Notre Dame Journal of Formal Logic
Volume 37, Number 3, Summer 1996

Dual-Intuitionistic Logic

IGOR URBAS

Abstract The sequent system **LDJ** is formulated using the same connectives as Gentzen's intuitionistic sequent system **LJ**, but is dual in the following sense: (i) whereas **LJ** is singular in the consequent, **LDJ** is singular in the antecedent; (ii) whereas **LJ** has the same sentential counter-theorems as classical **LK** but not the same theorems, **LDJ** has the same sentential theorems as **LK** but not the same counter-theorems. In particular, **LDJ** does not reject all contradictions and is accordingly paraconsistent. To obtain a more precise mapping, both **LJ** and **LDJ** are extended by adding a "pseudo-difference" operator $\dot{-}$ which is the dual of intuitionistic implication. Cut-elimination and decidability are proved for the extended systems $\mathbf{LJ}^{\dot{-}}$ and $\mathbf{LDJ}^{\dot{-}}$, and a simply consistent but ω-inconsistent Set Theory with Unrestricted Comprehension Schema based on **LDJ** is sketched.

1 Introduction Intuitionistic logic differs from classical logic most notably in not affirming every instance of $A \vee \neg A$, the classical Law of Excluded Middle. It has accordingly been advanced as more appropriate for reasoning in *incomplete* situations, where for some sentence A neither A nor its negation $\neg A$ holds. The intuitionistic sequent system **LJ** of Gentzen [3] is obtained by restricting sequents of the classical sequent system **LK** to being (at most) singular in the consequent. It follows from a well-known result of Glivenko [4] that **LJ** and **LK** share the same sentential counter-theorems, i.e., sentences A not containing quantifiers such that the sequent $A \vdash$ (with empty consequent) is derivable; they do not, of course, share the same sentential theorems.

It is possible to obtain a *dual-intuitionistic* sequent system by restricting sequents of **LK** instead to being (at most) singular in the antecedent. The resulting logic then has the dual property of sharing all sentential theorems of **LK** while not sharing all counter-theorems. In particular, not every contradiction $A \,\&\, \neg A$ is rejected, which indicates that such a logic should be more appropriate than either classical or intuitionistic logic for reasoning in *inconsistent* situations, where for some sentence A both A and its negation $\neg A$ hold. Moreover, since the sequent $A \,\&\, \neg A \vdash B$ is also

Received August 22, 1990; revised October 4, 1994

rejected, dual-intuitionistic logic can be used as the basis for nontrivial inconsistent theories and is therefore paraconsistent.

Such sequent systems have been constructed in Czermak [2] and Goodman [5]. The former takes a purely proof-theoretic approach, whereas the latter proceeds by considering Brouwerian algebras (the algebraic duals of intuitionistic Heyting algebras) and moves on to a similarly proof-theoretic treatment. Although both offer sequent systems which are (at most) singular in the antecedent, neither uses exactly Gentzen's original connectives &, $\vee$, $\supset$, and $\neg$, and quantifiers $\forall$ and $\exists$. Czermak's system lacks rules for $\supset$ and $\exists$; Goodman's uses &, $\vee$, $\forall$, and $\exists$, as well as a connective $\dot{-}$ representing "pseudo-difference" and a sentential constant T. Consequently, it is not immediately clear in what exact sense each is dual to intuitionistic **LJ**. We begin by presenting a sequent system **LDJ** which is singular in the antecedent and which employs exactly Gentzen's connectives and quantifiers.

2 *The system* LDJ

The system **LDJ** has the following components:

Basic Sequents: all sequents of the form $A \vdash A$

Structural Rules:

$$\frac{\vdash \Delta}{A \vdash \Delta}\ (\text{Thin} \vdash) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ (\vdash \text{Thin})$$

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (\vdash \text{Cont}) \qquad \frac{\Gamma \vdash \Delta, A, B, \Theta}{\Gamma \vdash \Delta, B, A, \Theta}\ (\vdash \text{Int})$$

$$\frac{\Gamma \vdash \Delta, A \quad A \vdash \Theta}{\Gamma \vdash \Delta, \Theta}\ (\text{Cut})$$

Connective Rules:

$$\frac{A \vdash \Delta}{A \,\&\, B \vdash \Delta}\ (\& \vdash) \qquad \frac{B \vdash \Delta}{A \,\&\, B \vdash \Delta}\ (\& \vdash)$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \,\&\, B}\ (\vdash \&)$$

$$\frac{A \vdash \Delta \quad B \vdash \Delta}{A \vee B \vdash \Delta}\ (\vee \vdash)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vdash \vee) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vdash \vee)$$

$$\frac{\vdash \Delta, A \quad B \vdash \Theta}{A \supset B \vdash \Delta, \Theta}\ (\supset \vdash)$$

$$\frac{A \vdash \Delta}{\vdash \Delta, A \supset B}\ (\vdash \supset) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}\ (\vdash \supset)$$

$$\frac{\vdash \Delta, A}{\neg A \vdash \Delta}\ (\neg \vdash) \qquad \frac{A \vdash \Delta}{\vdash \Delta, \neg A}\ (\vdash \neg)$$

Quantifier Rules: $\dfrac{Fa \vdash \Delta}{\forall x Fx \vdash \Delta}$ $(\forall \vdash)$ $\qquad$ $\dfrac{\Gamma \vdash \Delta, Fa}{\Gamma \vdash \Delta, \forall x Fx}$ $(\vdash \forall)$

$\dfrac{Fa \vdash \Delta}{\exists x Fx \vdash \Delta}$ $(\exists \vdash)$ $\qquad$ $\dfrac{\Gamma \vdash \Delta, Fa}{\Gamma \vdash \Delta, \exists x Fx}$ $(\vdash \exists)$

Restrictions on variables: In the rules $(\vdash \forall)$ and $(\exists \vdash)$, the object variable a must not occur in the lower sequents (i.e., in Γ, Δ, or Fx).

Most of the rules of **LDJ** require no comment, as they are simply the result of restricting the rules of Gentzen's classical sequent system **LK** to being (at most) singular in the antecedent. The only exception is the pair of rules $(\vdash\supset)$. In their place, Gentzen's **LK** employs a single rule, the restricted version of which would be:

$$\frac{A \vdash \Delta, B}{\vdash \Delta, A \supset B} \; (\vdash\supset')$$

The essential difference between $(\vdash\supset)$ and $(\vdash\supset')$ is that in the latter rule both constituents A and B of the principal formula $A \supset B$ are explicit in the upper sequent, whereas in each of the former pair of rules only one constituent is explicit. Following Curry [1], we will call connective rules such as $(\vdash\supset')$ "Ketonen rules" and those such as $(\vdash\supset)$ "non-Ketonen rules."

Classical logic can be formulated indifferently using the Ketonen or non-Ketonen rules for introducing & into antecedent sequences and $\vee$ or $\supset$ into consequent sequences. The difference becomes crucial only when restrictions to singularity are considered. For example, the Ketonen rule for $\vee$-introduction in the consequent is:

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \; (\vdash \vee')$$

(In **LK** the sequence Γ may of course be multiple). Although $(\vdash \vee)$ and $(\vdash \vee')$ are interderivable in **LK** and in **LDJ** using the structural rules, only the non-Ketonen rules $(\vdash \vee)$ can be restricted to singularity in the consequent. Accordingly, Gentzen formulated **LK** using $(\vdash \vee)$ so as to be able to obtain **LJ** by imposing this restriction. Similarly, only the non-Ketonen rules $(\& \vdash)$ can be retricted to singularity in the antecedent as in **LDJ**; happily, Gentzen also used these rules in formulating **LK**.

The case of $\supset$-introduction in the consequent is somewhat different, as both the Ketonen and non-Ketonen rules for this introduction can be retricted to singularity in the antecedent and the consequent. However, the deductive strength of the systems so obtained varies according to the rules employed. In **LJ**, which is formulated using the singular-in-the-consequent version of $(\vdash\supset')$, the corresponding $(\vdash\supset)$ can also be derived; but $(\vdash\supset')$ could not be similarly derived if **LJ** were formulated using $(\vdash\supset)$ instead. Dually, $(\vdash\supset')$ is easily derived from $(\vdash\supset)$ in **LDJ**, but $(\vdash\supset)$ would not be derivable if **LDJ** were formulated using $(\vdash\supset')$ instead.

Exactly such a formulation, obtained by simply restricting **LK** to singularity in the antecedent, was considered by Czermak [2]. In view of the preceding remarks, it is not surprising that it proved to be somewhat weaker than **LDJ**, notably lacking such classical implicational theorems as $A \supset (B \supset A)$. As the dual to Glivenko's result could therefore not be established for this formulation, Czermak instead considered the &-$\vee$-$\neg$-$\forall$ fragment, for which it does hold. (The formulation of Goodman [5] also lacks the connective $\supset$.) So a first test to distinguish **LDJ** as a better formulation of dual-intuitionistic logic is to deliver the dual to Glivenko's result.

Theorem 2.1 (dual-Glivenko) **LDJ** *has the same sentential theorems as* **LK**.

Proof: That all of the sentential theorems of **LDJ** are theorems of **LK** is obvious, since the former is a subsystem of the latter. To establish the converse, it suffices to derive the axioms and rules of any Hilbert-style formulation of classical sentential logic in **LDJ**. This is made easier by first deriving ($\vdash\supset'$) in this system.

$$\cfrac{\cfrac{\cfrac{A \vdash \Delta, B}{A \vdash \Delta, A \supset B}\,(\vdash\supset)}{\vdash \Delta, A \supset B, A \supset B}\,(\vdash\supset)}{\vdash \Delta, A \supset B}\,(\vdash \mathrm{Cont})$$

The derivation of classical axioms is then straightforward. We illustrate with $A \supset (B \supset A)$.

$$\cfrac{\cfrac{A \vdash A}{A \vdash B \supset A}\,(\vdash\supset)}{\vdash A \supset (B \supset A)}\,(\vdash\supset')$$

The sole rule of a standard Hilbert axiomatics is *modus ponens*, from $\vdash A$ and $\vdash A \supset B$ to $\vdash B$. This is derived in **LDJ** simply as follows:

$$\cfrac{\overset{\textit{(ass.)}}{\vdash A \supset B} \qquad \cfrac{\overset{\textit{(ass.)}}{\vdash A} \qquad B \vdash B}{A \supset B \vdash B}\,(\supset\vdash)}{\vdash B}\,(\mathrm{Cut})$$

□

Thus, **LDJ** has the same sentential theorems as **LK**. In order to determine whether this result also holds for theorems involving quantifiers, we next consider an extension of both **LDJ** and **LJ** which will allow an isomorphic mapping between the resulting systems. This extension involves the connective $\dot{-}$ which figures in the formulation of dual-intuitionistic logic presented in [5]. Goodman suggests interpreting this "pseudo-difference" operator as "but not." However, this is somewhat misleading, as it suggests an equivalence between $A \dot{-} B$ and $A \mathbin{\&} \neg B$. Such an equivalence will prove to hold in the extended system $\mathbf{LJ}^{\dot{-}}$, but not in $\mathbf{LDJ}^{\dot{-}}$. We prefer to address the question of interpretation later, and proceed instead to introduce the connective $\dot{-}$ operationally. (It will become clear why a Ketonen rule is used in one extension while non-Ketonen rules are used in the other).

3 The system $\mathbf{LDJ}^{\dot{-}}$

The system $\mathbf{LDJ}^{\dot{-}}$ is obtained by adding the following rules to **LDJ**:

$$\frac{A \vdash \Delta, B}{A \dot{-} B \vdash \Delta}\,(\dot{-}\vdash') \qquad\qquad \frac{\Gamma \vdash \Delta, A \qquad B \vdash \Theta}{\Gamma \vdash \Delta, \Theta, A \dot{-} B}\,(\vdash \dot{-})$$

The system $\mathbf{LJ}^{\dot{-}}$ is obtained by adding the following rules to **LJ**:

$$\frac{A, \Gamma \vdash \Delta}{A \dot{-} B, \Gamma \vdash \Delta}\ (\dot{-} \vdash) \qquad\qquad \frac{\Gamma \vdash B}{A \dot{-} B, \Gamma \vdash}\ (\dot{-} \vdash)$$

$$\frac{\Gamma \vdash A \qquad B, \Delta \vdash}{\Gamma, \Delta \vdash A \dot{-} B}\ (\vdash \dot{-})$$

The mapping $\star$ (which builds upon one described in [2]) is defined as follows.

Formulas:

$$\begin{array}{rcl} p^\star & = & p \text{ for atomic } p \\ (\neg A)^\star & = & \neg A^\star \\ (A \,\&\, B)^\star & = & B^\star \vee A^\star \\ (A \vee B)^\star & = & B^\star \,\&\, A^\star \\ (A \supset B)^\star & = & B^\star \dot{-} A^\star \\ (A \dot{-} B)^\star & = & B^\star \supset A^\star \\ (\forall x F x)^\star & = & \exists x (F x)^\star \\ (\exists x F x)^\star & = & \forall x (F x)^\star \end{array}$$

Sequences: if Γ is the sequence $A_1, \ldots A_n (n \geq 0)$, then

$$\Gamma^\star = (A_n)^\star, \ldots (A_1)^\star$$

Sequents: $(\Gamma \vdash \Delta)^\star = \Delta^\star \vdash \Gamma^\star$

Rules: if R is a rule of the form $\dfrac{S_1 \ldots S_{n-1}}{S_n}$ $(n \geq 2)$, where $S_1, \ldots S_n$ are sequents, then

$$R^\star = \frac{(S_{n-1})^\star \ldots (S_1)^\star}{(S_n)^\star}$$

Theorem 3.1 *The mapping $\star$ is an isomorphism such that S is a derivable sequent in* $\mathbf{LDJ}^{\dot{-}}$ *if and only if $S^\star$ is derivable in* $\mathbf{LJ}^{\dot{-}}$*. Moreover, $\star$ is an involution: $S^{\star\star} = S$.*

Proof: It is easy to verify that $\star$ is an isomorphic mapping from the basic sequents and rules of $\mathbf{LDJ}^{\dot{-}}$ to, respectively, basic sequents and rules of $\mathbf{LJ}^{\dot{-}}$. For example, *Cut* in $\mathbf{LDJ}^{\dot{-}}$ is mapped to:

$$\frac{\Theta^\star \vdash A^\star \qquad A^\star, \Delta^\star \vdash \Gamma^\star}{\Theta^\star, \Delta^\star \vdash \Gamma^\star}$$

which is just an instance of *Cut* in $\mathbf{LJ}^{\dot{-}}$. Similarly, the Ketonen rule $(\dot{-} \vdash')$ of $\mathbf{LDJ}^{\dot{-}}$ is mapped to the Ketonen rule $(\vdash \supset')$ of $\mathbf{LJ}^{\dot{-}}$, and the non-Ketonen rules $(\vdash \supset)$ of $\mathbf{LDJ}^{\dot{-}}$ are mapped to the non-Ketonen rules $(\dot{-} \vdash)$ of $\mathbf{LJ}^{\dot{-}}$. Thus, applying $\star$ to every sequent occurring in a derivation of S in $\mathbf{LDJ}^{\dot{-}}$ produces a derivation of $S^\star$ in $\mathbf{LJ}^{\dot{-}}$. Conversely, if $S^\star$ is derivable in $\mathbf{LJ}^{\dot{-}}$, then $S^{\star\star}$ is derivable in $\mathbf{LDJ}^{\dot{-}}$. It remains only to verify that $\star$ is an involution on sequents: $S^{\star\star} = S$. This follows straightforwardly from the fact that $\star$ is an involution on formulas, which can be proved by induction on the number of connectives occurring in a formula A.

Base Case: In this case, A is an atomic formula p. But $p^{\star\star} = p^\star = p$.

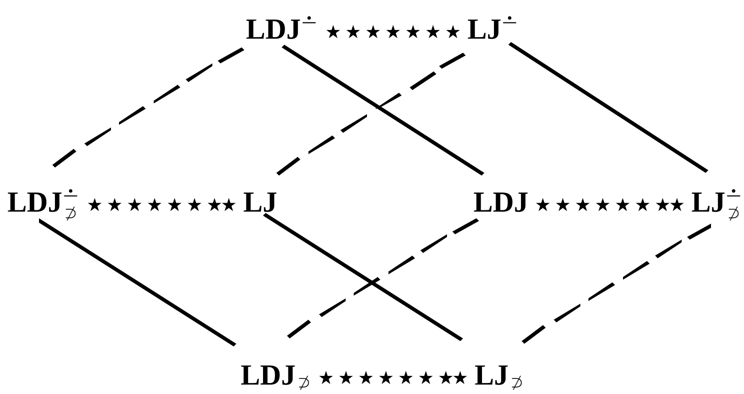
Inductive Step: Assume that A is of the form $B \,\&\, C$. Then $(B \,\&\, C)^{\star\star} = (C^{\star} \vee B^{\star})^{\star} = B^{\star\star} \,\&\, C^{\star\star} = B \,\&\, C$ (on inductive hypothesis). All other cases are dealt with similarly. □

The mapping $\star$ provides a more precise way of establishing the correspondences between intuitionistic and dual-intuitionistic logics and their fragments. For example, using the subscript $\not\supset$ to denote fragments not involving the connective $\supset$, it is clear that **LDJ** is in fact a definitional extension of $\mathbf{LDJ}_{\not\supset}$, with $A \supset B$ defined as $\neg A \vee B$. The rules corresponding to **(**$\supset\vdash$**)** and **(**$\vdash\supset$**)** are derived in the latter fragment as follows, where the double lines represent necessary applications of Structural Rules:

$$\frac{\dfrac{\dfrac{\vdash \Delta, A}{\neg A \vdash \Delta}\ (\neg\vdash)}{\neg A \vdash \Delta, \Theta} \qquad \dfrac{B \vdash \Theta}{B \vdash \Theta, \Delta}}{\neg A \vee B \vdash \Delta, \Theta}\ (\vee\vdash)$$

$$\frac{\dfrac{A \vdash \Delta}{\vdash \Delta, \neg A}\ (\vdash\neg)}{\vdash \Delta, \neg A \vee B}\ (\vdash\vee) \qquad\qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, \neg A \vee B}\ (\vdash\vee)$$

Similarly, $\mathbf{LDJ}^{\dot{-}}$ is a definitional extension of $\mathbf{LDJ}^{\dot{-}}_{\not\supset}$; whereas dually, $\mathbf{LJ}^{\dot{-}}$ and $\mathbf{LJ}^{\dot{-}}_{\not\supset}$ are respectively definitional extensions of **LJ** and $\mathbf{LJ}_{\not\supset}$, with $\dot{-}$ defined as $A \,\&\, \neg B$. The total picture that emerges is as follows, with unbroken lines representing proper extensions, broken lines representing definitional extensions, and horizontal lines of $\star$s representing duals according to the $\star$-mapping:



Thus, it turns out that the exact dual of Gentzen's **LJ** is, according to the $\star$-mapping, $\mathbf{LDJ}^{\dot{-}}_{\not\supset}$ rather than **LDJ**. However, the latter is easily extended to $\mathbf{LDJ}^{\dot{-}}$, which is itself a definitional extension of $\mathbf{LDJ}^{\dot{-}}_{\not\supset}$. Both **LDJ** and $\mathbf{LDJ}^{\dot{-}}$ are dual to **LJ** in the sense that they are singular in the antecedent rather than in the consequent; moreover, **LDJ** has the dual-Glivenko property of sharing sentential theorems but not counter-theorems with **LK** while being formulated with the same connectives. For the remainder of our investigation, we will concentrate on the maximally expressive system $\mathbf{LDJ}^{\dot{-}}$; though the results obtained will apply directly or with obvious modifications to the remaining dual-intuitionistic systems.

We now prove a *Cut*-elimination Theorem for $\mathbf{LJ}^{\dot{-}}$, from which will follow *Cut*-elimination and decidability for $\mathbf{LDJ}^{\dot{-}}$ via the $\star$-mapping. Afterwards, we illustrate the deductive features of the dual-intuitionistic systems with some derivable and underivable sequents, and we return to the question of whether the dual-Glivenko result can be extended to theorems involving quantifiers. Finally, we investigate the paraconsistent nature of the dual-intuitionistic systems and sketch a dual-intuitionistic set theory with Unrestricted Comprehension Schema which is nonetheless simply consistent (though ω-inconsistent).

4 *Cut-elimination and decidability*

Theorem 4.1 (Cut-elimination) *The rule* Cut *is eliminable from* $\mathbf{LJ}^{\dot{-}}$*; that is, every sequent which is derivable in this system has a derivation in which* Cut *does not figure.*

Proof: It suffices to add to the *Cut*-elimination proof for **LJ** of Gentzen [3] the following sections dealing with $\dot{-}$. (Note that the symbol * occurring in these sections is Gentzen's notation and has nothing to do with the $\star$-mapping).

3.113.37. Suppose the terminal symbol of the *Mix*-formula M is $\dot{-}$. Then the end of the derivation has one of the following forms:

$$\dfrac{\dfrac{\Gamma \vdash A \qquad B, \Delta \vdash}{\Gamma, \Delta \vdash A \dot{-} B}\,(\vdash \dot{-}) \qquad \dfrac{A, \Theta \vdash \Lambda}{A \dot{-} B, \Theta \vdash \Lambda}\,(\dot{-} \vdash)}{\Gamma, \Delta, \Theta \vdash \Lambda}\,(\text{Mix})$$

$$\dfrac{\dfrac{\Gamma \vdash A \qquad B, \Delta \vdash}{\Gamma, \Delta \vdash A \dot{-} B}\,(\vdash \dot{-}) \qquad \dfrac{\Theta \vdash B}{A \dot{-} B, \Theta \vdash}\,(\dot{-} \vdash)}{\Gamma, \Delta, \Theta \vdash \Lambda}\,(\text{Mix})$$

These are respectively transformed into:

$$\dfrac{\dfrac{\Gamma \vdash A \qquad A, \Theta \vdash \Lambda}{\Gamma, \Theta^* \vdash \Lambda}\,(\text{Mix}),}{\overline{\Gamma, \Delta, \Theta \vdash \Lambda}}$$

$$\dfrac{\dfrac{\Theta \vdash B \qquad B, \Delta \vdash}{\Theta, \Delta^* \vdash}\,(\text{Mix})}{\overline{\Gamma, \Delta, \Theta \vdash}}$$

where the double lines represent possible applications of Structural Rules excluding *Cut/Mix*. In both cases, the *Mix*-formula of the remaining *Mix* is of lower degree. These *Mix*es can therefore be eliminated on the inductive hypothesis.

3.121.22. This section applies without modification to $(\dot{-} \vdash)$.

We add the following sections to 3.121.23 to deal with $(\vdash \dot{-})$.

3.121.234. Suppose the *Mix* occurs after a $(\vdash \dot{-})$. Then the end of the derivation is of the form:

$$\dfrac{\Theta \vdash M \qquad \dfrac{\Gamma \vdash A \qquad B, \Delta \vdash}{\Gamma, \Delta \vdash A \dot{-} B}\,(\vdash \dot{-})}{\Theta, \Gamma^*, \Delta^* \vdash A \dot{-} B}\,(\text{Mix})$$

The *Mix*-formula M must occur in Γ or in Δ or in both. We distinguish these three cases.

3.121.234.1. Suppose M occurs in Γ but not in Δ. Then the end of the derivation is transformed into:

$$\dfrac{\dfrac{\Theta \vdash M \qquad \Gamma \vdash A}{\Theta, \Gamma^* \vdash A}\ (\text{Mix}) \qquad \Theta \vdash M}{\Theta, \Gamma^*, \Delta^* \vdash A \mathbin{\dot{-}} B}\ (\vdash \dot{-})$$

Since M does not occur in Δ, it follows that Δ^* is just Δ, and the lower sequent is the same as the lower sequent of the original derivation. Moreover, since the new derivation is of lower rank the *Mix* occurring in it may be eliminated on the inductive hypothesis.

3.121.234.2. Suppose M occurs in Δ but not in Γ. This is dealt with as in case 3.121.234.1.

3.121.234.3. Suppose M occurs in both Γ and Δ. Then the end of the derivation is transformed initially into:

$$\dfrac{\Theta \vdash M \qquad \Gamma \vdash A}{\Theta, \Gamma^* \vdash A}\ (\text{Mix}) \qquad\qquad \dfrac{\Theta \vdash M \qquad B, \Delta \vdash}{\Theta, B^*, \Delta^* \vdash}\ (\text{Mix})$$

Here B^* represents either B or nothing depending on whether B is different from M or the same. If B^* is B, the derivation continues as follows.

$$\dfrac{\dfrac{[\Theta, \Gamma^* \vdash A] \qquad \dfrac{[\Theta, B^*, \Delta^* \vdash\ ]}{B, \Theta, \Delta^* \vdash}\ (\text{Ints} \vdash)}{\Theta, \Gamma^*, \Theta, \Delta^* \vdash A \mathbin{\dot{-}} B}\ (\vdash \dot{-})}{\Theta, \Gamma^*, \Delta^* \vdash A \mathbin{\dot{-}} B}$$

where the double lines again represent possible applications of Structural Rules excluding *Cut/Mix*. If B^* is nothing, then the derivation continues as follows:

$$\dfrac{\dfrac{[\Theta, \Gamma^* \vdash A] \qquad \dfrac{[\Theta, B^*, \Delta^* \vdash\ ]}{B, \Theta, \Delta^* \vdash}\ (\text{Thin} \vdash)}{\Theta, \Gamma^*, \Theta, \Delta^* \vdash A \mathbin{\dot{-}} B}\ (\vdash \dot{-})}{\Theta, \Gamma^*, \Delta^* \vdash A \mathbin{\dot{-}} B}$$

In both cases, the new derivation is of lower rank. The *Mix*es occurring therein can therefore be eliminated on inductive hypothesis.

3.122. This section does not apply to $(\vdash \dot{-})$ and requires no modification to apply to $(\dot{-} \vdash)$.

All other sections remain exactly as in Gentzen [3]. □

Theorem 4.2 *The rule* (Cut) *is eliminable from* $\mathbf{LDJ}^{\dot{-}}$.

Proof: Let S be a derivable sequent of $\mathbf{LDJ}^{\dot{-}}$. By Theorem 3.1, $S^\star$ is derivable in $\mathbf{LJ}^{\dot{-}}$. By Theorem 4.1, $S^\star$ has a *Cut*-free derivation in $\mathbf{LJ}^{\dot{-}}$. Applying the mapping $\star$ to every sequent occurring in this derivation produces a *Cut*-free derivation in $\mathbf{LDJ}^{\dot{-}}$ of $S^{\star\star}$, which, again by Theorem 3.1, is just S. □

As in [3], a derivation has the Subformula Property if every formula occurring in it is a subformula of some formula occurring in the final sequent of the derivation. Since *Cut* is the only rule of the sentential fragment of $\mathbf{LDJ}^{\dot{-}}$ which does not preserve this property, Theorem 4.2 has the following consequence.

Corollary 4.3 *Every derivable sequent of the sentential fragment of* $\mathbf{LDJ}^{\dot{-}}$ *has a* (Cut*-free) derivation with the Subformula Property.*

As for Gentzen's **LJ**, this fact can be used to establish that the sentential fragment of $\mathbf{LDJ}^{\dot{-}}$ is decidable.

Corollary 4.4 *Sentential* $\mathbf{LDJ}^{\dot{-}}$ *is decidable.*

Moreover, since every derivable sequent of sentential $\mathbf{LDJ}^{\dot{-}}$ not involving the connective $\dot{-}$ thus has a derivation in which the rules $(\dot{-}\vdash')$ and $(\vdash\dot{-})$ do not figure, it follows that sentential $\mathbf{LDJ}^{\dot{-}}$ is a conservative extension of sentential **LDJ**. And since $\mathbf{LDJ}^{\dot{-}}$ and **LDJ** share the same quantifier rules, this result can be generalized to the full systems.

Corollary 4.5 $\mathbf{LDJ}^{\dot{-}}$ *is a conservative extension of* **LDJ**.

All of these results, with obvious modifications, hold for all of the remaining dual-intuitionistic systems. We now proceed to contrast some of those sequents which are derivable in these systems with some of those that are not, and we return to the question of whether the dual-Glivenko result can be extended beyond the sentential level.

5 Derivable sequents and theorems

Theorem 5.1 *The following sequents are derivable in* **LDJ** *(and in* $\mathbf{LDJ}^{\dot{-}}$*); those not involving the connective* $\supset$ *are also derivable in* $\mathbf{LDJ}_{\not\supset}$ *(and in* $\mathbf{LDJ}^{\dot{-}}_{\not\supset}$*):*

$$\begin{array}{rclcrclcrcl}
A & \vdash & A & \quad & A\supset B & \vdash & \neg A\vee B & \quad & \forall xFx & \vdash & \exists xFx\\
A & \vdash & B\supset A & & \neg A\vee B & \vdash & A\supset B & & \neg\forall xFx & \vdash & \exists x\neg Fx\\
A\,\&\,B & \vdash & A & & \neg A\vee\neg B & \vdash & \neg(A\,\&\,B) & & \neg\forall x\neg Fx & \vdash & \exists xFx\\
A & \vdash & A\vee B & & \neg(A\vee B) & \vdash & \neg A\,\&\,\neg B & & \neg\exists xFx & \vdash & \forall x\neg Fx\\
\neg\neg A & \vdash & A & & A\supset\neg B & \vdash & B\supset\neg A & & \neg\exists x\neg Fx & \vdash & \forall xFx\\
\neg A & \vdash & A\supset B & & \neg A\supset\neg B & \vdash & B\supset A & & & \vdash & (\exists xFx)\ \vee\ (\exists x\neg Fx).
\end{array}$$

In addition, the following sequents involving the connective $\dot{-}$ *are derivable in* $\mathbf{LDJ}^{\dot{-}}$ *(and in* $\mathbf{LDJ}^{\dot{-}}_{\not\supset}$*):*

$$\begin{array}{rclcrclcrcl}
A\dot{-}A & \vdash & & \quad & \neg A\dot{-}B & \vdash & \neg B\dot{-}A & \quad & A\dot{-}B & \vdash & A\,\&\,\neg B\\
A\dot{-}A & \vdash & B & & \neg A\dot{-}\neg B & \vdash & B\dot{-}A & & A & \vdash & B,\ A\dot{-}B.
\end{array}$$

Theorem 5.2 *The following sequents are not derivable in any of the systems of Theorem 5.1:*

$$
\begin{array}{ccc}
A \vdash \neg\neg A & A \,\&\, \neg A \;\vdash\; \neg B & \forall x F x \;\vdash\; \neg\exists x \neg F x \\
A \vdash \neg A \supset B & A \vee B \;\vdash\; \neg(\neg A \,\&\, \neg B) & \forall x \neg F x \;\vdash\; \neg\exists x F x \\
A \,\&\, (A \supset B) \;\vdash\; B & A \,\&\, B \;\vdash\; \neg(\neg A \vee \neg B) & \exists x F x \;\vdash\; \neg\forall x \neg F x \\
A \,\&\, (\neg A \vee B) \;\vdash\; B & \neg A \,\&\, \neg B \;\vdash\; \neg(A \vee B) & \exists x \neg F x \;\vdash\; \neg\forall x F x \\
A \,\&\, \neg A \;\vdash & A \supset B \;\vdash\; \neg B \supset \neg A & \exists x (F x \;\&\; \neg F x) \vdash \\
A \,\&\, \neg A \;\vdash\; B & \neg A \supset B \;\vdash\; \neg B \supset A & \vdash \neg\exists x (F x \;\&\; \neg F x).
\end{array}
$$

The following sequents involving the connective $\dot{-}$ are also not derivable in any of the systems of Theorem 5:

$$
\begin{array}{ccc}
A \dot{-} \neg\neg A \vdash & A \dot{-} B \;\vdash\; \neg B \dot{-} \neg A & A \,\&\, \neg B \vdash A \dot{-} B \\
A \,\&\, (B \dot{-} A) \vdash & A \dot{-} \neg B \;\vdash\; B \dot{-} \neg A & (A \,\&\, \neg A) \dot{-} B \vdash .
\end{array}
$$

The last sequent involving the existential quantifier $\exists$ listed in Theorem 5.2 is particularly noteworthy, as the formula $\neg\exists x(Fx \,\&\, \neg Fx)$ is in fact a theorem of classical logic **LK**. This answers the question of whether Theorem 2.1 (the dual-Glivenko result) can be extended beyond the sentential level.

Theorem 5.3 *The dual-intuitionistic systems of Theorem 5.1 lack some theorems which involve the quantifier $\exists$.*

In fact, the dual-Glivenko result can be extended to theorems involving the quantifier $\forall$ but not $\exists$, as is done for the system $\mathbf{LDJ}_{\not\supset}$ in [2]. The latter remarks on, but does not demonstrate, the failure of this result to extend to theorems involving $\exists$; Goodman [5] makes a similar observation, also giving $\neg\exists x(Fx \,\&\, \neg Fx)$ as a counterexample (even though $\neg$ is not a primitive connective in Goodman's formulation).

Theorems 5.2 and 5.3 clarify the extent to which the dual-intuitionistic systems are suitable for paraconsistent purposes. In rejecting the sequent $A \,\&\, \neg A \vdash B$, they can evidently support theories which are inconsistent but not trivial (that is, which contain some sentence A together with its negation $\neg A$ but not every sentence B). Moreover, the absence of $\vdash \neg\exists x(Fx \,\&\, \neg Fx)$ allows theories based on these systems to contain a sentence of the form $\neg\exists x(Fx \,\&\, \neg Fx)$ without thereby being inconsistent, notwithstanding that every instance of $\neg(Fx \,\&\, \neg Fx)$ is a theorem. Such theories thus exhibit a variant of the property known as ω-inconsistency, which does not in general amount to simple inconsistency.

For example, Goodman [5] describes a dual-intuitionistic set theory which incorporates an unrestricted Comprehension Schema $\vdash \exists y \forall x(x \in y \equiv Fx)$, where y is not free in Fx and $A \equiv B$ abbreviates $(A \supset B) \,\&\, (B \supset A)$. Russell's Paradox is reproducible in this set theory, with the result that it contains the sequent $\vdash \exists x(x \in x \,\&\, x \notin x)$, but is nonetheless simply consistent. The same holds for similar set theories based on **LDJ** and $\mathbf{LDJ}^{\dot{-}}$, possibly augmented by postulates for equality = and an Axiom of Extensionality $\vdash \forall x \forall y(\forall z(z \in x \equiv z \in y) \supset x = y)$. We note that Curry's Paradox cannot be reproduced in these set theories as the sequent $A \supset (A \supset B) \vdash A \supset B$ corresponding to Contraction, but not the sequent $A \,\&\, (A \supset B) \vdash B$ corresponding to *modus ponens*, is derivable in the underlying logics.

Evidently, then, our dual-intuitionistic systems bear out their motivation as logics appropriate for reasoning in inconsistent (or ω-inconsistent) situations. It may be, however, that they do so only at the expense of otherwise desirable systemic properties. For example, an argument of [5] shows that these systems lack a definable implication connective that is the analogue of deducibility. Clearly $\supset$, which serves this purpose in classical and intuitionistic logic, fails to do so in **LDJ** and **LDJ**$^{\dot{-}}$, where there are theorems of the form $A \supset B$ such that $A \vdash B$ is not derivable. Essentially, this amounts to a failure of the rule of *modus ponens* for $\supset$ in these systems. Although the theorematic form of this rule, from $\vdash A$ and $\vdash A \supset B$ to $\vdash B$, is derivable (as in the Proof of Theorem 2.1), the absence of $A \,\&\, (A \supset B) \vdash B$ as a derivable sequent ensures that a sentence B cannot be derived from premises A and $A \supset B$ in general. Thus, though the sets of theorems of **LDJ** and **LDJ**$^{\dot{-}}$ are closed under *modus ponens*, theories based on these systems are in general not. Indeed, this failure is essential to the paraconsistent character of these systems, given the derivability of $\vdash (A \,\&\, \neg A) \supset B$.

The failure of *modus ponens* for dual-intuitionistic $\supset$ argues against interpreting this connective as a kind of implication. (It might also be said that the conformity of classical, intuitionistic, and dual-intuitionistic $\supset$ to the non-Ketonen rules ($\vdash\supset$) argues against interpreting this connective as implication in any of these logics.) Moreover, no connective definable in **LDJ** or **LDJ**$^{\dot{-}}$ fares any better as an implicational analogue of deducibility.

Theorem 5.4 *There is no connective $\oplus$ definable in* **LDJ** *or* **LDJ**$^{\dot{-}}$ *such that $A \vdash B$ is a derivable sequent if and only if $A \oplus B$ is a theorem.*

Proof: The following is a syntactic version of Goodman's algebraic argument. If there were such a connective $\oplus$, it would likewise be definable in classical **LK**. Since $A \vdash A$ is derivable in **LDJ** and **LDJ**$^{\dot{-}}$, $A \oplus A$ would be a theorem of these systems, and therefore also of **LK**. But the result of substituting $\neg\neg A$ for any occurrence of a sentence A in any theorem of **LK** is also a theorem; thus, $A \oplus \neg\neg A$ would also be a classical theorem. By Theorem 2.1, $A \oplus \neg\neg A$ would then also be a theorem of **LDJ** and therefore of **LDJ**$^{\dot{-}}$, but then $A \vdash \neg\neg A$ would be derivable in these systems, contrary to Theorem 5.2. □

Theorem 5.4 shows that there can be no standard axiomatics for our dual-intuitionistic systems, in the (strong) sense that $A \,/\, B$ is a derivable rule in the axiomatics if and only if $A \vdash B$ is derivable in the sequent system. It does not quite establish, however, that there is absolutely no syntactic analogue of deducibility in **LDJ** and **LDJ**$^{\dot{-}}$; in the latter system at least there is such a connective, though it is clearly not a kind of implication.

Theorem 5.5 *$A \vdash B$ is a derivable sequent in* **LDJ**$^{\dot{-}}$ *if and only if $A \dot{-} B$ is a counter-theorem; that is, the sequent $A \dot{-} B \vdash$ is derivable. Generally, $A \vdash \Delta, B$ is derivable in* **LDJ**$^{\dot{-}}$ *if and only if $A \dot{-} B \vdash \Delta$ is also derivable.*

Proof: If $A \vdash \Delta, B$ is derivable in **LDJ**$^{\dot{-}}$, then $A \dot{-} B \vdash \Delta$ follows by $(\dot{-} \vdash')$. The converse derivation is as follows.

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \vdash B, A \dot{-} B}\,(\vdash \dot{-}) \qquad A \dot{-} B \vdash \Delta}{\dfrac{A \vdash B, \Delta}{A \vdash \Delta, B}\,(\vdash \text{Ints})}\,(\text{Cut})$$

□

Theorem 5.5 is the best we can do for an analogue of the (Strong) Deduction Theorem for $\mathbf{LDJ}^{\dot{-}}$. In view of the central role of the connective $\dot{-}$ both as the $\star$-correlate of $\supset$ and as the syntactic correlate of deducibility in $\mathbf{LDJ}^{\dot{-}}$, we close with some remarks on its interpretation. The suggestion of Goodman in [5] that $A \dot{-} B$ be interpreted as "A but not B" is not quite satisfactory as it suggests an equivalence with $A \mathbin{\&} \neg B$. This equivalence holds classically (for **LK**) and intuitionistically (for $\mathbf{LJ}^{\dot{-}}$ and $\mathbf{LJ}^{\dot{-}}_{\not\supset}$), but it fails for $\mathbf{LDJ}^{\dot{-}}$. Instead, we suggest that $A \dot{-} B$ be interpreted as "A excludes B." This is the natural dual of logical implication: whereas A implies B just in case the logical content of A must include that of B, dually A excludes B just in case the logical content of A cannot include that of B. Thus, $\mathbf{LDJ}^{\dot{-}}$ turns out to be a very liberal sort of logic—it allows B to be deduced from A just in case it is absurd (a counter-theorem) for A to exclude B.

Acknowledgments The author wishes to thank the following for their generous hospitality and support during various stages of writing this paper: Michael McRobbie and the Centre for Information Science Research, *ANU*, Canberra; Lorenzo Peña and the Instituto de Filosofia, *CSIC*, Madrid; André Fuhrmann and the Fakultät Philosophie, Universität Konstanz; the Alexander von Humboldt Foundation, Bonn.

REFERENCES

[1] Curry, H. B., *Foundations of Mathematical Logic*, Dover, New York, 1976. Zbl 0396.03001 MR 55:7715

[2] Czermak, J., "A remark on Gentzen's calculus of sequents," *Notre Dame Journal of Formal Logic*, vol. 18 (1977), pp. 471–474. Zbl 0314.02026 MR 58:21500 1, 2, 3, 5

[3] Gentzen, G., "Investigations into logical deduction," *The Collected Papers of Gerhard Gentzen*, edited by M. E. Szabo, North-Holland, Amsterdam, 1969. Zbl 0209.30001 MR 41:6660 1, 4, 4

[4] Glivenko, V., "Sur quelques points de la logique de M. Brouwer," *Académie Royale de Belgique, Bulletins de la classe des sciences*, ser. 5, vol. 15 (1929), pp. 183–188. 1

[5] Goodman, N. D., "The logic of contradiction," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 27 (1981), pp. 119–126. Zbl 0467.03019 MR 82g:03043 1, 2, 2, 5, 5, 5, 5

Department of Philosophy
Universität Konstanz
Postfach 5560-D24
Konstanz 78434
Germany