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A Lemma in the Logic of Action

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Abstract In this paper, a result is proved that has two consequences for Segerberg's Logic of Action. First, in [1] and [2] his general frames can be replaced by full frames without change to the logic; secondly, a certain rule in [2] is proved to be sound.

Introduction The ultimate goal of this paper is to show that within the imperative logic described in Segerberg [2] the rule

$$\frac{\vdash [\alpha]\mathbf{P} \equiv [\beta]\mathbf{P}}{\vdash !\alpha \equiv !\beta} \quad \text{where } \mathbf{P} \text{ is a propositional variable not in either } \alpha \text{ or } \beta \tag{II}$$

is sound. In showing this we establish a result in the underlying logic of action, namely that Segerberg's restriction of the set of propositions in a frame is unnecessary. Essentially what we will show is that, given a standard frame $\mathfrak{F} = (U, A, D, P)$ with $D: P \to A$ satisfying

(D1) $D(X)(x) \subseteq X$, for all $X \in P$, $x \in U$
(D2) $D(X)(x) \subseteq Y \Rightarrow D(X)(x) \subseteq D(Y)(x)$, for all $X, Y \in P$, $x \in U$

we can find an extension D' of D to the whole of $\mathcal{P}(U)$, where D' still maintains these conditions. With this result, we will be able to show that given any countermodel to $!\alpha \equiv !\beta$ we can construct another model in which $[\alpha]\mathbf{P} \equiv [\beta]\mathbf{P}$ fails to hold.

1 Frames We take as our standard frames those outlined in Segerberg [1]. For a function f with range $\mathcal{P}(U \times U)$ we take $f(X)(x) = \{y : \langle x, y\rangle \in f(X)\}$, and use $f|_P$ to mean the restriction of f to P.

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Theorem 1.1 *If $\mathfrak{F} = (U,A,D,P)$ is a standard frame, then there is a frame $\mathfrak{F}' = (U,\mathcal{P}(U \times U),D',\mathcal{P}(U))$ with the property that $D'|_P = D$.*

Proof: Let $\mathfrak{F} = (U,A,D,P)$ be a standard frame. We construct $D' : \mathcal{P}(U) \to \mathcal{P}(U \times U)$ as follows:

$$D'(X)(x) = \cup\{D(Q)(x) : Q \in P \text{ and } D(Q)(x) \subseteq X\}.$$

All that remains is to establish the validity of the following claims.

Claim 1 $D'|_P = D$. To see this, let $X \in P$ and $x \in U$. Immediately from condition (D1) we have that $D(X)(x) \subseteq X$ and hence by the above definition $D(X)(x) \subseteq D'(X)(x)$. For the reverse inclusion let $Q \in P$ and suppose that $D(Q)(x) \subseteq X$. Since D satisfies (D2) we have that $D(Q)(x) \subseteq D(X)(x)$. Thus $D'(X)(x) \subseteq D(X)(x)$.

Claim 2 *D' satisfies conditions (D1) and (D2).* Let $X, Y \subseteq U$ and $x \in U$. Immediately from the definition $D'(X)(x) \subseteq X$, so (D1) is satisfied. For (D2), suppose that $D'(X)(x) \subseteq Y$ and let $y \in D'(X)(x)$. Hence there is a $Q \in P$ such that $y \in D(Q)(x)$ and $D'(X)(x) \subseteq Y$. So from the definition of $D'(Y)(x)$, $y \in D'(Y)(x)$. Thus $D'(X)(x) \subseteq D'(Y)(x)$.

2 The imperative logic Soundness of the rule [II] essentially comes down to the need to separate any two points in a frame by means of a proposition. For a full frame, this is a straightforward process, but for a frame with a restricted set of propositions we may find it necessary to "fill up the frame". Let us first fix some notation. Let $\mathcal{R}$ denote the set of all propositional variables, and for A a term or variable let $F(A)$ represent the set of all propositional variables occurring in A.

By induction, using as the induction order laid out in Segerberg [1], the proof of the following theorem can be readily obtained.

Theorem 2.1 *Let $R \subseteq \mathcal{R}$, and $\mathfrak{F} = (U,A,D,P)$ be a standard frame. Let $\mathfrak{F}' = (U,\mathcal{P}(U \times U),D,\mathcal{P}(U))$ be a standard frame with the property that $D'|_P = D$. Let $\mathfrak{M} = (U,A,D,P,V_1)$ and $\mathfrak{M}' = (U,\mathcal{P}(U \times U),D,\mathcal{P}(U),V_2)$ be models on $\mathfrak{F}$ and $\mathfrak{F}'$ respectively, satisfying $V_1|_R = V_2|_R$. Then $\|\mathbf{A}\|^{\mathfrak{M}} = \|\mathbf{A}\|^{\mathfrak{M}'}$ and $\|\alpha\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}'}$ for all formulas* $\mathbf{A}$ and terms α *such that $F(\mathbf{A}),F(\alpha) \subseteq R$.*

Now the soundness of [II] can be established.

Theorem 2.2 *The rule:* $\dfrac{\vdash[\alpha]\mathbf{P} \equiv [\beta]\mathbf{P}}{\vdash !\alpha \equiv !\beta}$ $\mathbf{P} \notin F(\alpha) \cup F(\beta)$ *is sound.*

Proof: Assume that $\models[\alpha]\mathbf{P} \equiv [\beta]\mathbf{P}$, $\mathbf{P} \notin F(\alpha) \cup F(\beta)$, and that *not* $\models !\alpha \equiv !\beta$. Hence there is a model $\mathfrak{M} = (U,A,D,P,V)$ such that *not* $\models^{\mathfrak{M}} !\alpha \equiv !\beta$, so we have $\|\alpha\|^{\mathfrak{M}} \neq \|\beta\|^{\mathfrak{M}}$. Without loss of generality take $\langle x,y\rangle \in \|\alpha\|^{\mathfrak{M}} - \|\beta\|^{\mathfrak{M}}$. Consider $\mathfrak{M}' = (U,\mathcal{P}(U \times U),D,\mathcal{P}(U),V')$ where D' is as in the previous theorems and

$$V'(\mathbf{P}_k) = \begin{cases} V(\mathbf{P}_k), & \text{if } \mathbf{P}_k \in F(\alpha) \cup F(\beta) \\ \|\beta\|^{\mathfrak{M}}(x), & \text{if } \mathbf{P}_k = \mathbf{P} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Now from Theorem 2.1 $\|\alpha\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}'} \neq \|\beta\|^{\mathfrak{M}} = \|\beta\|^{\mathfrak{M}'}$. Clearly in the model $\mathfrak{M}' \vDash_x [\beta]\mathbf{P}$; however, since $y \notin \|\beta\|^{\mathfrak{M}}(x) = \|\mathbf{P}\|^{\mathfrak{M}'}$ we have that in $\mathfrak{M}'$ *not* $\vDash_x [\alpha]\mathbf{P}$, which contradicts $\vDash [\alpha]\mathbf{P} \equiv [\beta]\mathbf{P}$.

REFERENCES

[1] Segerberg, K., "Bringing it about," *Journal of Philosophical Logic*, to appear.

[2] Segerberg, K., "Validity and satisfaction in imperative logic," *Notre Dame Journal of Formal Logic*, vol. 31 (1990), pp. 203–221.

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