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# Multi-Dimensional Semantics for Modal Logics

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**Abstract** We show that every modal logic (with arbitrary many modalities of arbitrary arity) can be seen as a multi-dimensional modal logic in the sense of Venema. This result shows that we can give every modal logic a uniform "concrete" semantics, as advocated by Henkin et al. This can also be obtained using the unravelling method described by de Rijke. The advantage of our construction is that the obtained class of frames is easily seen to be elementary and that the worlds have a more uniform character.

***1 Introduction*** Multi-dimensional (MD) modal logics are the topic of Venema [10]. We intuitively describe what is meant by this term. A MD modal logic is a semantically given modal logic in which the worlds of the frames are $\alpha$-long sequences, for some fixed ordinal $\alpha$, and the accessibility relations are defined in a uniform set-theoretic manner. A typical example of such a relation is "forgetting the $i$-th coordinate," i.e., for $s$, $t$ sequences, we define

$$s \equiv_i t \overset{\text{def}}{\Longleftrightarrow} (\forall j \neq i) s_j = t_j.$$

Then,

$$\mathfrak{M}, s \Vdash \Diamond_i \varphi \overset{\text{def}}{\Longleftrightarrow} (\exists t)(s \equiv_i t \;\&\; \mathfrak{M}, t \Vdash \varphi).$$

Note the similarity of this modality with the existential quantifier in first-order logic: $\exists x \varphi$ is satisfied at an assignment $s$ iff there exists an assignment $t$ which agrees with $s$ on every variable except maybe $x$, and $\varphi$ is satisfied at $t$. The simplest and probably best known MD modal logic is **S5**, in which the worlds are sequences of length 1, and the accessibility relation is defined as above. Recall that this logic is equivalent to monadic first-order logic. If we make the sequences longer, say $n$ long, and we add modalities for every coordinate, we arrive in cylindric modal logic, the modal counterpart of first-order logic with $n$ variables (cf. [11]). In this paper we start from

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another natural operation on sequences: composition of binary relations. For $V \subseteq U \times U$ and $\{s, r, t\} \subseteq V$, define

$$C_V srt \overset{\text{def}}{\iff} s_0 = r_0, r_1 = t_0 \;\&\; t_1 = s_1$$
$$\mathfrak{M}, s \Vdash \varphi \circ^V \psi \overset{\text{def}}{\iff} (\exists rt) : C_V srt \;\&\; \mathfrak{M}, r \Vdash \varphi \;\&\; \mathfrak{M}, t \Vdash \psi.$$

A well-investigated modal logic with $\circ^V$ as its main connective is *arrow logic* (cf., van Benthem [8], [9], Venema [10], Marx [6]). Traditionally, only the "square" version of arrow logic (in which one considers only frames whose universe is a full Cartesian product, that is, the modal counterpart of Representable Relation Algebras) was investigated. Recently, the nonsquare versions (in which the universe can be any subset of a Cartesian product) received a good deal of attention (cf., e.g., Maddux [5], Kramer [4], Marx [6]). If one considers a modal logic with one binary modality, the weakest derivation system (the generalization of the $K$ system for unary modal logic) turns out to be strongly sound and complete with respect to the class of multi-dimensional frames $\{\mathfrak{F} = \langle V, C_V \rangle : V \subseteq U \times U$ for some set $U\}$. This indicates that a natural interpretation for a binary modality is indeed ("nonsquare") relation composition. This result formed the basis for the theorem to be proved here and is an instance of the more general formulation.

We start with the necessary definitions. After that we look at modal logics with just one modality and show how to give them a multi-dimensional semantics. Having established these results we are ready for the general theorem which deals with modal logics with arbitrary many modalities. Then we show that the proposed frame-classes are elementary, and we finish with a discussion of the results.

***1.1 Preliminaries*** Arbitrary modalities are denoted by $\nabla$; their duals $\neg\nabla\neg$ by $\underline{\nabla}$. A *modal similarity type* $S$ is a pair $(O, \rho)$ with $O$ a set of logical connectives and $\rho : O \longrightarrow \omega$ a function assigning to each symbol in $O$ a finite rank or arity. We call $\mathcal{L}$ a *modal logic* of type $S = (O, \rho)$ if $\mathcal{L}$ is a tuple $(Fml, \mathsf{L}, \Vdash)$ in which,

- *Fml* is the smallest set containing countably many propositional variables and which is closed under the Boolean connectives and the connectives in $O$.
- $\mathsf{L}$ is a class of frames of the form $(W, R^\nabla)_{\nabla \in O}$, in which $W$ is a set, and each $R^\nabla$ is a subset of $W^{\rho\nabla+1}$.
- $\Vdash$ is the usual truth-relation from modal logic between models over frames in $\mathsf{L}$, worlds, and formulas. For the modal connectives it is defined as

$$\mathfrak{M}, x \Vdash \nabla(\varphi_1, \ldots, \varphi_{\rho\nabla}) \overset{\text{def}}{\iff} (\exists x_1 \ldots x_{\rho\nabla}) : R^\nabla x x_1 \ldots x_{\rho\nabla} \;\& \\ \mathfrak{M}, x_1 \Vdash \varphi_1 \;\&\ldots\&\; \mathfrak{M}, x_{\rho\nabla} \Vdash \varphi_{\rho\nabla}.$$

The *minimal derivation system* $K_S$ for a similarity type $S$ is defined as having only *CT* and *DB* as its axioms and only *MP*, *UG*, and *SUB* as its derivation rules.

| | | |
|---|---|---|
| (*CT*) | all classical tautologies | |
| (*DB*) | $\underline{\nabla}(p_1, \ldots, p_{i-1}, p \to p', p_{i+1}, \ldots, p_n) \leftrightarrow$ $\underline{\nabla}(p_1, \ldots, p_{i-1}, p, p_{i+1}, \ldots, p_n) \to \underline{\nabla}(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_n)$ | |
| (*MP*) | $\{\varphi, \varphi \to \psi\} \;/\; \psi$ | (Modus Ponens) |

| | | |
|---|---|---|
| $(UG)$ | $\varphi \,/\, \underline{\nabla}(\varphi_1, \dots, \varphi_{i-1}, \varphi, \varphi_{i+1}, \dots, \varphi_n)$ | (Universal Generalization) |
| $(SUB)$ | $\varphi \,/\, \sigma\varphi,$ | (Substitution) |

where $\nabla$ is an $n$-adic operator and $\sigma$ a substitution.

We recall the standard modal logical result that $K_S$ is strongly sound and complete with respect to the class of all frames of type $S$. We denote this class by $\mathsf{K}_S$.

If $\mathfrak{F}$ is a frame, we use $F$ to denote its universe. A modal logic $(Fml, \mathsf{L}, \Vdash)$ is called *$\alpha$-dimensional* if

- $(\forall \mathfrak{F} \in \mathsf{L}) : F \subseteq {}^{\alpha}U$ for some set $U$, and
- $(\forall \mathfrak{F}, \mathfrak{G} \in \mathsf{L}) : F = G \Rightarrow \mathfrak{F} = \mathfrak{G}$.

A frame $\mathfrak{F}_a$ is called *rooted*, if $a \in F$ and the subframe of $\mathfrak{F}_a$ generated by $a$ is $\mathfrak{F}_a$. For $\mathfrak{F}$, $\mathfrak{G}$, two frames of type $S = (O, \rho)$, we say that $\mathfrak{F}$ and $\mathfrak{G}$ are *$S$-bisimular* (notation: $\mathfrak{F} \rightleftharpoons_S \mathfrak{G}$) if there exists a nonempty relation $B \subseteq F \times G$ such that, for every $\nabla \in O$,

- if $R^{\nabla}_{\mathfrak{F}} x_0 x_1 \dots x_{\rho\nabla}$ and $x_0 B y_0$, then there exists $y_1, \dots, y_{\rho\nabla} \in G$ such that $R^{\nabla}_{\mathfrak{G}} y_0 y_1 \dots y_{\rho\nabla}$ and $x_i B y_i$ (forward condition),
- similarly in the other direction (backward condition).

The relation $B$ is called an *$S$-bisimulation*. A function $f : F \longrightarrow G$ is called an *$S$-zigzagmorphism* (also called *p*- or *bounded morphism*) from $\mathfrak{F}$ onto $\mathfrak{G}$ (notation: $\mathfrak{F} \overset{f}{\twoheadrightarrow} \mathfrak{G}$), if $f$ is surjective and $f$ is an $S$-bisimulation. Note that the forward condition then states that $f$ is a homomorphism. For $\mathsf{K}$ a class of frames, we use $\mathbf{Zig}\mathsf{K}$ to denote the class of all zigzagmorphic images of members of $\mathsf{K}$.

## *2 Multi-dimensional semantics for modal logics*

***2.1 Logics with one modality*** The major connective in this article is a generalization of dyadic composition to $n$-ary relations. We use $\bullet$ to denote this operator (the context provides its specific arity). The $n$-adic connective $\bullet$ has the following definition on $n$-dimensional frames. Let $V \subseteq U^n$, $\mathfrak{M} = (V, \mathsf{v})$ be a model, and $(x_0, x_1, \dots, x_{n-1}) \in V$. Then

$$\begin{array}{llll} & \mathfrak{M}, (x_0, x_1, \dots, x_{n-1}) & \Vdash & \bullet(\varphi_0, \varphi_1, \dots, \varphi_{n-1}) \quad \stackrel{\text{def}}{\Longleftrightarrow} \\ (\exists \mathbf{z}) & \mathfrak{M}, (\mathbf{z}, x_1, \dots, x_{n-1}) & \Vdash & \varphi_0 \quad \& \\ & \mathfrak{M}, (x_0, \mathbf{z}, x_2, \dots, x_{n-1}) & \Vdash & \varphi_1 \quad \& \\ & \quad\vdots & & \\ & \mathfrak{M}, (x_0, \dots, x_{n-2}, \mathbf{z}) & \Vdash & \varphi_{n-1}. \end{array}$$

For $n = 1$, we get the universal **S5** modality; for $n = 2$, $\bullet$ can be seen as the composition modality defined in the introduction (only the order of the arguments is reversed, i.e., $\varphi \bullet \psi = \psi \circ^V \varphi$).

For convenience, we use the substitution function $(\cdot)^i_z : U^n \longrightarrow U^n$, which is defined as follows: for $s \in U^n$, $z \in U$ and $0 \leq i, j < n$, we set

$$s^i_z(j) = \begin{cases} z & \text{if } i = j \\ s(j) & \text{otherwise.} \end{cases}$$

Then we can define $\bullet$ easily as follows: for $s \in V$, we have

$$s \Vdash \bullet(\varphi_0, \ldots, \varphi_{n-1}) \iff (\exists z) : s^0_z \Vdash \varphi_0 \,\&\ldots\&\, s^{n-1}_z \Vdash \varphi_{n-1}. \tag{1}$$

**Definition 2.1** For $n \geq 2$, $\mathsf{GC}_n$ denotes the class of all $n$-dimensional frames whose universe is a subset of $U^n$, for some base set $U$. The accessibility relation for $\bullet$ in a $\mathsf{GC}_n$-frame with universe $V$ is denoted by $C_V$. It is defined as $C_V sr^0 \ldots r^{n-1} \iff (\exists z)(\forall i) : s^i_z = r^i$.

We are ready to formulate the first theorem, dealing with modalities of rank higher than one.

**Theorem 2.2** *Let $S = (\bullet, n)$ be any similarity type in which the rank of $\bullet$ is higher than 1. Then $K_S$ is strongly sound and complete with respect to the class $\mathsf{GC}_{\rho(\bullet)}$.*

*Proof:* The theorem is an immediate consequence of the standard modal logical completeness result mentioned above and the fact that $\mathsf{K}_{(\bullet,n)} = \mathbf{Zig}\mathsf{GC}_n$, which follows simply from the next claim.

**Claim 2.3** *Let $\mathfrak{F}_a = (W, R)$, with $R \subseteq W^{n+1}$, for $n \geq 2$, be a rooted frame. Then $\mathfrak{F}_a$ is a zigzagmorphic image of a (rooted) frame $\mathfrak{G}_{a'} = (V, C_V)$, with $V \subseteq {}^nU$, for some set $U$, and $C_V$ is generalized composition.*

*Proof of claim:* Let $\mathfrak{F}_a$ be as stated in the lemma. We define the rooted frame $(V, C_V)$, its base $U$, and the zigzagmorphism simultaneously. The set $U$ will consist of strings of symbols, denoted by $\ulcorner \ldots \urcorner$. In these strings we code why we add an element to $U$. Define the binary relation $B$ as the smallest set such that

- $((\ulcorner a1 \urcorner, \ldots, \ulcorner an \urcorner), a) \in B$, and
- if $(s, x) \in B$ and $Rxy_0 \ldots y_{n-1}$, then $(s^i_{\ulcorner (s,x)Rxy_0\ldots y_{n-1}\urcorner}, y_i) \in B$.

Let $V$ be the domain of $B$ and $\mathfrak{G}_{(a1,\ldots,an)} = (V, C_V)$. We claim that $B$ is a zigzagmorphism from $\mathfrak{G}_{(a1,\ldots,an)}$ onto $\mathfrak{F}_a$. By the definition of $B$, $(*)$ below holds:

(*) $(r, w) \in B \Rightarrow$ $r = (\ulcorner a1 \urcorner, \ldots, \ulcorner an \urcorner)$ & $w = a$, or there exists unique $s \in dom(B)$, $\{x, y_0, \ldots, y_{n-1}\} \subseteq W$ and $i$ such that either $r = s$ & $w = x$ or $r = s^i_{\ulcorner (s,x)Rxy_0\ldots y_{n-1}\urcorner}$ & $w = y_i$.

So, $B$ is a function. It clearly obeys the backward condition of a bisimulation. Because $B$ is a function, the forward condition states that $B$ is a homomorphism, which follows simply from $(*)$. It is surjective, because $\mathfrak{F}_a$ is generated by $a$. So $B$ is a zigzagmorphism. Hence, the claim.

This finishes the proof of the theorem. □

Now we consider modal logics with one monadic modality.

**Definition 2.4** Consider a similarity type with one modality $\Diamond$ of rank 1. $\mathsf{GC}_{\Diamond}$ denotes the class of all two-dimensional frames, where $\Diamond$ is interpreted as the domino operator, i.e., for pairs $(a, b)$ in the domain of $\mathfrak{M}$,

$$\mathfrak{M}, (a, b) \Vdash \Diamond\varphi \iff (\exists z) : \mathfrak{M}, (b, z) \Vdash \varphi.$$

**Theorem 2.5** *Let $\Diamond$ be a monadic modality. The basic derivation system $K_\Diamond$ is strongly sound and complete with respect to the class of frames* $\mathsf{GC}_{\Diamond}$.

*Proof:* Soundness is immediate. For completeness, suppose $\Sigma \not\vdash_{K_\Diamond} \varphi$. By standard modal reasoning we find a frame $\mathfrak{F}_d = (W, R)$ such that

1. $\mathfrak{F}_d, d \models \Sigma$, but $\mathfrak{F}_d, d \not\models \varphi$,
2. $\mathfrak{F}_d$ is point-generated by $d$, and
3. $\mathfrak{F}_d \models \forall xyz((Ryx \wedge Rzx) \rightarrow y = z)$ ($\mathfrak{F}_d$ is "unravelled").

Let $\mathfrak{F}_{(d',d)}$ denote the $\mathsf{GC}_{\Diamond}$-frame with universe $\{(d', d)\} \cup \{(a, b) \in W \times W : Rab\}$. Using conditions (2) and (3) above it is easy to see that the function $h$ defined by $h(a, b) = b$ is an isomorphism between $\mathfrak{F}_d$ and $\mathfrak{F}_{(d',d)}$. But then, by (1), we are done. $\square$

Finally we consider the (trivial) case of modal logics with just one modal constant.

**Definition 2.6** Consider a similarity type with one modal constant $\nu$. $\mathsf{GC}_0$ denotes the class of all $\mathsf{GC}_2$-frames, where $\nu$ is interpreted as the identity constant, i.e.,

$$\mathfrak{M}, (a, b) \Vdash \nu \iff a = b.$$

The accessibility relation of $\nu$ in a $\mathsf{GC}_0$-frame with universe $V$ is denoted by $D^{01}_V$.

**Theorem 2.7** *Let $\nu$ be a modal constant. The basic derivation system $K_\nu$ is strongly sound and complete with respect to the class of frames* $\mathsf{GC}_0$.

*Proof:* This follows immediately from the following simple fact:

$$\text{every frame } (W, R) \text{ with } R \subseteq W \text{ is isomorphic to a frame in } \mathsf{GC}_0. \tag{2}$$

(Hint: note that every frame with one unary accessibility relation is a disjoint union of one-element frames.) $\square$

***2.2 Logics with arbitrary many modalities*** We are almost ready to prove the generalization of the last three theorems to arbitrary similarity types $S$: a multi-dimensional semantics for which the basic derivation system $K_S$ is sound and complete. The idea of the semantics is that we interpret the modalities as described in the previous section, and different modalities are interpreted on disjoint parts of the sequences. For instance, the semantics of a modal logic with two dyadic modalities will be given by 4-dimensional frames, on which one modality is interpreted as composition on the first two coordinates, and the other modality is interpreted on the last two coordinates.

Thus we will use generalized composition of rank $n$ on sets of relations with rank higher than $n$, say $\alpha$. The idea is that the connective works only on a specific subsequence of length $n$. On that part, it behaves just like $n$-adic composition. We define these modalities as follows. Let $V \subseteq U^\alpha$, $j, j + (n - 1) < \alpha$, and let $\kappa = (j, j + 1, \ldots, j + (n - 1))$ be a sequence of consecutive numbers. Let $\overset{\kappa}{\bar{s}}$ denote any sequence obtained from $s$ by changing some of the coordinates *outside* $\kappa$. Then for $s \in V$, we define

$$\mathfrak{M}, s \Vdash \bullet_\kappa(\varphi_0, \ldots, \varphi_{n-1}) \iff (\exists z) : \mathfrak{M}, \overset{\kappa}{\bar{s}}{}^{\kappa(0)}_z \Vdash \varphi_0 \,\&\ldots\&\, \mathfrak{M}, \overset{\kappa}{\bar{s}}{}^{\kappa(n-1)}_z \Vdash \varphi_{n-1}. \tag{3}$$

The accessibility relation of $\bullet_\kappa$ on an $\alpha$-dimensional frame with universe $V$ is denoted by $C_V^{\kappa(0),\rho(\bullet_\kappa)}$. Note that if $s$ and $\kappa$ are sequences of the same length, then we get the definition from the previous section. In the other cases, we hid the existential quantification over coordinates outside $\kappa$ in $\overset{\kappa}{s}$. As an example, the definition of $\bullet_{(2,3,4)}$ on sets of 7-ary relations is given below: (a "$-$" indicates that any element is allowed at this place)

$$
\begin{array}{lll}
& \mathfrak{M}, (x, y, a, b, c, v, w) & \Vdash \quad \bullet_{(2,3,4)}(\varphi_0, \varphi_1, \varphi_2) \\
& & \Updownarrow \\
& (x, y, a, b, c, v, w) & \in \quad V \ \& \\
(\exists \mathbf{z}) & \mathfrak{M}, (-, -, \mathbf{z}, b, c, -, -) & \Vdash \quad \varphi_0 \ \& \\
& \mathfrak{M}, (-, -, a, \mathbf{z}, c, -, -) & \Vdash \quad \varphi_1 \ \& \\
& \mathfrak{M}, (-, -, a, b, \mathbf{z}, -, -) & \Vdash \quad \varphi_2 .
\end{array}
$$

In a similar way we extend the definition of the domino operator and the identity constant; for $i, i{+}1 < \alpha$, and $s \in V$ we set

$$
\begin{array}{lcl}
\mathfrak{M}, s \Vdash \diamondsuit_{(i,i+1)}\varphi & \Longleftrightarrow & \text{there exists an } r \text{ s.t. } r(i) = s(i+1) \text{ and } \mathfrak{M}, r \Vdash \varphi \\
\mathfrak{M}, s \Vdash \iota\delta_{ij} & \Longleftrightarrow & s_i = s_j .
\end{array}
$$

In the next definition, we specify a multi-dimensional semantics for any modal type. A concrete example is provided in the proof of Theorem 2.9.

**Definition 2.8** Let $S = \{O, \rho\}$ be an arbitrary similarity type. $\mathsf{GC}_S$ denotes[1] the class of all $\alpha$-dimensional frames whose universe is a subset of an $\alpha$-dimensional cube, and where

- $\alpha = 2 \cdot |\{\nabla \in O : 0 \leq \rho(\nabla) \leq 2\}| + \sum_{2<n}(n \cdot |\{\nabla \in O : \rho(\nabla) = n\}|)$, and
- the modal constants are interpreted as diagonals $\iota\delta_{ij}$,
- the monadic modalities are interpreted as in $\mathsf{GC}_{\diamondsuit}$, but now using the domino operator $\diamondsuit_{(i,i+1)}$ on $\alpha$-sequences, as described above,
- for $n > 1$, all modalities of rank $n$ are interpreted as $\bullet_\kappa$ of rank $n$, and
- the $\iota\delta_{ij}$, the $\diamondsuit_{(i,i+1)}$, and $\bullet_\kappa$ are chosen such that all modalities are interpreted on pairwise disjoint parts of the $\alpha$-sequences.

We defined $\alpha$ precisely large enough such that every modality can be interpreted as in the previous section on separate subsequences of $\alpha$. In this way we ensured that there is no interaction between the different modalities.

The following theorem is a joint result with István Németi and Ildikó Sain.

**Theorem 2.9** *Let $S$ be an arbitrary similarity type. Then $K_S$ is strongly sound and complete with respect to the (multi-dimensional) class $\mathsf{GC}_S$.*

*Proof:* If $S$ consists of just one modality, the theorem follows from one of Theorems 2.2, 2.5, or 2.7. If we have more than one modality, we have to do additional work. Using the argument for monadic modalities given in the proof of Theorem 2.5 and standard modal reasoning, it is easy to prove the theorem from the following claim.

**Claim 2.10** *Let $\mathfrak{F} = (W, R_\nabla)_{\nabla \in S}$ be a frame where all relations are either unary or have rank higher than two. Then $\mathfrak{F}$ is a zigzagmorphic image of a $\mathsf{GC}_S$-frame.*

$$
\begin{array}{ccccl}
 & & \mathfrak{F} = (W,\, R_{\triangle},\, R_{\nabla},\, R_{\nu}) & & \\
 & \swarrow & \downarrow & \searrow & \text{split the frame} \\
(W,\, R_{\triangle}) & & (W,\, R_{\nabla}) & & (W,\, R_{\nu}) \\
\uparrow h_1 & & \uparrow h_2 & & \uparrow h_3 \quad \text{zigzagmorphisms} \\
(V_1, C_{V_1}) & & (V_2, C_{V_2}) & & (V_3,\, D^{01}_{V_3}) \\
 & \searrow & \downarrow & \swarrow & \text{glue together} \\
 & & \mathfrak{G} = (V,\, C^{0,2}_V,\, C^{2,3}_V,\, D^{56}_V) & &
\end{array}
$$

Figure 1: Road map of the proof of theorem 2.9

*Proof of claim:* We describe the proof of this claim for the case of three modalities $\triangle$, $\nabla$, and $\nu$ with $\rho(\triangle) = 2$, $\rho(\nabla) = 3$, and $\rho(\nu) = 0$. It will be clear from the proof how to extend it to any set of modalities. A "road map" of this proof is given in Figure 1. Let $\mathfrak{F} = (W,\, R_{\triangle},\, R_{\nabla},\, R_{\nu})$. We show that this frame is a zigzagmorphic image of a frame $\mathfrak{G} = (V,\, C^{0,2}_V,\, C^{2,3}_V,\, D^{56}_V)$, with $V \subseteq U^7$, for some set $U$.

First we split $\mathfrak{F}$ into three frames, one for each relation. We apply $\mathsf{K}_{(\bullet,n)} = \mathbf{Zig}\mathsf{GC}_n$ and (2) to the three frames $(W,\, R_{\triangle})$, $(W,\, R_{\nabla})$, and $(W,\, R_{\nu})$ and obtain three frames $(V_1, C_{V_1})$, $(V_2, C_{V_2})$, and $(V_3,\, D^{01}_{V_3})$ in which $V_1$ and $V_3$ are binary relations on sets $U_1$ and $U_3$, respectively, and $V_2$ is a ternary relation on some set $U_2$. The relations $C_V$ and $D^{ij}_V$ are defined as stated before. By $\mathsf{K}_{(\bullet,n)} = \mathbf{Zig}\mathsf{GC}_n$ and (2), the frames $(W,\, R_{\triangle})$, $(W,\, R_{\nabla})$, and $(W,\, R_{\nu})$ are zigzagmorphic images of the frames $(V_1, C_{V_1})$, $(V_2, C_{V_2})$, and $(V_3,\, D^{01}_{V_3})$ by the functions $h_1, h_2$, and $h_3$, respectively. For convenience, we denote the $i$-the coordinate of a sequence $s$ by $s_i$.

We define,

$$
\begin{aligned}
V \;\stackrel{\text{def}}{=}\; & \{s \in (U_1 \cup U_2 \cup U_3)^7 : (s_0, s_1) \in V_1 \;\&\; (s_2, s_3, s_4) \in V_2 \;\&\\
& (s_5, s_6) \in V_3 \;\&\; h_1((s_0, s_1)) = h_2((s_2, s_3, s_4)) = h_3((s_5, s_6))\}.
\end{aligned}
$$

We define the frame $\mathfrak{G} = (V,\, C^{0,2}_V,\, C^{2,3}_V,\, D^{56}_V)$.

By writing out definitions, we see that $C^{0,2}_V srt$ if and only if $C_{V_1}(s_0, s_1)(r_0, r_1)(t_0, t_1)$, and similarly for the two other relations. Now, we define a function $h : V \longrightarrow W$ as $h(s) = h_1(s_0, s_1)$. Note that, for all $s \in V$, we have $h(s) = h_1(s_0, s_1) = h_2(s_2, s_3, s_4) = h_3(s_5, s_6)$. The reader might have expected that

$$
h \text{ is a zigzagmorphism from } \mathfrak{G} \text{ onto } \mathfrak{F}. \tag{4}
$$

We now prove (4).

*h is surjective*. Let $w \in W$, then (because $h_1, h_2, h_3$ are surjective) there exists $s \in V_1$, $r \in V_2$ and $t \in V_3$, such that $h_1(s) = h_2(r) = h_3(t) = w$. Thus, $(s_0, s_1, r_0, r_1, r_3, t_0, t_1)$ is in $V$, and its $h$-image equals $w$.

*h is a homomorphism.* Suppose that $C_V^{0,2}srt$ holds:

$$\begin{array}{lcl} C_V^{0,2}srt & \iff & \text{(by definition of } V \text{ and } C_V^{0,2}) \\ C_{V_1}(s_0,s_1)(r_0,r_1)(t_0,t_1) & \Rightarrow & (h_1 \text{ is a homomorphism}) \\ R_\triangle h_1(s_0,s_1)h_1(r_0,r_1)h_1(t_0,t_1) & \stackrel{\text{def}}{\iff} & R_\triangle h(s)h(r)h(t). \end{array}$$

Because $h(s) = h_1(s_0,s_1) = h_2(s_2,s_3,s_4) = h_3(s_5,s_6)$, the proofs for $C_V^{2,3}$ and $D_V^{56}$ are similar.

*h satisfies the zigzag condition.* Suppose $R_\nabla h(s)y_1y_2y_3$ holds. Then, since $h_2$ is zigzag and $h(s) = h_2(s_2,s_3,s_4)$, we find $y_1', y_2', y_3' \in V_2$, such that $C_{V_2}(s_2,s_3,s_4)$, $(y_1',y_2',y_3')$ and $h_2(y_j') = y_j$. Choose $r,t,v \in V$ which agree on the second, third and fourth coordinate with $y_1', y_2', y_3'$, respectively. Since all labelling functions are surjective, we can find such $r,t,v$. By definition of $C_V^{2,3}$ and $h$, we have $C_V^{2,3}srtv$ and $h(r) = y_1,\ h(t) = y_2$ & $h(v) = y_3$. The proofs for $C_V^{0,2}$ and $D_V^{56}$ are similar. This finishes the proof of 4.

We have proved the theorem for this special case. Note that $\alpha = 7 = 2\cdot|\{\nu,\triangle\}| + 3\cdot|\{\nabla\}|$. Looking at the road map of this proof, it is easily verified that we can extend the proof to any set of modalities. □

***2.3 Elementarity*** In this subsection we show that for finite similarity types $S$ the classes $\mathsf{GC}_S$ are elementary.

**Theorem 2.11** *Let S be a modal similarity type consisting of finitely many modal operators. Then the class $\mathsf{GC}_S$ is elementary.*

*Proof:* We first show the theorem for types with just one modality. For $\mathsf{GC}_0$ this is immediate by (2). A first order definition of $\mathsf{GC}_\oplus$ can be found in a similar way as we will show for $\mathsf{GC}_2$; we leave the details to the reader. For $n > 1$, we give only the proof for $n = 2$. The other cases are similar. Let $\mathsf{SQ} \stackrel{\text{def}}{=} \{\mathfrak{F} = \langle V, C_V\rangle : V = U \times U \text{ for some set } U\}$. Because $\mathsf{GC}_2$ consists of all substructures of $\mathsf{SQ}$, it suffices to show that $\mathsf{SQ}$ is elementary. (Because then $\mathsf{GC}_2$ can be axiomatized by all the universal consequences of the first-order theory of $\mathsf{SQ}$.) We need four axioms. We use $Ix$ as an abbreviation for $Cxxx$. We also term-define two functions: $x_l = y \stackrel{\text{def}}{\iff} Cxyx \,\&\, Iy$ and $x_r = y \stackrel{\text{def}}{\iff} Cxxy \,\&\, Iy$ (by the first axiom $(\cdot)_l$ and $(\cdot)_r$ are total functions).

$$\forall x \exists! y(Cxyx \,\&\, Iy), \quad \forall x \exists! y(Cxxy \,\&\, Iy) \tag{5}$$

$$x_l = y_l \,\&\, x_r = y_r \Rightarrow x = y \tag{6}$$

$$Cxyz \iff x_l = y_l, y_r = z_l \,\&\, z_r = x_r \tag{7}$$

$$\forall xy((Ix \,\&\, Iy) \Rightarrow \exists z(x = z_l \,\&\, y = z_r)) \tag{8}$$

Any frame $\mathfrak{F} = \langle W, C\rangle$ satisfying (5)–(8) is isomorphic to the frame $(I \times I, C_{I\times I})$, by the function $h$ defined as $h(x) = (x_l, x_r)$. This finishes the proof for $\mathsf{GC}_n$.

This proof can easily be adapted to the case with finitely many modalities. By way of illustration we show how to modify the last proof for the similarity type $\{(\triangle_1, 2), (\triangle_2, 2)\}$. We show that any frame $\mathfrak{F} = \langle W, C_1, C_2\rangle$, with the $C_i$ ternary, which satisfies $Ax$ below is isomorphic to a frame $\mathfrak{G} = \langle V, C_V^{0,2}, C_V^{2,2}\rangle$ with $V = U_1 \times U_1 \times U_2 \times U_2$ for $U_1$ and $U_2$ some sets. Then the conclusion follows by the same argument as given above. The set of axioms $Ax$ is given as follows. We construct predicates $I_1$ and $I_2$ and functions $(\cdot)_{l_1}, (\cdot)_{r_1}, (\cdot)_{l_2}, (\cdot)_{r_1}$ as above from $C_1$ and $C_2$, respectively. The set $Ax$ consists of the indexed versions of (5) and (7) plus the following two:

$$x_{l_1} = y_{l_1} \;\&\; x_{r_1} = y_{r_1} \;\&\; x_{l_2} = y_{l_2} \;\&\; x_{r_2} = y_{r_2} \Rightarrow x = y, \tag{9}$$

$$\forall xyvw((I_1 x \;\&\; I_1 y \;\&\; I_2 v \;\&\; I_2 w) \Rightarrow$$
$$\exists z(x = z_{l_1} \;\&\; y = z_{r_1} \;\&\; v = z_{l_2} \;\&\; w = z_{r_2})). \tag{10}$$

It is now straightforward to show that the function $h : W \longrightarrow I_1 \times I_1 \times I_2 \times I_2$ defined by $h(x) = (x_{l_1}, x_{r_1}, x_{l_2}, x_{r_2})$ is an isomorphism from $\mathfrak{F}$ to the frame $(V, C_V^{0,2}, C_V^{2,2})$ with $V = I_1 \times I_1 \times I_2 \times I_2$. $\square$

We do not know whether the class $\mathsf{GC}_S$ is elementary for infinite $S$.

***3 Final remarks.*** In Henkin et al. [3] (remark 2.7.46) it is argued that a Kripke-style semantics for modal logic is not satisfactory, because the relations in the frames are "abstract." Instead, they advocate a "geometrical" or "concrete" semantics. A "concrete" semantics for modal logic should, in their terminology, consist of a class of frames in which the relations are defined in *straightforward set-theoretical terms, the definitions are uniform for all frames involved, and as a consequence, each of the frames is uniquely determined by its universe*. The "standard" *unravelling* (cf., Sahlqvist [7], Bull and Segerberg [1]) of modal frames gives a representation which is satisfactory from the above point of view. This result can be extended to any modal logic, as is shown by de Rijke [2] (Proposition 6.3.5). We think however that the representation given here is closer to the spirit of the remarks in [3]. The main advantage of our representation is that the worlds in the frames are uniform: each world in each frame is an $\alpha$-long sequence (for some *fixed* $\alpha$), whereas in the standard unravelling, the worlds are sequences of arbitrary length. (Their length is important, because it determines to which worlds it is related.) Moreover, it is not clear whether the class obtained by unravelling is elementary. Finally, we think that the generalized composition relations $C_V$ are rather intuitive, even if they do become awkward to draw once the sequences are longer than 2.

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## NOTE

1. We kept this definition deliberately imprecise in order to keep the intuituive idea of $\mathsf{GC}_S$. We can make it precise in the following way. $\mathsf{GC}_S$ is the class of all $\alpha$-dimensional frames where $\alpha$ is as above. Now let $f$ be a function assigning to each operator in $O$ a finite subsequence of $\alpha$ in such a way that the concatenation of all $f$-images is precisely $\alpha$. The modal constants $\nu$ are then interpreted as $\iota\delta_{f(\nu)}$, the monadic modalities $\Diamond$ as in $\mathsf{GC}_{\Diamond}$, using $\Diamond_{(f(i), f(i+1))}$, and the higher-adic modalities $\nabla$ as generalized composition $\bullet_{f(\nabla)}$.

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