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A Strengthening of Scott's $ZF^{\neq}$ Result

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Abstract Scott's proof that ZF is not interpretable in ZF-minus-extensionality can be transformed into a proof that a theory much weaker than ZF is not interpretable in ZF-minus-extensionality.

In [6], Scott established that ZF is not interpretable in $ZF^{\neq}$ (ZF-minus-Extensionality).[1] In particular, he showed:

Theorem (Scott) $\quad ZF \vdash \exists M(M \models ZF^{\neq})$.

Scott's model is of the form $\langle Q_{\omega+\omega}, \eta_{\omega+\omega}\rangle$, where both $Q_{\omega+\omega}$ and $\eta_{\omega+\omega}$ are subsets of $R(\omega+\omega)(R(\alpha)$ being, as usual, the result of iterating the power set operation up to $\alpha)$. The full strength of the replacement scheme is scarcely tapped here, Replacement being called upon merely to warrant recursive constructions up to $\omega+\omega$. So if we let RPL(α) be the scheme

$$\forall x,y\forall\beta \in \alpha((\varphi(\beta,x) \wedge \varphi(\beta,y)) \rightarrow x = y) \rightarrow \exists z\forall x(x \in z \leftrightarrow \exists\beta \in \alpha\varphi(\beta,x))$$

it is easy to establish:

Theorem $\quad Z + \mathrm{RPL}(\omega) \vdash \exists M(M \models ZF^{\neq})$.

It follows that Z + RPL(ω) is not interpretable in $ZF^{\neq}$. This result is of some interest because Z + RPL(ω) is considerably weaker than ZF—as the following elementary theorem indicates.

Theorem $\quad ZFC \vdash \exists M(M \models Z + AC + \mathrm{RPL}(\omega))$.

Proof: Let $\epsilon(\alpha) = \{\langle x,y\rangle \in R(\alpha) : x \in y\}$, and $M = \langle R(\omega_1), \epsilon(\omega_1)\rangle$. We need only verify that $M \models \mathrm{RPL}(\omega)$. Suppose that

$$\forall x,y \in R(\omega_1)\forall n \in \omega((\varphi^M(n,x) \wedge \varphi^M(n,y)) \rightarrow x = y).$$

Let $f(n) = \alpha$ if and only if $\exists x \in R(\omega_1)(\varphi^M(n,x) \wedge \mathrm{rank}(x) = \alpha)$. f cannot map ω cofinally into ω_1. So we may pick a $\beta \in \omega_1$ such that $\forall\alpha \in \mathrm{Range}(f)$, $\alpha < \beta$. Then $\{x \in R(\omega_1) : \exists n \in \omega\ \varphi^M(n,x)\} \subset R(\beta) \in R(\omega_1)$.

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It follows that ZFC is not interpretable in $Z + AC + RPL(\omega)$. But then neither is ZF (since ZFC is interpretable in ZF). And, a fortiori, ZF is not interpretable in $Z + RPL(\omega)$. So there is a theory considerably weaker than ZF which is not interpretable in $ZF^{\neq}$. (Contrast this with the interpretability of Z in $Z^{\neq}$ and with the many interpretability results of this sort given in [3].)

NOTE

1. There have been too few investigations into the role of extensionality in Zermelian set theories (Z, ZF, VNB/GB, Quine–Morse, Montague–Scott, etc.). [1] and [5] are pathbreaking works. Recent studies are [2], [3], and [4]. The current surge of interest in property theories (cf. the references in [7]) could make a deepened understanding of extensionality essential.

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