused on set theory. Pollard's answer appeals to the special foundational role which set theory plays in twentieth century mathematics: "It is the primary mechanism for ideological and theoretical unification in modern mathematics." He does note that set theory is not the only contender for this role. Category theory can also assume a foundational role; it too can be used to supply a unitary and coherent vision of the mathematical enterprise. But, Pollard argues, set theory currently rules the mathematical roost. Even though category theory has shown that it is a viable contender, it has not decisively demonstrated its superiority. Be this as it may, I would suggest that if Pollard's arguments for structuralism succeed, and if they were then followed to their logical conclusion, a re-assessment of category theory would be required. Category theory arose precisely out of a realization that the concepts and constructions which most frequently arise in connection with mathematical structures possess a universality which is independent of their set theoretic origin. Category theory then gives precise expression to the idea that the essence of a mathematical structure is to be sought, not in its internal constitution as a set theoretic entity, but in the form of its relationships with other structures. Its claim to be a superior unifying language for mathematics is based on the fact that it gives direct expression to the centrality of form and structure in mathematics, whereas set theory can express this only indirectly. If we were to become convinced that every mathematical sen-

Received November, 1990

tence is an assertion about an abstract structure some of whose occurrences appear within iterative hierarchies (p.152), then it would seem that we should also take the claims of category theory seriously and should require that any adequate philosophy of mathematics include an informed discussion of the relations between category theory and set theory.

The remainder of the book has an architectonic which does not really become apparent until Chapter 6. The strategy is to mount a case for structuralism by (1) establishing that a formalist account of "$\in$" (the mathematical set-membership relation) is the only one possible. In this case our understanding of "$\in$" derives only from our knowledge of the axioms of set theory and their logical vocabulary. Then (2) arguing for the plausibility of structuralism by arguing that we need not be similarly formalist about the reading of logical constants. The case for (1) requires that the two standardly suggested sources of interpretations for "$\in$" are not able to ground the mathematical notions of "set" and "set-membership". This involves arguing that neither the history of the mathematical origins of set theory (Chapter 2) nor our commonsense set talk (Chapter 3) can provide a basis for the interpretation of "$\in$".

The case for (2) is urged by introducing a monadic second-order version of Zermelo set theory, where the second-order qualifiers are treated as ranging over pluralities (Chapters 7 and 8). Such axiomatizations are said to be categorical and hence English interpretations of their quantified formulas could be claimed to provide a basis for specifications of the truth conditions for formulas containing "$\in$". It would then have to be shown that the patterns of ordinary English usage here appealed to for the interpretation of the formulas of monadic second-order set theory really do serve as the basis for a reading of "$\in$". This problem is addressed in Chapter 9, where Resnik's mathematical structuralism is discussed. But, as its concluding remarks acknowledge, there is no conclusive argument provided to show that the truth conditions of set theoretic propositions can be fixed by something other than overt set theoretic proof practice. Hence Pollard admits that he has no conclusive structuralist refutation of formalism as a philosophy of mathematics, although he has ruled out realism. Nonetheless he could be counted as having made a significant gain by formulating a precise challenge issued to all those who would resist formalism. Whether it can be agreed that this is a correct formulation of the challenge depends on whether the arguments leading up to its formulation are found to be persuasive.

Pollard's preliminary arguments, designed to rule out appeals either to common sense or to history for the basis of the mathematical notion of "set" are crucial to his project. If these fail, then his subsequent arguments will lack motivation. It is therefore worth considering them in more detail. He sets out to argue against two myths.

(1) The development of mathematical set theory was significantly influenced by notions borrowed directly from everyday thought.
(2) One can be expected to have an essentially sound notion of what a mathematical set is prior to learning anything about mathematical set theory.

The upshot is claimed to be that commonsense notions of set not only did not historically inform the mathematical notion, they could not have done. (It should be noted in passing that although Penelope Maddy is not mentioned as a target, her recent book [4] would be an example of the kind of position attacked here.) Clearly a full historical account of the origins of set theory cannot be compressed into a single chapter, so there is inevitably scope for complaint about omissions from the account offered in Chapter 2. Pollard's purpose, however, is to include sufficient history to make the point that the mathematically distinctive features of set theory, its extension of arithmetic into the transfinite and its commitment to the iterated formation of sets of sets, arose as mathematical responses to mathematical problems, primarily within the theory of functions and real numbers. He argues that set theory arose as a response to the uncommonsensical demands made by studies of relations of functional dependence between mathematical continua. He also appeals to Jacob Klein's arguments [2] concerning the transformation that Arabic algebra wrought on Greek conceptions of number. For the Greeks, numbers were either pluralities of given objects, or abstract representatives of such pluralities. Acquaintance with pluralities is therefore acquaintance with numbers. This is very much the position Maddy defends when she says that we perceive sets, and when we perceive them we also perceive their number properties ([4], pp. 88–89). Pollard, however, points out that the modern notion of number is so closely linked to mastery of arithmetical techniques that we would not credit anyone with a grasp of number concepts prior to their mastery of a significant body of arithmetical techniques. This would seem effectively to undermine Maddy's claim that the epistemology of arithmetic and that of finite set theory are inseparable and such that neither can be given epistemological priority, but it will not decisively damage it without the further argument that such conceptions of finite sets as arise pre-set theoretically do not, and cannot, form the basis for the set theoretic concept of "set".

The argument of Chapter 3 concerning what we do and do not derive from common sense is therefore crucial to Pollard's position. Here he not only needs to establish the strong negative thesis that the mathematical notion of set could not have been derived from reflection on commonsense notions but also the positive thesis that there is a commonsense understanding of mechanisms of plural reference which can subsequently be appealed to as ground for the interpretation of formulas of monadic second-order logic. Recent work on plural reference by Simons and Boolos has origins in Leśniewski's systems of Ontology and of Mereology, although Pollard does not make anything of this history. Instead he draws most heavily on the work of Boolos [1].

Since sets are pluralities one might think it obvious that set talk could be used to paraphrase plural references. But this is not always the case, since ordinary language contains contexts in which plural reference is to collectives, such as an orchestra, whose identity through time is not determined by its members and which therefore would not satisfy the axiom of extensionality. The more crucial question, however, from Pollard's point of view, is whether distributive plural reference can paraphrase or otherwise ground set-talk. To do this it would have to ground talk of sets of sets, of the empty set, and of unit sets, together with the distinction between set membership and set inclusion. Sometimes we do make plural reference to families as groups of families. But in this case membership is regarded as transitive—if you belong to one of the families, you also belong to the group of families. So the situations which might help ground talk of sets of sets do so in such a way as to yield a concept of set membership which is transitive. Pollard might have strengthened the argument at this point by calling on the history of logic and the theory of classes. One only has to read the work of older logicians, such as Schröder, to realize that the separation of class membership from class inclusion is far from dictated by ordinary language, where "part of" and "member of" are not strongly distinguished. Ordinary language enhanced by traditional Aristotelian logic makes the distinction even harder to discern. In any case, Pollard presents sufficient evidence that a mastery of English as presently constituted will not equip one with anything close to the mathematical concept of set. This does not, however, establish that ordinary language could not be reformed in such a way as to incorporate a basis for the mathematical concept which would be all that Maddy's counterargument would require.

All that Maddy's set theoretic realism requires here is that knowledge of finite sets and their properties can be grounded in everyday experience, that a concept of finite set can be learned prior to learning the axioms of ZF, and that the concept of set so learned is identical with that which provides the intended interpretation of the axioms of ZF. To make this identity claim plausible, however, Maddy has to urge that set theory is the study of the iterative hierarchy with physical objects as ur-elements. This is taking set theory back in the direction of Russell's hierarchy of simple types and for very much Russell's reasons. From Pollard's point of view, the move could be criticized for not taking seriously the mathematical notion of set which is that embodied in ZF and which is a universe of "abstract" sets all "generated" from the empty set. The idea of including physical objects as ur-elements in a mathematical set theory seeks to eliminate, or downplay, the difference between mathematical and physical objects. But physical objects are mutable temporary existents whose identity criteria, part-whole relations, etc. are complex. To imagine that they can be treated as individual set theoretic atoms reveals the *chosiste* tendency typical of realism—the tendency to place priority on objects and to assume their absolute giveness as individuals. If, on the other hand, we are investigating set theory with a view to understanding mathematics, it seems fair to insist, with Pollard, that we stick to the standard mathematicians' set theory, with its universe of abstract sets. Pollard could then be seen as insisting that a grasp of the notion of "abstract set" cannot be gained otherwise than via familiarity with the axioms of set theory.

Nonetheless, the arguments he offers leave an alternative open. Note the disparity between what the argument of Chapter 2 achieves—it dispels the myth that the development of mathematical set theory was significantly influenced by notions borrowed from everyday thought—and what the argument is said in Chapter 9 to require: that the history of the mathematical origins of set theory cannot provide a basis for the interpretation of "$\in$". Chapter 2 leaves open the possibility that the mathematical concept of set can be gained by doing mathematics in areas other than set theory, and can thus be grasped prior to acquaintance with axiomatic set theory. This is a lacuna which needs, I think, to be filled if Pollard's "Dummettian" argument in favor of formalism is to be compelling. The argument, briefly, is that our grasp of a concept cannot outrun our ability to display that grasp; otherwise concepts would be incommunicable. What is the range of behavior in which our grasp of the mathematical concept of "set" can be exhibited? Both Dummett and Pollard presume that such grasp as we have is fully displayed in the production and assessment of proofs in axiomatic set theory. Our grasp of the concept is, and can only be, manifested in our grasp of assertability conditions, i.e. in our grasp of set theoretic proof procedures. But since we know that neither CH nor GCH are provable in ZF, we cannot in this case, if we stick to classical logic, equate assertability conditions with truth conditions. But why might it not be the case, as realists like Gödel have suggested, that the set-theoretic "intuition" which outruns the proof procedures of ZF is grounded in the employment of naive set theory in other areas of mathematics.

Pollard might be able to to block this objection using his positive account of the basis of the mathematical notion of set. This account goes via the use of monadic second-order qualifiers, interpreted as ranging over pluralities. Pluralities are the sorts of things that form the referents of plural referring terms. They are distinguished from sets in that they are not themselves individual objects and thus cannot belong to further pluralities. Pollard argues that if we adopt a realist reading of the ordinary first-order existential quantifier, we should also adopt a realist reading of the monadic second-order existential quantifier read as ranging over pluralities. A plural existential generalization might be "There are objects which are R-related to a". This could be formalised as "$\exists X \forall y(Xy \rightarrow Rya)$". Pollard argues that a realist reading of such quantifiers commits us to the idea that the existence of a plurality of objects does not depend on our ability to construct a formula which characterizes them. This in turn, he suggests, should lead us to regard the following as a *logical* truth:

$$\text{(A)}\quad \forall w((\forall x(\phi xw \rightarrow \exists y \phi yx) \ \&\ \forall xyz((\phi xw \ \&\ \phi yw \ \&\phi zx \ \&\ \phi zy) \rightarrow x = y)) \\ \rightarrow \exists X \forall x(\phi xw \rightarrow \exists! y(\phi yx \ \&\ Xy))).$$

If "$\in$" is substituted for "ϕ" in (A), this results in a weakened form of AC—it asserts the existence of a plurality where AC asserts the existence of a set. But the Separation Axiom in Pollard's monadic second order Zermelo set theory (Z^2) is

Axiom of Separation $\forall X \forall x \exists y \forall z((z \in y \leftrightarrow (z \in x \ \&\ Xz).$

This effectively ensures the existence of a set corresponding to each plurality drawn from an existing set. If the weakened form of AC is a logical truth then the full AC becomes a theorem of Z^2. Since Z^2 is categorical, either CH or not-CH is Z^2-true, even though neither is provable. This would seem to suggest that the notion of plurality does the extra work to supply content to the set concept and can underwrite the claim that CH is true or false even though we may be unable to determine which is the case.

Now I find the claim the the ordinary language mechanisms of plural reference can ground acceptance of (A) highly dubious. As Leśniewski and Lejewski have argued [3], it is far from clear that ordinary language unambiguously grounds a realist interpretation even of contexts in which the first-order existential quantifier would be used. Moreover, the Boolos readings of monadic second-order sentences are hardly readings based on standard English. I would suggest that if (A) is accepted it is as a result of the mathematical use and mathematical exploration of "pluralities". What Pollard would then have shown is that this notion can serve as the basis on which axiomatic set theory is built and can underpin its realist interpretations.

However, this isn't the conclusion which Pollard wants. His move to structuralism requires, he says, that there be no extralogical source of meaning for "$\in$". He cites Resnik's argument to the effect that because mathematical theories are incapable to distinguishing their isomorphic models, they specify, and are hence theories only of, structures. The particular constituents of the model are of no significance. But Pollard concedes that this would be so only if the the truth conditions of sentences of a mathematical theory are fixed only by its logical vocabulary. If one were able to attach an extralogical significance to "$\in$" this would restrict the contents of possible models of set theory. It is therefore essential to his case for structuralism that there be no viable interpretations of this sort available.

> We now have a better feeling for what was at stake in chapters 2 and 3. We there made significant progress toward establishing that an appropriate extralogical content for "$\in$" is to be found neither in the history of set theory nor in ordinary set talk. If these twin projects could be completed in full detail , it would render set theoretic structuralism unavoidable—at least among philosophers who are both mathematical platonists and mainstream semantic unitarians (a mainstream semantic unitarian being someone who embraces a standard semantics for first order logic). (p. 149)

Although Pollard's separation axiom does not, and cannot, force identification of sets with pluralities, it does suggest an intended interpretation; sets are pluralities treated as objects. The understanding of pluralities is grounded in pure logic only if the understanding of monadic second-order quantifiers can be said to derive from the ordinary mechanisms of plural reference. My unsubstantiated guess is that this is not the case and that the status of monadic second-order logic as "pure logic" remains dubious (see [5]). In any case, we can be grateful to Pollard for raising the question.

REFERENCES

[1] Boolos, G., "To be is to be a value of a variable (or to be some values of some variables)," *Journal of Philosophy*, vol. 81 (1984), pp. 430–449.

[2] Klein, J., *Greek Mathematical Thought and the Origins of Algebra*, MIT Press, Cambridge, 1968.

[3] Lejewski, C., "Logic and existence," *British Journal for the Philosophy of Science*, vol. V, (1954), pp. 104–119.

[4] Maddy, P., *Realism in Mathematics*, Oxford University Press, Oxford, 1990.

[5] Resnik, M, "Second-order logic still wild," *The Journal of Philosophy*, vol. 75 (1988) pp. 75–87.

Mary Tiles
Department of Philosophy
University of Hawaii
Honolulu