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Quantified Modal Logics of Positive Rational Numbers and Some Related Systems

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Abstract The quantified modal logics **QK4.3.D.X** and **QS4.3** are shown to be characterized by the Kripke models based on the extended frames with nested domains $\langle Q^+, <, D\rangle$ and $\langle Q^+, \leq, D\rangle$, respectively, i.e., the set of positive rational numbers ordered by the numerical relation 'less than' ('less than or equal to'). Moreover, for each $n \geq 1$, the logics $\mathbf{C}_n$**.QK4.3.D.X** ($\mathbf{C}_n$**.QS4.3**) are shown to be characterized by the Kripke models based on (reflexive) towers of rank at most n and with nested domains. Other quantified extensions of **QK4.3** are considered and proved Kripke complete as well.

0 Introduction In Corsi [1] the author introduced the method of diagrams and applied it to quantified intermediate logics, in particular to Dummett's logic **LC** quantified. The same method turned out to be extremely useful in proving completeness results of various quantified modal logics. It is well known that "completeness" is rare in quantified modal logics and several incompleteness results have been established, see Ghilardi [4],[5] and Shehtman and Skvortsov [6]. Now, if we limit ourselves to logics correct with respect to some class of connected, transitive, and reflexive frames, we are confronted with the fact that at the propositional level Bull's Theorem tells us that any such logic is Kripke complete, whereas, at the quantified level, quite the opposite seems to be the case. In this paper, I show how to characterize classes of Kripke models with nested domains based either on the positive rational numbers or on some subset of them. The notion of diagram was first introduced for **QS5** in Fine [3].

1 The logics QK4.3, QK4.3.D.X and QS4.3 Let $\mathcal{L}$ be a first-order modal language, $\bot \in \mathcal{L}$, $\neg\alpha =_{\mathrm{df}} \alpha \to \bot$ and $\top =_{\mathrm{df}} \bot \to \bot$. **QK4.3** is the quantified modal calculus obtained by adding to the normal propositional modal logic K the following axioms and rules:

4 $\quad \Box\alpha \to \Box\Box\alpha$

3 $\quad \Box(\Box\alpha \wedge \alpha \to \beta) \vee \Box(\Box\beta \wedge \beta \to \alpha)$

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∀ $\forall x\alpha(x) \to \alpha(x/y)$, where y is free for x in α.

∃ $\alpha(x/y) \to \exists x\alpha(x)$, where y is free for x in α.

UG $\dfrac{\beta \to \alpha(x)}{\beta \to \forall x\alpha(x)}$ where x is not free in β.

EP $\dfrac{\alpha(x) \to \beta}{\exists x\alpha(x) \to \beta}$ where x is not free in β.

As is well known, Axiom **4** corresponds to transitivity and Axiom **3** to weak connectedness. **QK4.3.D.X** is the logic **QK4.3** plus axioms

D $\Box\alpha \to \Diamond\alpha$, and

X $\Box\Box\alpha \to \Box\alpha$.

QS4.3 is the logic **QK4.3** plus axiom

T $\Box\alpha \to \alpha$.

D corresponds to seriality, **X** to density, and **T** to reflexivity.

We recall that a relation R is said to be *weakly connected* iff $(vRw \wedge vRz \to wRz \vee zRw \vee w = z)$, *connected* iff $(wRz \vee zRw \vee w = z)$. R is a *linear order* if R is reflexive, transitive, connected, and antisymmetric; R is a *strict linear order* if R is transitive, connected, and irreflexive.

QK4.3 is the least logic considered in this paper, and we denote by **L** any logic which extends **QK4.3**. '$\vdash_{\mathbf{L}}\alpha$' means that α is a theorem of **L** and $A \vdash_{\mathbf{L}} \alpha$ that $\vdash_{\mathbf{L}}\beta_1 \wedge \ldots \wedge \beta_n \to \alpha$, for some finite subset of wffs $\beta_1, \ldots, \beta_n$ of A. When no confusion arises, we omit the '**L**'.

Here is a list of theorems that will be useful in what follows.

Lemma 1.1 *The following are theorems of* **QK4.3**:

(a) $\Box\forall\vec{x}\alpha \to \forall\vec{x}\Box\alpha$

(b) $\Box(\beta \to \Box\forall\vec{x}\alpha(\vec{x})) \to \Box\forall\vec{x}(\beta \to \Box\alpha(\vec{x}))$, *where $\vec{x}$ do not occur in β.*

(c) $\Box\forall\vec{x}((\beta(\vec{x}) \wedge \Box\alpha) \to \partial(\vec{x})) \to (\Box\alpha \to \Box\forall\vec{x}(\beta(\vec{x}) \to \partial(\vec{x})))$, *where $\vec{x}$ do not occur in α.*

(d) $\Box\forall\vec{x}(\alpha \wedge \gamma \to \partial) \to (\Box\alpha \to \Box\forall\vec{x}(\gamma \to \partial))$, *where $\vec{x}$ do not occur in α.*

(e) $\Box(\alpha \to \partial) \to [\Box(\Box\partial \wedge \partial \to \Box\neg\alpha \wedge \neg\alpha) \to \Box(\Box\partial \to \Box\neg\alpha \wedge \neg\alpha)]$

(f) $\Box(\alpha \to \neg\alpha) \to \Box\neg\alpha$

(g) $(\Box(\alpha \to \Box\partial) \wedge \Box(\Diamond\alpha \to \partial) \wedge \Box(\alpha \to \partial)) \to (\Diamond\alpha \to \Box\partial)$

(h) $\Box[\alpha \to \Box\forall\vec{x}(\beta(\vec{x}) \wedge \Box\neg\alpha \wedge \neg\alpha \to \gamma(\vec{x}))] \to [\Diamond\alpha \to \Box\forall\vec{x}(\beta(\vec{x}) \wedge \Box\neg\alpha \wedge \neg\alpha \to \gamma(\vec{x}))]$, *where no variable of $\vec{x}$ occurs in α.*

Proof: (a)–(f) are theorems of **QK4**, and (h) is derivable from (g) by substituting $\forall\vec{x}(\beta(\vec{x}) \wedge \Box\neg\alpha \wedge \neg\alpha \to \gamma(\vec{x}))$ for ∂. Notice that, once the substitution is made, **QK4** $\vdash \Box(\Diamond\alpha \to \partial) \wedge \Box(\alpha \to \partial)$, where $\vec{x}$ do not occur in α. The following is a proof of (g).

1. $\vdash\Box[\Box(\Box\partial \wedge \partial) \wedge (\Box\partial \wedge \partial) \to \Box\neg\alpha \wedge \neg\alpha] \vee \Box[\Box(\Box\neg\alpha \wedge \neg\alpha) \wedge (\Box\neg\alpha \wedge \neg\alpha) \to (\Box\partial \wedge \partial)]$ by Ax. **3**
2. $\vdash\Box(\Box\partial \wedge \partial \to \Box\neg\alpha \wedge \neg\alpha) \vee \Box(\Box\neg\alpha \wedge \neg\alpha \to \Box\partial \wedge \partial)$ from 1 and the fact that **K4** $\vdash \Box\beta \to \Box(\Box\beta \wedge \beta)$
3. $\vdash\Box(\alpha \to \partial) \to [\Box(\Box\partial \wedge \partial \to \Box\neg\alpha \wedge \neg\alpha) \to \Box(\Box\partial \to \Box\neg\alpha \wedge \neg\alpha)]$ from (e)

4. $\vdash\Box(\neg\partial\to\neg\alpha)\to[\Box(\Box\neg\alpha\wedge\neg\alpha\to\Box\partial\wedge\partial)\to\Box(\Box\neg\alpha\to\Box\partial\wedge\partial)]$ from (e)
5. $\vdash\Box(\alpha\to\partial)\to[\Box(\Box\neg\alpha\wedge\neg\alpha\to\Box\partial\wedge\partial)\to\Box(\Box\neg\alpha\to\Box\partial\wedge\partial)]$ from 4
6. $\vdash\Box(\alpha\to\partial)\to[\Box(\Box\partial\to\Box\neg\alpha\wedge\neg\alpha)\vee\Box(\Box\neg\alpha\to\Box\partial\wedge\partial)]$ from 2, 3 and 5
7. $\vdash\Box(\Box\partial\to\Box\neg\alpha\wedge\neg\alpha)\to[\Box(\alpha\to\Box\partial)\to\Box(\alpha\to\Box\neg\alpha\wedge\neg\alpha)]$ transitivity
8. $\vdash\Box(\alpha\to\Box\neg\alpha\wedge\neg\alpha)\to\Box\neg\alpha$ from (f)
9. $\vdash\Box(\Box\partial\to\Box\neg\alpha\wedge\neg\alpha)\to[\Box(\alpha\to\Box\partial)\to\Box\neg\alpha]$ from 7 and 8
10. $\vdash\Box(\Box\neg\alpha\to\Box\partial\wedge\partial)\to[\Box(\neg\partial\to\Box\neg\alpha)\to\Box\partial]$ from 9
11. $\vdash\Box(\alpha\to\partial)\to[\Box(\alpha\to\Box\partial)\wedge\Box(\neg\partial\to\Box\neg\alpha)\to(\Box\neg\alpha\vee\Box\partial)]$ from 6, 9 and 10
12. $\vdash\Box(\alpha\to\partial)\wedge\Box(\alpha\to\Box\partial)\wedge\Box(\Diamond\alpha\to\partial)\to(\Diamond\alpha\to\Box\partial)$ from 11.

Definition 1.2 A *Kripke model* $\mathcal{M}$ *with nested domains* for the first-order modal language $\mathcal{L}$ is a quadruple $\langle W,R,D,I\rangle$ where W is a non-empty set; R is a binary relation on W; D is a domain function such that for every $w\in W$, $D_w\neq\varnothing$ and if vRw then $D_v\subseteq D_w$; I is an interpretation function such that for all $w\in W$, $I_w(c)\in D_w$, where c is an individual constant of $\mathcal{L}$ and if wRv, then $I_w(c)=I_v(c)$, $I_w(P^n)\subseteq(D_w)^n$ where P^n is an n-ary predicate letter, $1\leq n<\omega$, of $\mathcal{L}$.

$\langle W,R\rangle$ ($\langle W,R,D\rangle$) is said to be the *frame* (*extended frame*) on which $\mathcal{M}$ is based. Let $w\in W$; a w-assignment μ is a function from the terms (individual variables and constants) of $\mathcal{L}$ into D_w and is such that $\mu(c)=I_w(c)$. If μ is a w-assignment and $d\in D_w$, by $\mu^{(x/d)}$ we denote the w-assignment such that if $y\neq x$, then $\mu^{(x/d)}(y)=\mu(y)$, if $y=x$, then $\mu^{(x/d)}(y)=d$. Note that if wRv, any w-assignment is a v-assignment.

Given a w-assignment μ and an element $w\in W$, the *truth of a wff* α *in* $\mathcal{M}$ *at* w *under* μ, $\mathcal{M}^\mu\models_w\alpha$, is defined in the usual way. We recall only three clauses:

$$\mathcal{M}^\mu\models_w P^n(t_1,\ldots,t_n) \text{ iff } \langle\mu(t_1),\ldots,\mu(t_n)\rangle\in I_w(P^n)$$

$$\mathcal{M}^\mu\models_w \forall x\alpha(x) \text{ iff for all } d\in D_w,\ \mathcal{M}^{\mu^{(x/d)}}\models_w\alpha$$

$$\mathcal{M}^\mu\models_w \Box\alpha \text{ iff for all } v,\ wRv,\ \mathcal{M}^\mu\models_v\alpha.$$

The notions of *truth in* $\mathcal{M}$ *at* w, $\mathcal{M}\models_w\alpha$, *truth in* $\mathcal{M}$, $\mathcal{M}\models\alpha$, and *validity on a frame* $\mathcal{F}$, $\mathcal{F}\models\alpha$, are the standard ones.

Lemma 1.3 **QK4.3 (QK4.3.D.X − QS4.3)** *is correct with respect to strict linear frames* (*serial, dense, strict linear frames—linear frames*) *with nested domains.*

2 Diagrams

2 Diagrams A diagram Δ is, roughly speaking, a set of indices with closed formulas attached to each of them. If a sentence α is attached to an index w (i.e., the pair $\langle w,\alpha\rangle\in\Delta$), the intended meaning is that α is true at w. A diagram is built up step by step by adding either a new index (together with a sentence) to the diagram built so far or a sentence to an index which is already present in the diagram. The construction has to be such that in the end the final diagram is a Kripke model with nested domains.

Our privileged set of indices is the set Q^+ of positive rational numbers, zero included, ordered by the numerical relation "less than", $<$.

With each $w \in Q^+$ we associate a countable non-empty set C_w of individual constants such that for all $v, w \in Q^+$, if $v \neq w$, then $C_v \cap C_w = \emptyset$.

For any $w \in Q^+$, the language of $w, \mathcal{L}_w$, is $\mathcal{L} \cup \{c \in C_v : v \leq w\}$ and $Fm(\mathcal{L}_w)$ is the set of closed wffs of $\mathcal{L}_w$.

Note that if $v < w$, then $\mathcal{L}_v \subset \mathcal{L}_w$ and for each constant c there is a unique $z \in Q^+$ such that $c \in C_z$ and a first or least (with respect to set-theoretical inclusion) language, $\mathcal{L}_z$, to which c belongs. C_z is said to be the set of *proper constants of z*.

Definition 2.1

a. $\mathcal{P} = \bigcup_{w \in Q^+} \{\langle w, \alpha \rangle : \alpha \in Fm(\mathcal{L}_w)\}$ is the set of all pairs each of which is determined by a rational numbers $w \in Q^+$ and a closed wff of $\mathcal{L}_w$.

b. A *diagram* is a subset of $\mathcal{P}$.

c. Γ is said to be a *subdiagram* of a diagram Δ iff $\Gamma \subseteq \Delta$.

Definition 2.2 Let Δ be a diagram.

a. The *support* of Δ is

$$\begin{aligned}\mathrm{Supp}(\Delta) = \{w : \langle w, \alpha \rangle \in \Delta, \text{ for some wff } \alpha\} \\ \cup \{v : \text{ there is a } \langle z, \beta(c) \rangle \in \Delta \text{ and some constant } c \\ \text{occurring in } \beta \text{ is a proper constant of } C_v\}.\end{aligned}$$

This definition makes sure that if a constant occurs in a wff of a diagram, then the rational number of which it is a proper constant belongs to the support of the diagram.

b. For any $w \in \mathrm{Supp}(\Delta)$, the set of formulas 'attached to' w in Δ, is

$$\Delta(w) = \{\alpha : \langle w, \alpha \rangle \in \Delta\} \cup \{\top\}.$$

c. Δ is said to be *quasi-finite* iff

i. the support of Δ is finite, and

ii. for any $v \in Q^+$, the constants of C_v occurring in wffs of Δ are finitely many.

If Δ is quasi-finite, the $\mathrm{Supp}(\Delta)$ is denoted by $\langle v_1, \ldots, v_n \rangle$, where $v_{i-1} < v_i$. Moreover, if Δ is finite and $\mathrm{Supp}(\Delta) = \langle v_1, \ldots, v_n \rangle$, then

$$\Delta_i =_{\mathrm{df}} \bigwedge \Delta(v_i).$$

Now we introduce the notion of **L**-coherence which turns out to be crucial for our constructions and which reduces to **L**-consistency in the case of diagrams whose support consists of only one point.

Definition 2.3 Let Δ be a finite diagram whose support is $\langle v_1, \ldots, v_n \rangle$. Δ is said to be **L**-*coherent* iff

$$\begin{aligned}\mathbf{L} \nvdash \forall \vec{x}_1 [\Delta_1(\vec{c}_1/\vec{x}_1) \to \Box \forall \vec{x}_2 [\Delta_2(\vec{c}_1/\vec{x}_1, \vec{c}_2/\vec{x}_2) \\ \to \ldots \to \Box \forall \vec{x}_n [\Delta_n(\vec{c}_1/\vec{x}_1, \vec{c}_2/\vec{x}_2, \ldots, \vec{c}_n/\vec{x}_n) \to \bot] \ldots]],\end{aligned}$$

where for all k, $1 \leq k \leq n$, $\vec{c}_k$ is the list $c_{k1}, \ldots, c_{kj_k}$, $0 \leq j_k < \omega$, of all the constants of C_k occurring in wffs of Δ, $\Delta_k(\vec{c}_1/\vec{x}_1, \ldots, \vec{c}_k/\vec{x}_k)$ is the wff obtained by uniformly substituting for all h, $1 \leq h \leq k$, $\vec{x}_h$ for $\vec{c}_h$ in Δ_k, i.e., $x_{h1}, \ldots, x_{hj_h}$ for $c_{h1}, \ldots, c_{hj_h}$, respectively, where $\vec{x}_h$ is a list of variables $x_{h1}, \ldots, x_{hj_h}$, $0 \leq j_h < \omega$.

We will refer to $\vec{c}_k$ as the list of *proper constants* of v_k occurring in Δ.

It is always intended that the constants actually occurring in Δ_k are among $\vec{c}_1, \ldots, \vec{c}_k$ and that all bound variables are distinct from one another.

Whenever possible and when no confusion arises, we express the above condition by

$$\mathbf{L} \nvdash \forall\vec{x}_1[\Delta_1(\vec{x}_1) \to \Box\forall\vec{x}_2[\Delta_2(\vec{x}_1,\vec{x}_2) \to \ldots \to \Box\forall\vec{x}_n[\Delta_n(\vec{x}_1,\ldots,\vec{x}_n) \to \bot]\ldots]], \text{ or, for short,}$$

$$\mathbf{L} \nvdash \forall\vec{x}_1[\Delta_1 \to \Box\forall\vec{x}_2[\Delta_2 \to \ldots \to \Box\forall\vec{x}_n[\Delta_n \to \bot]\ldots]]$$

and it is always intended that for all k, $1 \leq k \leq n$, the variables $\vec{x}_k$ in $\Delta_k(\vec{x}_1,\ldots,\vec{x}_k)$, or in Δ_k, are substituted for $\vec{c}_k$ according to the definition above.

Definition 2.4 An infinite diagram Δ is said to be **L**-*coherent* iff all of its finite subdiagrams are **L**-coherent.

Lemma 2.5
(a) *If Δ is an* **L**-*coherent diagram, then any subdiagram Γ of Δ is* **L**-*coherent.*
(b) *If Δ is an* **L**-*coherent diagram, then for all $w \in$* Supp(Δ), *$\Delta(w)$ is* **L**-*consistent.*

Proof: The proof of (a) is slightly laborious because the support of a diagram Δ contains in general more points than the ones actually occurring in the elements of Δ. However, this kind of difficulty arises only for the present lemmas. (a) If Δ is not finite, then by Definition 2.4, Γ is **L**-coherent. So let Δ be finite and $\mathrm{Supp}(\Delta) = \langle v_1,\ldots,v_n\rangle$. By the definition of **L**-coherence and the fact that $\vdash\alpha \leftrightarrow (\alpha \wedge \top)$, $\Lambda = \Delta \cup \{\langle v_1,\top\rangle,\ldots,\langle v_n,\top\rangle\}$ is **L**-coherent.

Let $\{\langle z_1,\alpha_1\rangle,\ldots,\langle z_p,\alpha_p\rangle\}$ be $(\Lambda - \Gamma)$, of course, $z_1,\ldots,z_p$ are among $v_1,\ldots,v_n$. First we show that

$$\Gamma \cup \{\langle v_1,\top\rangle,\ldots,\langle v_n,\top\rangle\}, \text{ i.e.}$$

$$(*) \qquad (\Lambda - \{\langle z_1,\alpha_1\rangle,\ldots,\langle z_p,\alpha_p\rangle\}) \cup \{\langle v_1,\top\rangle,\ldots,\langle v_n,\top\rangle\}$$

is **L**-coherent by induction on p.

$p = 0$. Then $\Lambda = \Lambda - \{\langle z_1,\alpha_1\rangle,\ldots,\langle z_p,\alpha_p\rangle\}$ and since $\Lambda = \Lambda \cup \{\langle v_1,\top\rangle,\ldots,\langle v_n,\top\rangle\}$, $(*)$ is proved.

Suppose that for $i \geq 0$,

$$\Pi^* = (\Lambda - \{\langle z_1,\alpha_1\rangle,\ldots,\langle z_i,\alpha_i\rangle\}) \cup \{\langle v_1,\top\rangle,\ldots,\langle v_n,\top\rangle\}$$

is **L**-coherent, we show that

$$\Pi = (\Lambda - \{\langle z_1,\alpha_1\rangle,\ldots,\langle z_{i+1},\alpha_{i+1}\rangle\}) \cup \{\langle v_1,\top\rangle,\ldots,\langle v_n,\top\rangle\}$$

is **L**-coherent too.

Now, $\mathrm{Supp}(\Pi)$ is obviously $\langle v_1,\ldots,v_n\rangle$, so let z_{i+1} be v_k and $\alpha_{i+1} = \beta$. If Π is not **L**-coherent, then

$$\vdash\forall\vec{x}_1[\Pi_1(\vec{x}_1) \to \ldots \to \Box\forall\vec{x}_k[\Pi_k(\vec{x}_1,\ldots,\vec{x}_k) \to \ldots \to \Box\forall\vec{x}_n[\Pi_n(\vec{x}_1,\ldots,\vec{x}_n) \to \bot]\ldots]\ldots],$$

where for each j, $1 \leq j \leq n$, $\vec{x}_j$ are substituted for the proper constants $\vec{c}_j$ of v_j occurring in Π. For each h, $1 \leq h \leq k$, let $\vec{e}_h$ be the list of constants of C_{v_h} occurring in β and not in Π. (Since $\mathrm{Supp}(\Pi) = \mathrm{Supp}(\Delta)$ for each constant c occurring in β, there is a v_h of the support of Π of which c is a proper constant.) Whence

$$\vdash \forall \vec{x}_1 [\Pi_1 \to \ldots \to \Box \forall \vec{x}_k [\Pi_k \wedge \exists \vec{y}_1 \ldots \exists \vec{y}_k \beta(\vec{e}_1/\vec{y}_1, \ldots, \vec{e}_k/\vec{y}_k, \vec{x}_1, \ldots, \vec{x}_k) \\ \to \ldots \to \Box \forall \vec{x}_n [\Pi_n \to \bot] \ldots] \ldots].$$

So, by classical logic,

$$\vdash \forall \vec{x}_1 [\Pi_1 \to \ldots \to \Box \forall \vec{x}_k \forall \vec{y}_1 \ldots \forall \vec{y}_k [\Pi_k \wedge \beta(\vec{y}_1, \ldots, \vec{y}_k, \vec{x}_1, \ldots, \vec{x}_k) \\ \to \ldots \to \Box \forall \vec{x}_n [\Pi_n \to \bot] \ldots] \ldots].$$

Since $\vec{y}_1, \ldots, \vec{y}_k$ do not occur in $\Pi_1, \ldots, \Pi_k$, it follows, by Lemma 1.1(b), that

$$\vdash \forall \vec{x}_1 \forall \vec{y}_1 [\Pi_1 \to \ldots \to \Box \forall \vec{x}_{k-1} \forall \vec{y}_{k-1} [\Pi_{k-1} \\ \to \Box \forall \vec{x}_k \forall \vec{y}_k [\Pi_k \wedge \beta(\vec{y}_1, \ldots, \vec{y}_k, \vec{x}_1, \ldots, \vec{x}_k) \\ \to \ldots \to \Box \forall \vec{x}_n [\Pi_n \to \bot] \ldots]] \ldots],$$

contrary to the **L**-coherence of Π^*.

We are now ready to show that Γ is **L**-coherent. $\Gamma \cup \{\langle v_1, \top \rangle, \ldots, \langle v_n, \top \rangle\} = \Gamma \cup \{\langle t_1, \top \rangle, \ldots, \langle t_r, \top \rangle\} \cup \{\langle u_1, \top \rangle, \ldots, \langle u_s, \top \rangle\}$, where each $t_i \in \mathrm{Supp}(\Gamma)$ and each $u_i \notin \mathrm{Supp}(\Gamma)$. Let $\Gamma^* = \Gamma \cup \{\langle t_1, \top \rangle, \ldots, \langle t_r, \top \rangle\}$.

Suppose that for $j \leq s$, $\Gamma^* \cup \{\langle u_1, \top \rangle, \ldots, \langle u_j, \top \rangle\}$ is **L**-coherent, we show that $\Pi = \Gamma^* \cup \{\langle u_1 \top \rangle, \ldots, \langle u_{j-1}, \top \rangle\}$ is **L**-coherent too. Let $\mathrm{Supp}(\Pi) = \langle w_1, \ldots, w_{k-1}, w_k, \ldots, w_h \rangle$ and $w_{k-1} < u_j < w_k$. Π is not **L**-coherent iff

$$\vdash \forall \vec{x}_1 [\Pi_1(\vec{x}_1) \to \ldots \to \Box \forall \vec{x}_k [\Pi_k(\vec{x}_1, \ldots, \vec{x}_k) \\ \to \ldots \to \Box \forall \vec{x}_h [\Pi_h(\vec{x}_1, \ldots, \vec{x}_h) \to \bot] \ldots] \ldots],$$

where for each j, $1 \leq j \leq h$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of the proper constants of v_j occurring in Δ. Then, by Axiom **4**,

$$\vdash \forall \vec{x}_1 [\Pi_1(\vec{x}_1) \to \ldots \to \Box\Box \forall \vec{x}_k [\Pi_k(\vec{x}_1, \ldots, \vec{x}_k) \\ \to \ldots \to \Box \forall \vec{x}_h [\Pi_h(\vec{x}_1, \ldots, \vec{x}_h) \to \bot] \ldots] \ldots],$$

and so, by classical logic,

$$\vdash \forall \vec{x}_1 [\Pi_1(\vec{x}_1) \to \ldots \to \Box [\top \to \Box \forall \vec{x}_k [\Pi_k(\vec{x}_1, \ldots, \vec{x}_k) \\ \to \ldots \to \Box \forall \vec{x}_h [\Pi_h(\vec{x}_1, \ldots, \vec{x}_h) \to \bot] \ldots]] \ldots],$$

contrary to the **L**-coherence of $\Gamma^* \cup \{\langle u_1, \top \rangle, \ldots, \langle u_j, \top \rangle\}$. Therefore Γ^* is **L**-coherent and consequently, Γ is **L**-coherent too. Analogously if $u_j < w_1$ or $u_j > w_h$.

(b) If $\Delta(w)$ is not **L**-consistent, then $\vdash \alpha_1 \wedge \ldots \wedge \alpha_n \to \bot$ for some wffs $\alpha_1, \ldots, \alpha_n \in \Delta(w)$. Let $\vec{c}_1, \ldots, \vec{c}_k$ be the constants occurring in $(\alpha_1 \wedge \ldots \wedge \alpha_n)$ and for each i, $1 \leq i \leq k$, let $v_i \in Q^+$ be the rational number such that $\vec{c}_i \subset C_{v_i}$. Suppose that $v_1 < \ldots < v_k$; trivially $v_k \leq w$, consider the case in which $v_k < w$. So

$$\vdash \forall \vec{x}_1 [\top \to \ldots \to \Box \forall \vec{x}_k [\top \to \Box \forall \vec{y}_w [(\alpha_1 \wedge \ldots \wedge \alpha_n)(\vec{c}_1/\vec{x}_1, \ldots, \vec{c}_k/\vec{x}_k) \to \bot]] \ldots].$$

So the diagram $\Gamma = \{\langle w, \alpha_i \rangle : 1 \leq i \leq n\}$ is not **L**-coherent, contrary to the fact that it is a subdiagram of Δ.

Lemma 2.5(a) will often be assumed without mentioning it when we take for granted that any diagram that extends an **L**-incoherent diagram is **L**-incoherent too.

Lemmas 2.6–2.11 guarantee that a given diagram, if **L**-coherent, admits of being extended to an **L**-saturated one.

Lemma 2.6 *Let Γ be an* **L**-*coherent diagram such that $v_i \in$* Supp(Γ). *For any $\alpha \in \mathfrak{L}_{v_i}$, if $\Gamma(v_i) \vdash \alpha$ and all the constants occurring in α are proper constants of elements of the support of Γ, then $\Gamma \cup \{\langle v_i, \alpha \rangle\}$ is an* **L**-*coherent diagram.*

Proof: If $\Gamma(v_i) \vdash \alpha$, then for some $\beta_1, \ldots, \beta_k \in \Gamma(v_i)$, $\beta_1, \ldots, \beta_k \vdash \alpha$. $\Gamma \cup \{\langle v_i, \alpha \rangle\}$ is not **L**-coherent iff for some finite subdiagram Δ of Γ, $\Delta \cup \{\langle v_i, \alpha \rangle\}$ is not **L**-coherent. Let Supp$(\Delta \cup \{\langle v_i, \alpha \rangle\}) = \langle v_1, \ldots, v_i, \ldots, v_n \rangle$ and assume that $\beta_1, \ldots, \beta_k \in \Delta(v_i)$. Then

$$\vdash \forall \vec{x}_1 [\Delta_1(\vec{x}_1) \to \ldots \to \Box \forall \vec{x}_i [\Delta_i(\vec{x}_1, \ldots, \vec{x}_i) \wedge \alpha(\vec{x}_1, \ldots, \vec{x}_i) \to \ldots \to \Box \forall \vec{x}_n [\Delta_n(\vec{x}_1, \ldots, \vec{x}_n) \to \bot] \ldots] \ldots],$$

where for each j, $1 \leq j \leq n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in $\Delta \cup \{\langle v_j, \alpha \rangle\}$.

Since $\Delta(v_i) \vdash \alpha(\vec{c}_1, \ldots, \vec{c}_i)$, then $\Delta_i(\vec{x}_1, \ldots, \vec{x}_i) \vdash \alpha(\vec{x}_1, \ldots, \vec{x}_i)$ and so

$$\vdash \forall \vec{x}_1 [\Delta_1 \to \ldots \to \Box \forall \vec{x}_i [\Delta_i \to \ldots \to \Box \forall \vec{x}_n [\Delta_n \to \bot] \ldots] \ldots],$$

contrary to the **L**-coherence of Γ.

Lemma 2.7 *Let Δ be an* **L**-*coherent diagram. If $\langle w, \Box\alpha \rangle \in \Delta$ then*
(a) *$\Delta \cup \{\langle z, \Box\alpha \rangle\}$ is* **L**-*coherent, where $w \leq z$ and $z \in$* Supp(Δ),
(b) *$\Delta \cup \{\langle z, \alpha \rangle\}$ is* **L**-*coherent, where $w < z$ and $z \in$* Supp(Δ).

Proof: (a) The proof is by induction and it will be enough to show that

$$\forall v [w \leq v < z \to \Delta \cup \{\langle v, \Box\alpha \rangle\} \text{ is } \mathbf{L}\text{-coherent}] \text{ only if } \Delta \cup \{\langle z, \Box\alpha \rangle\} \text{ is } \mathbf{L}\text{-coherent}.$$

If $\Delta \cup \{\langle z, \Box\alpha \rangle\}$ is not **L**-coherent, then for some finite subdiagram Γ of Δ, $\Gamma \cup \{\langle z, \Box\alpha \rangle\}$ is not **L**-coherent. Let Supp$(\Gamma) = \langle v_1, \ldots, v_k, \ldots, v_n \rangle$ with $z = v_k$. Then

$$\vdash \forall \vec{x}_1 [\Gamma_1(\vec{x}_1) \to \ldots \to \Box \forall \vec{x}_{k-1} [\Gamma_{k-1}(\vec{x}_1, \ldots, \vec{x}_{k-1}) \to \Box \forall \vec{x}_k [\Gamma_k(\vec{x}_1, \ldots, \vec{x}_k) \wedge \Box\alpha \to \ldots \to \Box \forall \vec{x}_n [\Gamma_n(\vec{x}_1, \ldots, \vec{x}_n) \to \bot] \ldots]] \ldots],$$

where for each j, $1 \leq j \leq n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in Γ. Since $\Box\alpha$ is at most a formula of $\mathfrak{L}_w$ and $w < v_k$, no variables of $\vec{x}_k$ occur in $\Box\alpha$, whence by Lemma 1.1(c),

$$\vdash \forall \vec{x}_1 [\Gamma_1 \to \ldots \to \Box \forall \vec{x}_{k-1} [\Gamma_{k-1} \wedge \Box \alpha \to \Box \forall \vec{x}_k [\Gamma_k \to \ldots \to \Box \forall \vec{x}_n [\Gamma_n \to \bot] \ldots]] \ldots].$$

But this contradicts the induction hypothesis, whence $\Delta \cup \{\langle z, \Box\alpha \rangle\}$ is **L**-coherent.

(b) From (a) it follows that $\Delta \cup \{\langle v, \Box\alpha \rangle\}$ is **L**-coherent, for any $v \geq w$ and $v \in \mathrm{Supp}(\Delta)$. Now, $\Delta \cup \{\langle z, \alpha \rangle\}$ is not **L**-coherent iff for some finite subdiagram Γ of Δ such that $\mathrm{Supp}(\Gamma) = \langle v_1, \ldots, v_{k-1}, v_k, \ldots, v_n \rangle$, $z = v_k$ and $w \leq v_{k-1}$, $\Gamma \cup \{\langle z, \alpha \rangle\}$ is not **L**-coherent, i.e.,

$$\vdash \forall \vec{x}_1 [\Gamma_1(\vec{x}_1) \to \ldots \to \Box \forall \vec{x}_{k-1} [\Gamma_{k-1}(\vec{x}_1, \ldots, \vec{x}_{k-1}) \to \Box \forall \vec{x}_k [(\Gamma_k \wedge \alpha)(\vec{x}_1, \ldots, \vec{x}_k) \to \ldots \to \Box \forall \vec{x}_n [\Gamma_n(\vec{x}_1, \ldots, \vec{x}_n) \to \bot] \ldots]] \ldots],$$

where for each j, $1 \leq j \leq n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in Γ. So, by Lemma 1.1(d),

$$\vdash \forall \vec{x}_1 [\Gamma_1 \to \ldots \to \Box \forall \vec{x}_{k-1} [\Gamma_{k-1} \wedge \Box \alpha(\vec{x}_1, \ldots, \vec{x}_k) \to \Box \forall \vec{x}_k [\Gamma_k \to \ldots \to \Box \forall \vec{x}_n [\Gamma_n \to \bot] \ldots]] \ldots],$$

contrary to the **L**-coherence of $\Delta \cup \{\langle v_{k-1}, \Box\alpha \rangle\}$.

Lemma 2.8 *Let Δ be an **L**-coherent diagram. For any $w \in \mathrm{Supp}(\Delta)$, and any $\alpha \in \mathcal{L}_w$ such that all the constants occurring in α are proper constants of elements of the support of Δ, either $\Delta \cup \{\langle w, \alpha \rangle\}$ or $\Delta \cup \{\langle w, \neg\alpha \rangle\}$ is an **L**-coherent diagram.*

Proof: If neither $\Delta \cup \{\langle w, \alpha \rangle\}$ nor $\Delta \cup \{\langle w, \neg\alpha \rangle\}$ is **L**-coherent, then for some finite subdiagram Γ of Δ such that $\mathrm{Supp}(\Gamma) = \langle v_1, \ldots, v_k, \ldots, v_n \rangle$ and $w = v_k$,

$$\vdash \forall \vec{x}_1 [\Gamma_1(\vec{x}_1) \to \ldots \to \Box \forall \vec{x}_k [(\Gamma_k \wedge \alpha)(\vec{x}_1, \ldots, \vec{x}_k) \to \ldots \to \Box \forall \vec{x}_n [\Gamma_n(\vec{x}_1, \ldots, \vec{x}_n) \to \bot] \ldots] \ldots], \text{ and}$$

$$\vdash \forall \vec{x}_1 [\Gamma_1(\vec{x}_1) \to \ldots \to \Box \forall \vec{x}_k [(\Gamma_k \wedge \neg\alpha)(\vec{x}_1, \ldots, \vec{x}_k) \to \ldots \to \Box \forall \vec{x}_n [\Gamma_n(\vec{x}_1, \ldots, \vec{x}_n) \to \bot] \ldots] \ldots],$$

where for each j, $1 \leq j \leq n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in Γ. Therefore, by classical logic,

$$\vdash \forall \vec{x}_1 [\Gamma_1 \to \ldots \to \Box \forall \vec{x}_k [\Gamma_k \to \ldots \to \Box \forall \vec{x}_n [\Gamma_n \to \bot] \ldots] \ldots],$$

contrary to the **L**-coherence of Δ.

Lemma 2.9 *Let Δ be an **L**-coherent diagram.*

(a) *If $\langle w, \alpha \vee \beta \rangle \in \Delta$ then either $\Delta \cup \{\langle w, \alpha \rangle\}$ or $\Delta \cup \{\langle w, \beta \rangle\}$ is an **L**-coherent diagram.*
(b) *If $\langle w, \alpha \rangle \in \Delta$, then $\Delta \cup \{\langle w, \alpha \vee \beta \rangle\}$ is an **L**-coherent diagram if all the constants occurring in β are proper constants of elements of the support of Δ.*
(c) *$\Delta \cup \{\langle w, \alpha \wedge \beta \rangle\}$ is **L**-coherent iff $\Delta \cup \{\langle w, \alpha \rangle\} \cup \{\langle w, \beta \rangle\}$ is **L**-coherent.*

Proof: (a) If $\Delta \cup \{\langle w, \alpha \rangle\}$ is not **L**-coherent, then by Lemma 2.8, $\Delta \cup \{\langle w, \neg\alpha \rangle\}$ is **L**-coherent. Now, if $\Delta \cup \{\langle w, \neg\alpha \rangle\} \cup \{\langle w, \beta \rangle\}$ is not **L**-coherent, then by Lemma 2.8 again, $\Delta' = \Delta \cup \{\langle w, \neg\alpha \rangle\} \cup \{\langle w, \neg\beta \rangle\}$ is **L**-coherent, contrary to the fact that $\Delta'(w) \vdash \bot$.

(b) and (c) Immediate.

Lemma 2.10 *Let Γ be a quasi-finite and* **L**-*coherent diagram. If* $\langle w, \exists y\alpha(y)\rangle \in \Gamma$, *then* $\Gamma \cup \{\langle w, \alpha(y/d)\rangle\}$ *is an* **L**-*coherent diagram, for some constant* $d \in C_w$.

Proof: Since Γ is quasi-finite there is at least a constant of C_w that does not occur in any wff of Γ. Let d be any such constant. If $\Gamma \cup \{\langle w, \alpha(y/d)\rangle\}$ is not **L**-coherent, then for some finite subdiagram Δ of Γ whose support is $\langle v_1, \dots, v_k, \dots, v_n\rangle$ with $w = v_k$,

$$\vdash \forall \vec{x}_1 [\Delta_1(\vec{x}_1) \to \dots \to \Box \forall y \forall \vec{x}_k [\Delta_k(\vec{x}_1, \dots, \vec{x}_k) \wedge \alpha(\vec{x}_1, \dots, \vec{x}_k, y)$$
$$\to \dots \to \Box \forall \vec{x}_n [\Delta_n(\vec{x}_1, \dots, \vec{x}_n) \to \bot] \dots] \dots],$$

where for each j, $1 \le j \le n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in Δ, y is substituted for d and is not among $\vec{x}_1, \dots, \vec{x}_n$. Then, by classical logic,

$$\vdash \forall \vec{x}_1 [\Delta_1 \to \dots \to \Box \forall \vec{x}_k [\Delta_k \wedge \exists y \alpha(\vec{x}_1, \dots, \vec{x}_k, y)$$
$$\to \dots \to \Box \forall \vec{x}_n [\Delta_n \to \bot] \dots] \dots],$$

contrary to the **L**-coherence of Γ.

Lemma 2.11 *Let Γ be a quasi-finite and* **L**-*coherent diagram whose support is* $\langle v_1, \dots, v_i, \dots, v_n\rangle$. *If* $\langle v_i, \Diamond\alpha\rangle \in \Gamma$, *then for some rational number* s, $v_i < s$, $\Gamma \cup \{\langle s, \alpha\rangle\}$ *is an* **L**-*coherent diagram.*

Proof: Case 1. $\forall k (i \le k \le n)(\Gamma \cup \{\langle v_k, \Box\neg\alpha\rangle\}$ is not **L**-coherent). Whence, in particular, $\Gamma \cup \{\langle v_n, \Box\neg\alpha\rangle\}$ is not **L**-coherent and so, by Lemma 2.8, $\Gamma' = \Gamma \cup \{\langle v_n, \Diamond\alpha\rangle\}$ is **L**-coherent. Take any rational number $s > v_n$, we claim that $\Lambda = \Gamma \cup \{\langle v_n, \Diamond\alpha\rangle\} \cup \{\langle s, \alpha\rangle\}$ is **L**-coherent.

$$\begin{array}{cccc} v_1 & \dots & v_n & s \\ \bullet & & \bullet & \bullet \\ \Diamond\alpha & & \Diamond\alpha & \alpha \end{array}$$

Suppose it is not, hence for some finite subdiagram Δ of Γ,

$$\vdash \forall \vec{x}_1 [\Delta_1(\vec{x}_1) \to \dots \to \Box \forall \vec{x}_n [\Delta_n(\vec{x}_1, \dots, \vec{x}_n) \wedge \Diamond\alpha(\vec{x}_1, \dots, \vec{x}_i)$$
$$\to \Box[\alpha(\vec{x}_1, \dots, \vec{x}_i) \to \bot]] \dots],$$

where for each j, $1 \le j \le n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in Δ. Then

$\vdash \forall \vec{x}_1 [\Delta_1 \to \dots \to \Box \forall \vec{x}_n [\Delta_n \wedge \Diamond\alpha(\vec{x}_1, \dots, \vec{x}_i) \to \Box\neg\alpha(\vec{x}_1, \dots, \vec{x}_i)] \dots]$, hence

$\vdash \forall \vec{x}_1 [\Delta_1 \to \dots \to \Box \forall \vec{x}_n [\Delta_n \wedge \Diamond\alpha(\vec{x}_1, \dots, \vec{x}_i) \to \bot] \dots]$,

contrary to the **L**-coherence of Γ'. Whence Λ is **L**-coherent and so $\Gamma \cup \{\langle s, \alpha\rangle\}$ is **L**-coherent too, by Lemma 2.5(a).

Case 2. $\exists k (i \le k \le n)$ $(\Gamma \cup \{\langle v_k, \Box\neg\alpha\rangle\}$ is **L**-coherent).

Let h be the smallest index among $\{i, \dots, n\}$ such that $\Gamma \cup \{\langle v_h, \Box\neg\alpha\rangle\}$ is **L**-coherent. Hence $i + 1 \le h \le n$ and $\Gamma \cup \{\langle v_{h-1}, \Diamond\alpha\rangle\} \cup \{\langle v_h, \Box\neg\alpha\rangle\}$ is **L**-

coherent. If $\Gamma \cup \{\langle v_{h-1}, \Diamond\alpha\rangle\} \cup \{\langle v_h, \Box\neg\alpha\rangle\} \cup \{\langle v_h, \alpha\rangle\}$ is **L**-coherent, then the lemma is proved and $s = v_h$.

$$\begin{array}{ccccccccc}
v_1 & \cdots & v_i & \cdots & v_{h-1} & v_h & \cdots & v_n \\
\bullet & & \bullet & & \bullet & \bullet & & \bullet \\
 & & \Diamond\alpha & & \Diamond\alpha & \Box\neg\alpha & & \\
 & & & & & \alpha & &
\end{array}$$

If not, then $\Gamma' = \Gamma \cup \{\langle v_{h-1}, \Diamond\alpha\rangle\} \cup \{\langle v_h, \Box\neg\alpha\rangle\} \cup \{\langle v_h, \neg\alpha\rangle\}$ is **L**-coherent.

Take any rational number s such that $v_{h-1} < s < v_h$, s exists because Q^+ is dense. Notice that no constant of C_s occurs in Γ and so in Γ', otherwise, by the definition of a diagram, $s \in \mathrm{Supp}(\Gamma)$, which is not the case.

We now show that $\Gamma'' = \Gamma' \cup \{\langle s, \alpha\rangle\}$ is **L**-coherent.

$$\begin{array}{ccccccccc}
v_1 & \cdots & v_i & \cdots & v_{h-1} & s & v_h & \cdots & v_n \\
\bullet & & \bullet & & \bullet & \bullet & \bullet & & \bullet \\
 & & \Diamond\alpha & & \Diamond\alpha & & \Box\neg\alpha & & \\
 & & & & & & \neg\alpha & & \\
 & & & & & \alpha & & &
\end{array}$$

If not, then, for some finite subdiagram Δ of Γ,

$$\begin{aligned}
\vdash \forall\vec{x}_1[\Delta_1(\vec{x}_1) \to \ldots \to \Box\forall\vec{x}_{h-1}[\Delta_{h-1}(\vec{x}_1, \ldots, \vec{x}_{h-1}) \wedge \Diamond\alpha(\vec{x}_1, \ldots, \vec{x}_i)& \\
\to \Box[\alpha(\vec{x}_1, \ldots, \vec{x}_i)& \\
\to \Box\forall\vec{x}_h[(\Delta_h(\vec{x}_1, \ldots, \vec{x}_h) \wedge (\Box\neg\alpha \wedge \neg\alpha)(\vec{x}_1, \ldots, \vec{x}_i)& \\
\to \ldots \to \Box\forall\vec{x}_n[\Delta_n(\vec{x}_1, \ldots, \vec{x}_n) \to \bot] \ldots]]] \ldots],&
\end{aligned}$$

where for each j, $1 \leq j \leq n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in Δ. Since all the constants in α are among $\vec{c}_1, \ldots, \vec{c}_i$, $i < h$, $\vec{x}_h$ do not occur in α, and so by Lemma 1.1(h),

$$\begin{aligned}
\vdash \forall\vec{x}_1[\Delta_1 \to \ldots \to \Box\forall\vec{x}_{h-1}[\Delta_{h-1} \wedge \Diamond\alpha(\vec{x}_1, \ldots, \vec{x}_i) \to [\Diamond\alpha(\vec{x}_1, \ldots, \vec{x}_i)& \\
\to \Box\forall\vec{x}_h[\Delta_h \wedge (\Box\neg\alpha \wedge \neg\alpha)(\vec{x}_1, \ldots, \vec{x}_i)& \\
\to \ldots \to \Box\forall\vec{x}_n[\Delta_n \to \bot] \ldots]]] \ldots],&
\end{aligned}$$

then

$$\begin{aligned}
\vdash \forall\vec{x}_1[\Delta_1 \to \ldots \to \Box\forall\vec{x}_{h-1}[\Delta_{h-1} \wedge \Diamond\alpha(\vec{x}_1, \ldots, \vec{x}_i)& \\
\to \Box\forall\vec{x}_h[\Delta_h \wedge (\Box\neg\alpha \wedge \neg\alpha)(\vec{x}_1, \ldots, \vec{x}_i)& \\
\to \ldots \to \Box\forall\vec{x}_n[\Delta_n \to \bot] \ldots]] \ldots],&
\end{aligned}$$

contrary to the **L**-coherence of Γ'. Hence $\Gamma' \cup \{\langle s, \alpha\rangle\}$ is **L**-coherent and so, by Lemma 2.5(a), $\Gamma \cup \{\langle s, \alpha\rangle\}$ is **L**-coherent too.

Lemma 2.12 *If A is an* **L**-*consistent set of closed formulas of* $\mathfrak{L}$, *then the diagram* $\{\langle 0, \varphi\rangle : \varphi \in A\}$ *is* **L**-*coherent.*

3 Completeness results

Definition 3.1 Let Δ be a diagram. For all $v \in \mathrm{Supp}(\Delta)$, $\mathcal{L}_v^\Delta$ is the sublanguage of $\mathcal{L}_v$ that contains only those constants of $\mathcal{L}_v$ occurring in formulas of $\Delta(v)$.

Definition 3.2 Let Δ be an **L**-coherent diagram. Δ is *complete* iff for all $v, w \in \mathrm{Supp}(\Delta)$

(1) if $v < w$, then $\mathcal{L}_v^\Delta \subseteq \mathcal{L}_w^\Delta$, and
(2) for any $\alpha \in \mathcal{L}_v^\Delta$, $\langle v, \alpha \rangle \in \Delta$ or $\langle v, \neg\alpha \} \in \Delta$,

Δ is *rich* iff for all $v \in \mathrm{Supp}(\Delta)$,

(1) if $\exists x \alpha(x) \in \Delta(v)$ then there is a constant $d \in \mathcal{L}_v$ such that $\alpha(x/d) \in \Delta(v)$, and
(2) if $\langle v, \Diamond\alpha \rangle \in \Delta$ then there is a $w \in \mathrm{Supp}(\Delta)$ such that $v < w$ and $\langle w, \alpha \rangle \in \Delta$.

Δ is said to be **L**-*saturated* iff Δ is **L**-coherent, complete and rich.

Theorem 3.3 *If A is an* **L**-*consistent set of closed formulas of $\mathcal{L}$, then there is an* **L**-*saturated diagram Δ such that* $\mathrm{Supp}(\Delta) \subseteq Q^+$ *and* $\{\langle 0, \varphi \rangle : \varphi \in A\} \subseteq \Delta$.

Proof: Let $\langle v_1, \alpha_1 \rangle, \langle v_2, \alpha_2 \rangle \ldots$ be an enumeration of all the elements of $\mathcal{P}$ (see Definition 2.1). Let us define the following chain of quasi-finite diagrams.

$$\Delta_0 = \{\langle 0, \varphi \rangle : \varphi \in A\}$$

Let $k \geq 1$. First define

$$\Delta'_k = \begin{cases} (\Delta_{k-1}) \cup \{\langle v_k, \alpha_k \rangle\}, & \text{if } (\Delta_{k-1}) \cup \{\langle v_k, \alpha_k \rangle\} \text{ is } \mathbf{L}\text{-coherent}, \\ (\Delta_{k-1}) \cup \{\langle v_k, \neg\alpha_k \rangle\}, & \text{if } (\Delta_{k-1}) \cup \{\langle v_k, \alpha_k \rangle\} \text{ is not } \mathbf{L}\text{-coherent and} \\ & (\Delta_{k-1}) \cup \{\langle v_k, \neg\alpha_k \rangle\} \text{ is } \mathbf{L}\text{-coherent}, \\ \Delta_{k-1} & \text{otherwise.} \end{cases}$$

Then let

$$\Delta_k = \begin{cases} \Delta'_k \cup \{\langle v_k, \gamma(x/d) \rangle\}, & \text{if } \alpha_k = \exists x \gamma(x),\ \langle v_k, \alpha_k \rangle \in \Delta'_k, \text{ and } d \text{ is a} \\ & \text{constant of } C_{v_k} \text{ such that } \Delta'_k \cup \{\langle v_k, \gamma(x/d) \rangle\} \\ & \text{is an } \mathbf{L}\text{-coherent diagram;} \\ \Delta'_k \cup \{\langle s, \gamma \rangle\}, & \text{if } \alpha_k = \Diamond\gamma,\ \langle v_k, \alpha_k \rangle \in \Delta'_k,\ s \text{ is some element} \\ & \text{of } Q^+ \text{ such that } s > v_k, \text{ and } \Delta'_k \cup \{\langle s, \gamma \rangle\} \\ & \text{is } \mathbf{L}\text{-coherent;} \\ \Delta'_k, & \text{otherwise.} \end{cases}$$

$$\Delta = \bigcup_{k \in N} \{\Delta_k\}.$$

First, Δ is **L**-coherent. For, each Δ_k is an **L**-coherent diagram by construction; moreover, if Δ is not **L**-coherent, then there is a finite diagram Γ, $\Gamma \subseteq \Delta$, which is not **L**-coherent. Let k be the maximum index assigned to the formulas of Γ in the enumeration of $\mathcal{P}$. Now $\Gamma \subseteq \Delta_{k+1}$, so by Lemma 2.5(a), Δ_{k+1} should be **L**-incoherent, whereas it is not.

Second, Δ is complete.

(1) If $v \in \text{Supp}(\Delta)$, then by Lemma 2.6 $\langle v, \Box(\alpha \to \alpha)\rangle \in \Delta$ where α contains constants of $\mathcal{L}_v^\Delta$. By Lemma 2.7, $\langle w, (\alpha \to \alpha)\rangle \in \Delta$ for all $w > v$, and so $\mathcal{L}_v^\Delta \subseteq \mathcal{L}_w^\Delta$.

(2) We have to show that if $v \in \text{Supp}(\Delta)$ and $\alpha \in \mathcal{L}_v^\Delta$, then $\langle v, \alpha\rangle \in \Delta$ or $\langle v, \neg\alpha\rangle \in \Delta$. Let $\langle v, \alpha\rangle = \langle v_h, \beta_h\rangle$ in the enumeration we started with and suppose by *reductio* that neither $\langle v_h, \beta_h\rangle$ nor $\langle v_h, \neg\beta_h\rangle$ can be **L**-coherently added to Δ_{h-1}. Therefore, by Lemma 2.5(a) neither of them can ever be **L**-coherently added to any extension of Δ_{h-1}. Since $\alpha \in \mathcal{L}_v^\Delta$, there is a point k in the construction of Δ when all the rational numbers, of which the constants of α are proper constants, do belong to the support of Δ_k. Therefore, by Lemma 2.8, for any $h > k$, either $\Delta_h \cup \{\langle v_h, \beta_h\rangle\}$ or $\Delta_h \cup \{\langle v_h, \neg\beta_h\rangle\}$ is **L**-coherent, contrary to our supposition that neither of them can ever be **L**-coherently added to any extension of Δ_{h-1}.

Because of Lemmas 2.10 and 2.11, Δ is also rich. So the theorem is proved.

Definition 3.4 Let Δ be an **L**-saturated diagram. A Δ-*canonical model* $\mathcal{M}^\Delta = \langle W, R, D, I\rangle$ is defined as follows:

$W = \text{Supp}(\Delta)$, $R = <$, D is a function such that for all $v \in W$, D_v is the set of constants of $\mathcal{L}_v^\Delta$, I is the interpretation function such that for all $v \in W$, $c \in \mathcal{L}_v^\Delta$ and predicate letter P^n, $I_v(c) = c$ and $I_v(P^n) = \{\langle c_1, \dots, c_n\rangle : P^n(c_1, \dots, c_n) \in \Delta(v)\}$.

$\mathcal{M}^\Delta$, so defined, is a Kripke model with nested domains.

Lemma 3.5 *If $\mathcal{M}^\Delta = \langle W, R, D, I\rangle$ is a Δ-canonical model for some* **L**-*saturated diagram Δ, then, for all closed wffs $\alpha \in \mathcal{L}_v^\Delta$,*

$$\mathcal{M}^\Delta \models_v \alpha \text{ iff } \langle v, \alpha\rangle \in \Delta.$$

Proof: Standard.

Theorem 3.6 *If A is a* **QK4.3**-*consistent set of formulas, then there is a Kripke model $\mathcal{M}$ with nested domains based on a strict linear frame, such that $\mathcal{M} \models A$.*

Proof: Start with the diagram $\{\langle 0, \varphi\rangle : \varphi \in A\}$ and extend it to a **QK4.3**-saturated diagram Δ. The Δ-canonical model is the required model.

4 Completeness for QK4.3.D.X and QS4.3 A feature of **QK4.3** is that, given a **QK4.3**-coherent diagram we cannot add, in general, a new point w (say, for example, the pair $\langle w, \top\rangle$) and still obtain a **QK4.3**-coherent diagram. Analogously, we cannot add to a diagram Δ the pair $\langle w, \alpha\rangle$, if α contains constants that are proper constants of rational numbers not belonging to the support of Δ. This feature does not hold anymore for extensions of **QK4.3.D.X**, in the sense that given a diagram Δ whose support is $\langle v_1, \dots, v_n\rangle$ we can always add new points 'after v_1'; therefore, if the support contains the zero, we can add to Δ all of the positive rational numbers. The proviso 'after v_1' can be dropped for extensions of **QS4.3**.

In this section we denote by **L.D.X** any logic extending **QK4.3.D.X**.

Lemma 4.1 *Let Γ be an* **L.D.X**-*coherent diagram.*
(a) *For any $z \in \mathrm{Supp}(\Gamma)$ and $w \geq z$, $\Gamma \cup \{\langle w, \top \rangle\}$ is* **L.D.X**-*coherent.*
(b) *If $0 \in \mathrm{Supp}(\Gamma)$, then for any wff α, the diagram Γ_α is* **L.D.X**-*coherent, where $\Gamma_\alpha = \Gamma \cup \{\langle z, \top \rangle$: there is a constant $c \in C_z$ that occurs in $\alpha\}$.*

Proof:
(a) If $\Gamma = \varnothing$, then $\Gamma \cup \{\langle w, \top \rangle\}$ is not **L.D.X**-coherent iff **L.D.X** $\vdash \top \to \bot$ iff **L.D.X** $\vdash \bot$, which is not the case because of the consistency of **L.D.X**.

Let $\Gamma \neq \varnothing$. If $w \in \mathrm{Supp}(\Gamma)$, then $\Gamma \cup \{\langle w, \top \rangle\}$ is trivially **L.D.X**-coherent. So let $w \notin \mathrm{Supp}(\Gamma)$. $\Gamma \cup \{\langle w, \top \rangle\}$ is not **L.D.X**-coherent iff for some finite subdiagram Δ of Γ, $\Delta \cup \{\langle w, \top \rangle\}$ is not **L.D.X**-coherent. Let $\mathrm{Supp}(\Delta) = \langle v_1, \ldots, v_n \rangle$ and $v_{i-1} < w < v_i$. Then

$$\begin{aligned} \vdash \forall \vec{x}_1 [\Delta_1(\vec{x}_1) \to \ldots \to \Box \forall \vec{x}_{i-1} [\Delta_{i-1}(\vec{x}_1, \ldots, \vec{x}_{i-1}) \\ \to \Box [\top \to \Box \forall \vec{x}_i [\Delta_i(\vec{x}_1, \ldots, \vec{x}_i) \\ \to \ldots \to \Box \forall \vec{x}_n [\Delta_n(\vec{x}_1, \ldots, \vec{x}_n) \to \bot] \ldots]]] \ldots], \end{aligned}$$

where for each j, $1 \leq j \leq n$, $\vec{x}_j$ is substituted for the list $\vec{c}_j$ of proper constants of v_j occurring in Δ. Hence, by classical logic and Axiom **X**,

$$\vdash \forall \vec{x}_1 [\Delta_1 \to \ldots \to \Box \forall \vec{x}_{i-1} [\Delta_{i-1} \to \Box \forall \vec{x}_i [\Delta_i \to \ldots \to \Box \forall \vec{x}_n [\Delta_n \to \bot] \ldots]] \ldots],$$

contrary to the **L.D.X**-coherence of Γ.

If $v_n < w$, then $\Delta \cup \{\langle w, \top \rangle\}$ is not **L.D.X**-coherent iff

$$\vdash \forall \vec{x}_1 [\Delta_1 \to \ldots \to \Box \forall \vec{x}_n [\Delta_n \to \Box [\top \to \bot]] \ldots],$$

whence

$$\vdash \forall \vec{x}_1 [\Delta_1 \to \ldots \to \Box \forall \vec{x}_n [\Delta_n \to \Box \bot] \ldots].$$

Thus, by Axiom **D**,

$$\vdash \forall \vec{x}_1 [\Delta_1 \to \ldots \to \Box \forall \vec{x}_n [\Delta_n \to \bot] \ldots],$$

contrary to the **L.D.X**-coherence of Γ.

(b) Immediate from (a).

Corollary 4.2 *Let Δ be an* **L.D.X**-*coherent diagram such that* $0 \in \mathrm{Supp}(\Delta)$.
(a) *For any $w \in Q^+$, $\alpha \in \mathcal{L}_w$, if $\Delta(w) \vdash \alpha$, then $\Delta' = \Delta \cup \{\langle w, \alpha \rangle\}$ is* **L.D.X**-*coherent.*
(b) *For any $w \in Q^+$ and any $\alpha \in \mathcal{L}_w$, either $\Gamma \cup \{\langle w, \alpha \rangle\}$ is* **L.D.X**-*coherent or $\Gamma \cup \{\langle w, \neg\alpha \rangle\}$ is* **L.D.X**-*coherent.*
(c) *$\Delta \cup \{\langle w, \alpha \vee \beta \rangle\}$ is* **L.D.X**-*coherent iff either $\Delta \cup \{\langle w, \alpha \rangle\}$ is* **L.D.X**-*coherent or $\Delta \cup \{\langle w, \beta \rangle\}$ is* **L.D.X**-*coherent.*

Proof: By Lemma 4.1(b), Δ_α is **L.D.X**-coherent, whence by Lemma 2.6, $\Delta_\alpha \cup \{\langle w, \alpha \rangle\}$ is **L.D.X**-coherent; therefore by Lemma 2.5(a), Δ' is **L.D.X**-coherent.
Analogously for (b) and (c).

As to the construction of Theorem 3.3 for logics extending **QK4.3.D.X**, the third case in the definition of Δ'_k can never be the case; hence $\mathrm{Supp}(\Delta) = Q^+$, and for any $v \in Q^+$, $\mathcal{L}_v^\Delta = \mathcal{L}_v$.

Theorem 4.3 *If A is a* **QK4.3.D.X**-*consistent set of formulas, then there is a Kripke model $\mathcal{M}$ based on the extended frame $\langle Q^+, <, D\rangle$ with nested domains, such that $\mathcal{M} \models A$.*

If A is a **QS4.3**-*consistent set of formulas, then there is a Kripke model $\mathcal{M}$ based on the extended frame $\langle Q^+, \leq, D\rangle$ with nested domains, such that $\mathcal{M} \models A$.*

Notice that Axiom **T** is needed only in the proof of Lemma 3.4 in order to show that $\mathcal{M}^\Delta \not\models_v \Box\alpha$ only if $\Box\alpha \notin \Delta(v)$.[1]

5 The logics C_m.QK4.3.D.X and C_m.QS4.3, $m \geq 1$

Consider the propositional schema

$$\mathbf{C}_m^\circ : \beta_1 \vee \Box(\Box\beta_1 \wedge \beta_1 \rightarrow \beta_2) \vee \ldots \vee \Box(\Box\beta_m \wedge \beta_m \rightarrow \bot).$$

It is well known that **K4.3** + $\mathbf{C}_m^\circ$ (**S4.3** + $\mathbf{C}_m^\circ$) is characterized by the class of connected, transitive (and reflexive) frames of rank at most m.

Given a transitive frame $\mathcal{F} = \langle W, R\rangle$, for every $v \in W$, the *cluster of v* is

$$Cl_v = \{w \in W : (wRv \wedge vRw) \text{ or } (w = v)\}.$$

Clusters can be ordered by $Cl_v \leq_R Cl_w$ iff vRw. $\leq_R$ is transitive and antisymmetric; therefore, by putting

$$Cl_v <_R Cl_w \quad \text{iff} \quad Cl_v \leq_R Cl_w \quad \text{and} \quad Cl_v \neq Cl_w$$

we get a transitive and irreflexive ordering of the clusters. If R is connected, $<_R$ is connected too. A cluster is *degenerate* if it consists of one non-reflexive point.

A frame $\mathcal{F}$ is said to be *of rank m* iff $\mathcal{F}$ contains at most m clusters. A (*reflexive*) *tower of rank m* is a frame $\langle W, R\rangle$ of rank m, where R is (reflexive) transitive, connected, serial, and dense.

One could expect that, for example, **Q.S4.3** + $\mathbf{C}_m^\circ$ would be characterized by the class of Kripke models with nested domains based on reflexive towers of rank at most m. On the contrary, in order to axiomatize that class of models, a stronger version of Axiom $\mathbf{C}_m^\circ$ is needed, one which says that certain exchanges between universal quantifiers and box operators are admitted.

Given the formulas $\alpha_1(\vec{x}_0, \vec{x}_1), \ldots, \alpha_i(\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_i), \ldots, \alpha_{m+1}(\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_{m+1})$, define

$$\gamma_1 = \alpha_1$$

$$\gamma_{i+1} = \Box\neg\alpha_i \wedge \neg\alpha_i \wedge \alpha_{i+1}$$

and put:

$$\mathbf{C}_m = \forall\vec{x}_1[\gamma_1 \rightarrow \Box\forall\vec{x}_2[\gamma_2 \rightarrow \ldots \rightarrow \Box\forall\vec{x}_m[\gamma_m \rightarrow \Box\forall\vec{x}_{m+1}[\gamma_{m+1} \rightarrow \bot]] \ldots]].$$

Notice that for $\alpha_{m+1} = \top$,

$$\mathbf{C}_m = \forall\vec{x}_1[\gamma_1 \rightarrow \Box\forall\vec{x}_2[\gamma_2 \rightarrow \ldots \rightarrow \Box\forall\vec{x}_m[\gamma_m \rightarrow \Box[\Box\neg\alpha_m \wedge \neg\alpha_m \rightarrow \bot]] \ldots]].$$

Lemma 5.1 *The logics* $\mathbf{C}_m$.**QK4.3.D.X** ($\mathbf{C}_m$.**QS4.3.**), $m \geq 1$, *are correct with respect to the* (*reflexive*) *towers of rank at most m and with nested domains.*

In this section, by $\mathbf{C}_m$.**L.D.X** we denote any logic which extends $\mathbf{C}_m$.**QK4.3.D.X**. Let φ be a closed wff of $\mathcal{L}$ and suppose that $\mathbf{C}_m$.**L.D.X** $\not\vdash \varphi$,

$m \geq 1$. Let $n = \mu s(\mathbf{C}_s.\mathbf{L.D.X} \nvdash \varphi)$. Of course $n \leq m$. If $n = 1$, then, since $\mathbf{C}_1.\mathbf{L.D.X} \vdash (\Diamond\beta \to \Box\Diamond\beta) \wedge (\beta \to \Diamond\beta)$ (see proof below), $\mathbf{C}_1.\mathbf{L.D.X}$ is equivalent to **QS5**, and so φ can be falsified on a reflexive tower of rank 1, i.e., on one nondegenerate cluster.

$\alpha_1 \to \Box(\Box\neg\alpha_1 \wedge \neg\alpha_1 \wedge \alpha_2 \to \bot)$	$\mathbf{C}_1$
$\Diamond\beta \to \Box(\Diamond\Diamond\beta \vee \Diamond\beta)$	$\alpha_1/\Diamond\beta$; $\alpha_2/\top$
$\Diamond\beta \to \Box\Diamond\beta$	by Ax. **4**
$\Box(\neg\Diamond\beta \to \beta) \to (\Box\neg\Diamond\beta \to \Box\beta)$	Ax. **K**
$\Box(\Diamond\beta \vee \beta) \to (\Diamond\Diamond\beta \vee \Box\beta)$	
$\beta \to \Box(\Box\neg\beta \wedge \neg\beta \to \bot)$	from $\mathbf{C}_1$
$\beta \to \Box(\Diamond\beta \vee \beta)$	
$\beta \to (\Diamond\Diamond\beta \vee \Box\beta)$	by transitivity
$\beta \to (\Diamond\beta \vee \Box\beta)$	by Ax. **4**
$\beta \to \Diamond\beta$	by Ax. **D**.

If $n > 1$, we show that φ can be falsified on a tower of rank n.

Let $\mathrm{P}^{(k)}_{n-1}$ be the universal closure of the instance of $\mathbf{C}_{n-1}$ obtained by putting:

$$\gamma_1 = \neg P_1(x_1, y_1, \ldots, y_k)$$

$$\gamma_{i+1} = \Box P_i(x_i, y_1, \ldots, y_k) \wedge P_i(x_i, y_1, \ldots, y_k) \wedge \neg P_{i+1}(x_{i+1}, y_1, \ldots, y_k),$$
$$1 \leq i \leq n,$$

where $P_1, \ldots, P_n$ are $k + 1$-ary predicate letters not occurring in φ.

Now we show that

$$\mathbf{C}_n.\mathbf{L.D.X} \nvdash \varphi \vee \Box\mathrm{P}^{(k)}_{n-1}, \text{ for some } k \geq 0.$$

For, $\mathbf{C}_{n-1}.\mathbf{L.D.X} \vdash \varphi$ and so

$$(*) \qquad \mathbf{C}_n.\mathbf{L.D.X} \vdash \Box\psi_1 \wedge \ldots \wedge \Box\psi_r \to \varphi,$$

where $\psi_1, \ldots, \psi_r$ are universal closures of instances of Axiom $\mathbf{C}_{n-1}$. Suppose, by *reductio*, that

$$\mathbf{C}_n.\mathbf{L.D.X} \vdash \varphi \vee \Box\mathrm{P}^{(k)}_{n-1}, \text{ for all } k \geq 0.$$

Therefore, by uniform substitution we get

$$\mathbf{C}_n.\mathbf{L.D.X} \vdash \varphi \vee \Box\psi_1 \quad \text{and} \ldots \text{and} \quad \mathbf{C}_n.\mathbf{L.D.X} \vdash \varphi \vee \Box\psi_r;$$

hence

$$\mathbf{C}_n.\mathbf{L.D.X} \vdash \varphi \vee (\Box\psi_1 \wedge \ldots \wedge \Box\psi_r),$$

and so by (*)

$$\mathbf{C}_n.\mathbf{L.D.X} \vdash \varphi$$

contrary to the hypothesis.

Consider the rational interval $[1, n + 1) = \{w \in Q^+ : 1 \leq w < n + 1\}$. Let $C_1, \ldots, C_n$ be n countable sets of constants pairwise disjoint and for every $w \in [i, i + 1)$, $i = 1, \ldots, n$, let

$$\mathcal{L}_w = \mathcal{L} \cup C_1 \cup \ldots \cup C_i$$

and

$$\mathcal{P} = \{\langle w,\alpha\rangle : w \in [1,n+1) \text{ and } \alpha \in \mathcal{L}_w\}.$$

Define the diagram

$$\begin{aligned}
\Delta = \{&\langle 1,\Diamond\neg\varphi\rangle,\langle 1,\Diamond\neg P_1(c_1,\vec{e})\rangle,\\
&\langle 2,\Box P_1(c_1,\vec{e})\rangle,\langle 2,\Diamond\neg P_2(c_2,\vec{e})\rangle,\\
&\qquad\quad \ldots\\
&\langle n-1,\Box P_{n-2}(c_{n-2},\vec{e})\rangle,\langle n-1,\Diamond\neg P_{n-1}(c_{n-1},\vec{e})\rangle,\\
&\langle n,\Box P_{n-1}(c_{n-1},\vec{e})\rangle,\langle n,\Diamond\neg P_n(c_n,\vec{e})\rangle\},
\end{aligned}$$

where $c_i \in C_i$, $1 \le i \le n$, $\vec{e} = e_1,\ldots,e_k \in C_1$ and k is such that $\mathbf{C_n.L.D.X} \nvdash \alpha \vee \Box \mathrm{P}^{(k)}_{n-1}$.

We will refer to Δ as the *base diagram* and to $1,\ldots,n$ as the *base points*. We can easily see that Δ is $\mathbf{C_n.L.D.X}$-coherent. For, if not, then $\mathbf{C_n.L.D.X} \vdash \varphi \vee \Box \mathrm{P}^{(k)}_{n-1}$, which is not the case.

In the remainder of this section, Δ will always denote the diagram we have just defined and the index i will always vary on the set of natural numbers $\{1,\ldots,n\}$.

It is important to notice that since $\langle i,\Diamond\neg P_i(c_i,\vec{e})\rangle \in \Delta$ and $\langle i+1,\Box P_i(c_i,\vec{e})\rangle \in \Delta$, i and $i+1$ are not only distinct, but can never belong to the same cluster. This leads to the following definition:

Definition 5.2 For any $i < n$, i and $i+1$ are said to be *strongly distinct* if there is a wff γ such that

$$(*) \qquad \Delta \cup \{\langle i,\gamma\rangle\} \text{ and } \Delta \cup \{\langle i+1, \Box\neg\gamma \wedge \neg\gamma\rangle\}$$

are both necessary extensions of Δ, where $\Delta \cup \{\langle w,\alpha\rangle\}$ is said to be a *necessary extension* of Δ iff $\Delta \cup \{\langle w,\neg\alpha\rangle\}$ is not $\mathbf{C_n.L.D.X}$-coherent.

When $(*)$ obtains, γ is said to *separate i from $i+1$.*

Given that the following is a theorem of $\mathbf{C_n.L.D.X}$:

$$\forall x_1[\Diamond\neg\partial_1(x_1) \to \Box\forall x_2[(\Box\partial_1(x_1) \wedge \Diamond\neg\partial_2(x_2)) \to \ldots \to \Box\forall x_n[(\Box\partial_{n-1}(x_{n-1}) \wedge \Diamond\neg\partial_n(x_n)) \to \Box[\Box\partial_n(x_n) \to \bot]\ldots]],$$

any diagram containing a subdiagram of this kind:

$$\begin{array}{ccccccccc}
1 & \cdots & 2 & \cdots & n-1 & & n & & n+1\\
\bullet & & \bullet & & \bullet & & \bullet & & \bullet\\
\Diamond\neg\partial_1 & & \Box\partial_1 & \cdots & \Box\partial_{n-2} & & \Box\partial_{n-1} & & \Box\partial_n\\
 & & \Diamond\neg\partial_2 & \cdots & \Diamond\neg\partial_{n-1} & & \Diamond\neg\partial_n & &
\end{array}$$

is not $\mathbf{C_n.L.D.X}$-coherent. In general, any diagram containing at least $n+1$ strongly distinct points is not $\mathbf{C_n.L.D.X}$-coherent.

The formulas $\Diamond\neg P_1(c_1,\vec{e})$, $\Box P_1(c_1,\vec{e}) \wedge \Diamond\neg P_2(c_2,\vec{e}),\ldots,\Box P_{n-1}(c_{n-1},\vec{e}) \wedge \Diamond\neg P_n(c_n,\vec{e})$ we started with serve the purpose of labelling the points $1,\ldots,n$ and guarantee that the support of Δ cannot contain further strongly distinct points. In the following we will write simply P_i instead of $P_i(c_i,\vec{e})$. It is easily seen that in any $\mathbf{C_n.L.D.X}$-coherent extension Γ of the base diagram Δ, for all wffs β,

$$\langle n, \Box\beta\rangle \in \Gamma \quad \text{iff} \quad \langle w, \Box\beta\rangle \in \Gamma, \text{ where } n \leq w.$$

Now we want to show that we can build a $\mathbf{C_n.L.D.X}$-saturated diagram Γ based on $[1, n+1)$ which is an extension of Δ and in which the above property holds for any w, v such that $i \leq w$, $v < i+1$. This obtains from the following lemmas.

Lemma 5.3 *Let Γ be a finite and $\mathbf{C_n.L.D.X}$-coherent diagram such that $\Gamma \supseteq \Delta$ and for all $w \in \mathrm{Supp}(\Gamma)$, if $i \leq w < i+1$ then $\langle w, \Diamond\neg P_i\rangle \in \Gamma$. Then*

(1) *if there is at least a point z between i and $i+1$, then for all $w \in \mathrm{Supp}(\Gamma)$, if $i \leq w < i+1$ and $\langle w, \beta\rangle \in \Gamma$ then $\Gamma \cup \{\langle w, \Diamond\beta\rangle\}$ is a necessary extension of Γ.*

(2) *if $\langle w, \Diamond\beta\rangle \in \Gamma$, $i \leq w < i+1$, then there is a $k = i, \ldots, n$ and an s such that $w \leq s$ and $k \leq s < k+1$ and $\Gamma \cup \{\langle s, \beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent. Moreover $\Gamma \cup \{\langle s, \beta\rangle\} \cup \{\langle s, \Diamond\neg P_k\rangle\}$ is a necessary extension of $\Gamma \cup \{\langle s, \beta\rangle\}$.*

(3) *if $\langle i, \Diamond\beta\rangle \in \Gamma$, then for any w, $i \leq w < i+1$, $\Gamma \cup \{\langle w, \Diamond\beta\rangle\}$ is a necessary extension of Γ;*

(4) *if for some w, $i \leq w < i+1$, $\langle w, \Box\beta\rangle \in \Sigma$ then $\Gamma \cup \{\langle i, \Box\beta\rangle\}$ is a necessary extension of Γ.*

Proof:

(1) $w \neq i$. Then $\Gamma \cup \{\langle i, \Diamond\beta\rangle\}$ is a necessary extension of Γ. $\Gamma \cup \{\langle i, \Diamond\beta\rangle\} \cup \{\langle w, \Box\neg\beta\rangle\}$ is **L**-incoherent, since it contains $n+1$ strongly distinct points, hence $\Gamma \cup \{\langle w, \Diamond\beta\rangle\}$ is **L**-coherent.

$w = i$. Suppose by *reductio* that $\Gamma \cup \{\langle i, \Diamond\beta\rangle\}$ is not a necessary extension of Γ, then $\Gamma \cup \{\langle i, \Box\neg\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, so also $\Gamma \cup \{\langle z, \Box\neg\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, for some $z, i < z < i+1$. Moreover, $\Gamma \cup \{\langle z, \Box\neg\beta\rangle\} \cup \{\langle z, \neg\beta\rangle\}$ contains $n+1$ strongly distinct points and so $\Gamma \cup \{\langle z, \Box\neg\beta\rangle\} \cup \{\langle z, \beta\rangle\}$ is a necessary extension of $\Gamma \cup \{\langle z, \Box\neg\beta\rangle\}$; therefore $\Gamma \cup \{\langle i, \Diamond\beta\rangle\}$ is a necessary extension of Γ. So we get a contradiction.

(2) Subcase 1. For all base elements i of Γ, $\Gamma \cup \{\langle i, \Box\neg\beta\rangle\}$ is not $\mathbf{C_n.L.D.X}$-coherent. Then $\Gamma \cup \{\langle n, \Diamond\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent and by Lemma 2.11, for some $s > n$, $\Gamma' = \Gamma \cup \{\langle n, \Diamond\beta\rangle\} \cup \{\langle s, \beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent. But then also $\Gamma' \cup \{\langle s, \Diamond\neg P_n\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, since $\Gamma' \cup \{\langle s, \Box P_n\rangle\}$ contains $n+1$ strongly distinct points ($\Diamond\neg P_n$ separates n from s) and consequently is not $\mathbf{C_n.L.D.X}$-coherent.

Subcase 2. There is a base element i of Γ, $w < i$, such that $\Gamma \cup \{\langle i, \Box\neg\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent. Let $k+1$ be the least of them (i is obviously greater than 1). If $\Gamma \cup \{\langle k+1, \beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, then the lemma is proved. Otherwise, by Lemma 2.11, for some $s > w$ and $k \leq s < k+1$, $\Gamma' = \Gamma \cup \{\langle k, \Diamond\beta\rangle\} \cup \{\langle s, \beta\rangle\} \cup \{\langle k+1, \Box\neg\beta \wedge \neg\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent. Moreover $\Gamma'' = \Gamma' \cup \{\langle s, \Diamond\neg P_k\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, since $\Gamma' \cup \{\langle s, \Box P_k\rangle\}$ contains $n+1$ strongly distinct points ($\Diamond\neg P_k$ separates k from s and β separates s from $k+1$) and consequently is not $\mathbf{C_n.L.D.X}$-coherent. Thus $\Gamma \cup \{\langle s, \beta\rangle\} \cup \{\langle s, \Diamond\neg P_k\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent.

(3) Suppose, by *reductio*, that for some w, $i \leq w < i+1$, $\Gamma \cup \{\langle w, \Diamond\beta\rangle\}$ is not $\mathbf{C_n.L.D.X}$-coherent. Then $\Gamma' = \Gamma \cup \{\langle w, \Box\neg\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent. But then we have the following situation: $\Diamond\beta$ separates i from w and $\Diamond\neg P_i$ separates

w from $i+1$, because $\langle v, \Diamond\neg P_i\rangle \in \Gamma$ and $\langle i+1, \Box P_i\rangle \in \Gamma$. Hence Γ' contains $n+1$ strongly distinct points and so is not $\mathbf{C_n.L.D.X}$-coherent. Consequently $\Gamma \cup \{\langle w, \Diamond\beta\rangle\}$ is a necessary extension of Γ.

(4) Suppose, by *reductio*, that $\Gamma \cup \{\langle i, \Box\beta\rangle\}$ is not a necessary extension of Γ. Then $\Gamma \cup \{\langle i, \Diamond\neg\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent. Hence by (3), $\Gamma \cup \{\langle i, \Diamond\neg\beta\rangle\} \cup \{\langle w, \Diamond\neg\beta\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, contrary to the fact that $\langle w, \Box\beta\rangle \in \Gamma$.

Lemma 5.4 *Let Γ be a finite and $\mathbf{C_n.L.D.X}$-coherent diagram such that $\Gamma \supseteq \Delta$, and for all $w \in \mathrm{Supp}(\Gamma)$, if $i \le w < i+1$ then $\langle w, \Diamond\neg P_i\rangle \in \Gamma$.*

(1) *If w_k and w_{k+1}, $i \le w_k$, $w_{k+1} < i+1$ are two consecutive points, then the diagram Γ' obtained by exchanging the order of w_k and w_{k+1} is $\mathbf{C_n.L.D.X}$-coherent.*

(2) *If $\langle w, \exists y\alpha(y)\rangle \in \Gamma$, then $\Gamma \cup \{\langle w, \alpha(y/d)\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, for some constant $d \in \mathcal{L}_w$.*

Proof:

(1) Let $\langle w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\rangle$ be the support of Γ, where $i \le w_k < w_{k+1} < i+1$. The conjunction of the formulas of $\Gamma(w_k)$ is of the form $(\Diamond\neg P_i \wedge \gamma)$ and the conjunction of the formulas of $\Gamma(w_{k+1})$ is of the form $(\Diamond\neg P_i \wedge \partial)$. By Lemma 5.3(1), $\Gamma \cup \{\langle w_k, \Diamond(\Diamond\neg P_i \wedge \gamma)\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent. Moreover, $\Gamma \cup \{\langle w_{k+1}, \Diamond(\Diamond\neg P_i \wedge \gamma)\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, for $\Gamma \cup \{\langle w_{k+1}, \Box\neg(\Diamond\neg P_i \wedge \gamma)\rangle\}$ contains $n+1$ strongly distinct points and so is not $\mathbf{C_n.L.D.X}$-coherent ($\Diamond(\Diamond\neg P_i \wedge \gamma)$ separates w_k from w_{k+1}). By Lemma 5.3(2), $\Gamma^* = \Gamma \cup \{\langle w_{k+1}, \Diamond(\Diamond\neg P_i \wedge \gamma)\rangle\} \cup \{\langle s, (\Diamond\neg P_i \wedge \gamma)\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent, where $w_{k+1} < s < i+1$. But Γ' (with w_k instead of s) is a subdiagram of Γ^* and consequently is $\mathbf{C_n.L.D.X}$-coherent.

w_1	$\cdots$	w_k	w_{k+1}	s	$\cdots$	w_n
$\bullet$		$\bullet$	$\bullet$	$\bullet$		$\bullet$
		$\Diamond\neg P_i \wedge \gamma$	$\Diamond\neg P_i \wedge \partial$			
		$\Diamond(\Diamond\neg P_i \wedge \gamma)$	$\Diamond(\Diamond\neg P_i \wedge \gamma)$	$\Diamond\neg P_i \wedge \gamma$		

(2) If $w = 1, \ldots, n$, then the lemma is proved as for **QK4.3**. Let $i < w < i+1$, for some $i = 1, \ldots, n$ and $v_1, \ldots, v_r$ be all the points of the support of Γ such that $i < v_1 < \ldots < v_r < w$. Let Γ^* be the diagram which differs from Γ only in that the relation $<$ is replaced by $<^*$ which coincides with $<$ except that the points i, $v_1, \ldots, v_r$, w are ordered as follows: $w <^* i <^* v_1 <^* \ldots <^* v_r$. Then by $r+1$ applications of Lemma 5.4(1), we get the result that the diagram Γ^* is $\mathbf{C_n.L.D.X}$-coherent. So by Lemma 2.10, $\Gamma^* \cup \{\langle w, \alpha(y/d)\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent and by applying Lemma 5.4(1) again $r+1$ times, we get the result that $\Gamma \cup \{\langle w, \alpha(y/d)\rangle\}$ is $\mathbf{C_n.L.D.X}$-coherent.

We can now extend our base diagram Δ to a $\mathbf{C_n.L.D.X}$-saturated one, as we did for **QK4.3**, being careful to add $\langle w, \Diamond\neg P_i\rangle$ to any new point. We can do this in virtue of Lemma 5.3(2).

Let $\mathcal{M}^\Gamma = \langle \mathrm{Supp}(\Gamma), <, D, I\rangle$ be the Γ-canonical model as in Definition 3.4. By Lemma 3.5, $\mathcal{M}^\Gamma \vDash_w \alpha$ iff $\langle w, \alpha\rangle \in \Gamma$.

Consider now the model $\mathcal{M} = \langle \mathrm{Supp}(\Gamma), R, D, I\rangle$ which is like $\mathcal{M}^\Gamma$ except that wRv iff either $w < v$ or $w, v \in [i, i+1)$. $\mathcal{M}$ is based on a tower of rank n.

Lemma 5.5 *For any sentence γ of $\mathcal{L}_w$,*

$$\mathcal{M} \models_w \gamma \text{ iff } \mathcal{M}^\Gamma \models_w \gamma.$$

Proof: By induction on γ. Consider the case of $\gamma = \Box\partial$.

If $\mathcal{M} \not\models_w \Box\partial$, then there is a v, wRv, such that $\mathcal{M} \not\models_v \partial$ and so by the induction hypothesis there is a v, wRv, such that $\mathcal{M}^\Gamma \not\models_v \partial$. Hence $\mathcal{M}^\Gamma \models_v \neg\partial$.

If $w < v$, then $\mathcal{M}^\Gamma \not\models_w \Box\partial$. If $w, v \in [i, i+1)$ for some base point i, then $\mathcal{M}^\Gamma \not\models_i \Box\partial$, hence by Lemmas 5.3(4), $\mathcal{M}^\Gamma \not\models_w \Box\partial$. The other direction is trivial. Consequently $\mathcal{M} \not\models \varphi$, where φ is the wff we started with. It follows that

Theorem 5.6 *If $\not\vdash_{\mathbf{C}_m.\mathbf{QK4.3.D.X}} \varphi$, then there is a Kripke model $\mathcal{M}$ with nested domains based on a tower of rank at most m, such that $\mathcal{M} \not\models \varphi$.*

If $\not\vdash_{\mathbf{C}_m.\mathbf{QS4.3}} \alpha$, then there is a Kripke model with nested domains $\mathcal{M}$ based on a reflexive tower of rank at most m, such that $\mathcal{M} \not\models \alpha$.

6 *Other extensions*

We recall that

$$\begin{aligned} \mathbf{BF} &= \forall x\Box\alpha \to \Box\forall x\alpha \\ \mathbf{ML}^* &= \Box\forall x\Diamond\Box\alpha \to \Diamond\Box\forall x\alpha \\ \mathbf{ML} &= \forall x\Diamond\Box\alpha \to \Diamond\Box\forall x\alpha \\ \mathbf{1} &= \Box\Diamond\alpha \to \Diamond\Box\alpha. \end{aligned}$$

It is clear that if **L** is any logic that extends **QS4.3** and φ is a wff not provable in **L**, then there is a model $\mathcal{M}$ for **L**, based on the frame $\langle Q^+, \leq\rangle$ with nested domains, such that $\mathcal{M} \not\models \varphi$. Let us call the frame $\langle Q^+, \leq\rangle$ *the rational frame.*

Using the constructions of Corsi and Ghilardi [2], it can be shown that **ML*.QS4.3** is characterized by $\langle Q^+ \cup N, \leq\rangle$ with nested domains and **ML*.QS4.3.1** is characterized by $\langle Q^+ \cup \{\infty\}, \leq\rangle$ with nested domains, where $\langle Q^+ \cup N, \leq\rangle$ is the rational frame with maximum cluster and $\langle Q^+ \cup \{\infty\}, \leq\rangle$ is the rational frame with maximum element.

As is well known, **BF.QS4.3 (BF.QK4.3.D.X)** is characterized by the extended frame $\langle Q^+, \leq, D\rangle$ ($\langle Q^+, <, D\rangle$), where the domain function D is constant. The proof of this result by the method of diagrams is very simple. With each rational number the same set C of new constants is associated and the notion of coherence for a finite diagram Δ is so modified:

Definition 6.1 Let Δ be a finite diagram whose support is $\langle v_1, \ldots, v_n\rangle$. Δ is said to be *coherent* iff

$$\not\vdash \forall\vec{x}[\Delta_1(\vec{c}/\vec{x}) \to \Box[\Delta_2(\vec{c}/\vec{x}) \to \ldots \to \Box[\Delta_n(\vec{c}/\vec{x}) \to \bot]\ldots]],$$

where $\vec{c}$ is the list of all the constants occurring in Δ.

Lemmas 2.6–2.11 are proved in the same way as for **QK4.3**. In the proof of Lemma 2.10 Axiom **BF** is needed because of the new definition of coherence. In the presence of Axiom **BF**, Axiom $\mathbf{C}_m$ is equivalent to

$$\begin{aligned} &\forall x_1\forall x_2\ldots\forall x_n[\beta_1(x_1) \vee \Box[(\Box\beta_1(x_1) \to \beta_2(x_2)) \\ &\vee \ldots \vee \Box[(\Box\beta_m(x_m) \to \beta_{m+1}(x_{m+1})]\ldots]], \end{aligned}$$

and so we easily get

Theorem 6.2 *The logics* **BF.C$_m$.QS4.3**, $m \geq 1$, *are characterized by the class of reflexive towers of rank at most m and with constant domains.*

Following [2], we easily see that **ML.BF.QS4.3** is characterized by the rational frame with maximum cluster and constant domains and that **ML.BF.QS4.3.1** is characterized by the rational frame with maximum point and constant domains.

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NOTE

1. It is worthwhile to notice that in the case of **QS4.3** a different definition of support and coherence could be given, see [1], i.e.:

Definition $\mathrm{Supp}(\Delta) =_{\mathrm{df}} \{w : \langle w, \alpha \rangle \in \Delta, \text{ for some wff } \alpha\}$.

If Δ is a finite diagram whose support is $\langle v_1, \ldots, v_n \rangle$, Δ is **QS4.3**-*coherent* iff

$$\nvdash_{\mathbf{QS4.3}} \Box\forall\vec{x}_1[\Delta_1(\vec{c}_1/\vec{x}_1) \rightarrow \Box\forall\vec{x}_2[\Delta_2(\vec{c}_1/\vec{x}_1, \vec{c}_2/\vec{x}_2) \rightarrow \ldots \rightarrow \Box\forall\vec{x}_n[\Delta_n(\vec{c}_1/\vec{x}_1, \vec{c}_2/\vec{x}_2, \ldots, \vec{c}_n/\vec{x}_n) \rightarrow \bot] \ldots]],$$

where for all k, $1 \leq k \leq n$, $\vec{c}_k$ is the list $c_{k1}, \ldots, c_{kj_k}$, $0 \leq j_k < \omega$, of all the constants of $(\mathcal{L}_k - \mathcal{L}_{k-1})$ occurring in wffs of Δ and $\Delta_k(\vec{c}_1/\vec{x}_1, \ldots, \vec{c}_k/\vec{x}_k)$ is the wff obtained by uniformly substituting for all h, $1 \leq h \leq k$, $\vec{x}_h$ for $\vec{c}_h$ in Δ_k. When $k = 1$, $\mathcal{L}_{k-1}$ denotes the language $\mathcal{L}$ we started with.

These definitions lead to some simplifications. For example, we do not need to take into consideration the diagram Δ_α, once Δ and α are given, but at the same time the proof of Lemma 4.1(a) depends on the fact that $\Box\forall x\Box\alpha(x) \rightarrow \Box\forall x\alpha(x)$ is a theorem of **QS4.3** and so Lemma 4.1(a) does not hold anymore for **QK4.3.D.X**.

Moreover in the proof of Lemma 2.11, the world s has to be chosen in such a way that $C \cap \mathcal{L}_s = \varnothing$, where C is the set of constants $\{c_1, \ldots, c_m\}$ occurring in $\Delta(v_h), \ldots, \Delta(v_n)$ and not belonging to $\mathcal{L}_{v_{h-1}}$. To this aim it is enough to choose an s such that $v_{h-1} < s < \min(z, v_h)$, where z is the least rational number of which $c_1, \ldots, c_m$ are proper constants.

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