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Book Review

Peter Aczel. *Non-well-founded sets.* CSLI Lecture Notes Number 14, Stanford, 1988. 137 pages.

This book presents a positive point of view about non–well-founded sets. Indeed, such sets usually arise in (more or less) marginal set theories (such as Quine's NF), and are generally considered as curiosities or pathological objects, in any case as useless and anti-intuitive. Here, they are given by a new axiom AFA (anti-foundation axiom), which is added to ZFC^- (ZF with the axiom of choice but without the foundation axiom), and are used to represent non-well-founded structures by sets. Basically, the idea of modeling structures in set theory is an old one. Ordinals, for example, are a natural representation of well-ordering structures; another example is the Mostowski collapse, which allows any binary extensional well-founded relation to be represented by a transitive set. The anti-foundation axiom's intention is to drop the "well-foundedness" limitation. Some mathematicians have already considered the possibility of replacing the foundation axiom by some "universality" axiom; the universality axiom of Boffa is such an example (it states that every extensional structure is isomorphic to a transitive set). But the interesting facts about AFA are that it contains a uniqueness condition of the representation and that it has applications in different domains.

The book is divided into three parts, and contains two appendices, a foreword (by J. Barwise), a preface, and an introduction. Appendix B gives the necessary background set theory. It should be remarked that the book does not require very much familiarity with set theory, and is written in a clear and agreeable style. The essential facts are proved and the routine arguments are left as exercises. So this work can be directed to a large readership.

In Part One (containing Chapters 1, 2, and 3), the anti-foundation axiom AFA is introduced. This axiom has been (independently) investigated (among other anti-foundation axioms) by Forti and Honsell (1983), who gave a consistency proof (relatively to ZFC^-). The new features are:

(1) the elegant presentation of the axiom which allows it to be compared easily with other anti-foundation axioms and to uniformize and simplify the notations and the proofs
(2) the applications of the axiom.

Axiom AFA says that "every graph has a unique decoration (by sets)." A graph is simply a *set* of nodes, with a *set* of edges; the notation for "(x,y) is an edge" is $x \to y$; a decoration of a graph is an assignment d of a set $d(x)$ to each node x in such a way that

$$d(y) = \{d(x) | x \leftarrow y\}.$$

So AFA extends the Mostowski collapse to all graphs.

After some examples of equations which have a unique solution when AFA is assumed (for example, $x = \{x\}$), a general *solution lemma* is proved, which shows that, for a large class of systems of equations, AFA allows the existence and uniqueness of a solution. For example, the system

$$\begin{aligned} x_0 &= (a_0, x_1) \\ x_1 &= (a_1, x_2) \\ x_2 &= (a_2, x_3) \\ &\vdots \end{aligned}$$

with an infinity of equations (the a_i are given sets ($i \in \mathbb{N}$) and (u, v) is the usual Kuratowski ordered pair) has a unique solution, which is the (strange) set $x = (a_0, (a_1, (a_2, \ldots)))$. Furthermore, the equivalence of the "local" version of AFA and the "class" version of AFA is proved. The class version is as follows: Any *system* has a unique decoration (a system is a *class* of nodes, with a class of edges, and satisfies the condition that for any node x, $\{y | y \leftarrow x\}$ is a *set*).

In Chapter 2 AFA is studied with respect to its relation with the notion of "bisimulation". As mentioned in Appendix A ("Notes towards a history"), the notion of bisimulation appeared in the work of various authors, under different names and with different motivations; it appeared both in set theory and in the study of mathematical models for transitive (computational) systems. The similarities between constructions worked out in distinct domains gave the initial impulse for this interest in non-well-founded sets, and provided the impetus to Aczel to formulate axiom AFA (which he discovered to have already been investigated by Forti and Honsell).

A binary relation R on a system M is a bisimulation on M iff

$$R \subseteq R^+, \text{ with } aR^+b \Leftrightarrow [(\forall x \leftarrow a\ \exists y \leftarrow b\ xRy) \\ \wedge (\forall y \leftarrow b\ \exists x \leftarrow a\ xRy)].$$

The + operator is monotone and it is proved that every system M has a unique maximum bisimulation (written $\equiv_M$); this $\equiv_M$ is always an equivalence and the natural quotient $M/\equiv$ is extensional. This leads to a first definition of "strong extensionality" (other notions of "strong extensionality" appear later in the book): A system M is said to be "strongly extensional" iff $\equiv_M$ is the identity on M. This notion leads to a new (local) formulation of AFA:

> a graph G is isomorphic to a transitive set (seen as a graph, with $x \leftarrow y$ iff $x \in y$) iff G is strongly extensional.

This chapter ends with an application: a simplified proof of a completeness theorem (Kanger 1957) and of a variant due to Gordeev (1982).

Chapter 3 gives a proof of the consistency of AFA relatively to ZFC^-. The inner model is the quotient $\equiv_M$ of the class M of the apgs or "accessible pointed graphs". A graph G is an apg if every node x is accessible from the distinguished point a_G (this means that there is a finite path $x \leftarrow x_1 \leftarrow x_2 \ldots \leftarrow a_G$);

the edges in M are given by the rule $Ga \to Gb$ iff $a \to b$ (in G); for a system N, Nx is the apg whose nodes and edges are those of N that lie on (descending) paths of N starting from x; the distinguished point of Nx is x itself. The last theorem in this chapter states that this is the unique system (unique up to isomorphism) which is a full model (in the sense of Rieger) for $\mathrm{ZFC}^- + \mathrm{AFA}$.

Part Two is divided into two chapters (Chapters 4 and 5) and considers variants of the anti-foundation axiom. These variants are instances of a family of axioms $\mathrm{AFA}^\sim$, determined by a suitable equivalence relation $\sim$ on the system of the apgs; such a $\sim$ determines a notion of $\sim$-extensionality for systems: M is $\sim$-extensional iff $Ma \sim Mb \to a = b$. The axiom $\mathrm{AFA}^\sim$ can then be described as:

an apg G is isomorphic to a transitive set
iff
G is $\sim$-extensional.

AFA corresponds to $\mathrm{AFA}^\sim$ when $\sim$ is $\equiv_{\mathrm{M}}$ (where M is the system of the apgs).

The main theorems about AFA are generalized to $\mathrm{AFA}^\sim$ (for suitable $\sim$). In particular, axioms FAFA and SAFA are considered. FAFA is $\mathrm{AFA}^\sim$, where the $\sim$-extensionality is exactly the "downwards isomorphic" extensionality given by axiom SEXT ("strong extensionality" studied by Von Rimscha), and inspired by Finsler (1926). SAFA corresponds to a third notion of "strong extensionality", due to Scott (here the tree obtained by unfolding an apg determines the notion of extensionality). These three notions of strong extensionality are shown to be effectively different, so that AFA, FAFA, and SAFA are pairwise incompatible axioms.

Chapter 5 discusses anti-foundation axioms of a somewhat different kind, namely "universality" axioms introduced by Boffa. For example, the universality axiom BA states that every extensional graph is isomorphic to a transitive set. Here the representation of the graph by a transitive set is no longer unique.

Part Three (Chapters 6, 7, and 8) presents some interesting applications of AFA.

Chapter 6 studies fixed points of "set continuous" operators. A class operator Φ (i.e., ΦX is a class for each class X) is said to be *set continuous* iff

$$\forall X(\Phi X = \bigcup \{\Phi a \mid a \in V \wedge a \subseteq X\}).$$

Different characterizations are given for this notion, together with properties and examples. The main result states that every set continuous operator has a least and a largest fixed point (for the order $\subseteq$), with an explicit description of these extreme fixed points. It should be remarked that here the collection scheme is used (and this fully justifies the formulation of ZFC^- used in Appendix B).

Finally, a useful theorem is proved, which localizes the unique solution of a system of equations (as described in Part One, solution lemma) in the largest fixed point of a set continuous Φ. An application of this is given in Barwise and Etchemendy's *The Liar*, Oxford University Press (1987); perhaps this example should have been described in more detail instead of being just mentioned at the end of Appendix A. In this example the class of the *propositions* is seen as the largest fixed point X_0 of the (set continuous) operator Φ, where $\Phi(X) = B \cup (\{T\} \times X) \cup (\{F\} \times X) \cup (\{\wedge\} \times (X \times X)) \cup \ldots$. The symbol $\cup$ represents "disjunct union" here; T, F, $\wedge, \ldots$ are constants with the classical meaning (T = true, F = false, $\wedge$ = conjunction, ...); and B is a set of basic propositions. For example, if $p \in X_0$, p can be of type $(\wedge, p_1, p_2)$, so it is the conjunction "$p_1 \wedge p_2$" of p_1 and p_2; or p can be of type (T, q), so it is the proposition "q is true", etc. Assuming AFA, there is a unique solution to the equation $x = (\mathrm{F}, x)$ and by the theorem just mentioned this solution is an element of the largest fixed point of Φ, so it is still a proposition. This proposition x_0 is exactly "x_0 is false", and so is the liar's proposition.

Another important application of the last theorem of Chapter 6 is discussed in Chapter 8 and briefly described below.

Chapter 7 gives a very general description of the extreme fixed points for "standard" functors (which generalize the notion of "set continuous operator") in terms of "initial algebra" (= the least fixed point) and "final co-algebra" (= the largest fixed point), seen as objects in a (suitable) category. This shows that the phenomenon of the existence of a least and a largest fixed point is in fact a categorical one; it was discovered independently in the context of bisimulation-techniques by different authors (see, *inter alia*, Forti & Honsell, Gordeev, Hinnion) mentioned in the references of the book.

Chapter 8 presents an application to communicating systems. A transition system is a class X (of possible states) and a family of binary transition relations $\xrightarrow{a}$ between states, one for each possible "atomic action" a. When the collection Act of the atomic actions is a singleton, the corresponding "transition system" notion is exactly the "system" notion described before. In a transition system distinct states of processes may have the same "external" behavior. This notion of abstract behavior is captured by using the concept of a bisimulation relation on a transition system; this was first considered by Park (1981) and developed by Milner (1983). The maximal bisimulation is moreover an equivalence relation (Milner calls this a "strong congruence"), and the quotient of the class of "expressions" (intended to represent the possible states of systems that can communicate with each other) by this maximal bisimulation is a model of "abstract behaviors" for these computational systems. Here the same result is obtained very easily (assuming AFA), as a simple application of the final co-algebra theorem (Chapter 7) for a suitable "standard" functor

$$\Phi(X) = \mathrm{pow}(\mathrm{Act} \times X), \text{ where } \mathrm{pow}\, Y = \{b \mid b \in V \wedge b \subseteq Y\}.$$

One gets in this way a "complete" transition system P (= the largest fixed point for Φ) such that $x \xrightarrow{a} y$ iff $(a, y) \in x$ (for $x, y \in P$), and for each transition system X there is a (unique) map $\pi : X \to P$ such that $\forall x \in X$

$$\pi x = \{(a, \pi y) \mid x \xrightarrow{a} y \text{ in } X\}.$$

This map π is the "behavior map" for X. So, if a transition system X constitutes an operational semantics for a programming language, the behavior map for X gives a canonical representation of the abstract behavior of the programs of the language. Thus the complete P is a domain of mathematical objects which are denotations of programs for such a programming language. This chapter ends with the definition of operations on P, in particular of the four fundamental combinators used by Milner to define the "expressions" of SCCS (Synchronous Calculus of Communicating Systems).

Appendix A ("Notes towards a history") is a very interesting survey of the development in this century of the idea of a non–well-founded set. It is divided into quarter century periods. The fundamental distinction between the well-founded and the non–well-founded sets was formulated by Mirimanoff (1917). The foundation axiom appeared somewhat later (von Neumann, Zermelo) and in 1926 Finsler proposed an axiom system which corresponds to FAFA. Axiom SAFA appeared in 1960 (Scott) and AFA in 1983 (Forti & Honsell).

Other applications of non–well-founded set theory are mentioned, particularly for the rich period after 1950. Finally, the application of AFA to situation semantics is mentioned (Barwise & Etchemendy 1987): A situation is represented by a set of facts, and a fact is seen as a triple (R, a, σ), where R is a relation, a is a triple of objects appropriated for the relation R, and σ is 0 or 1 (false or true). As situations are themselves objects they can occur as components of facts, so circular situations arise, which can be set-theoretically modeled only in a non–well-founded set theory. AFA provides such a natural and sufficient context.

As mentioned, Appendix B gives the background set theory. Some remarks are made about the global choice and its relation to the quotients (over classes) which are worked out in the book. At last, Rieger's theorem (about full systems) is stated and proved.

The best conclusion about this book may be found in the foreword by J. Barwise: "The theory of circular and otherwise extra-ordinary sets presented in this book is an excellent example of this synergistic process". The synergistic process referred to is that of a convergence of similar ideas in distinct domains, brought together here in a clear synthesis which is surely a starting point for further investigations.

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