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An Unclassifiable Unidimensional Theory without OTOP

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Abstract A countable unidimensional theory without the *omitting types order property* (OTOP) has prime models over pairs and is hence classifiable. We show that this is not true for uncountable unidimensional theories.

1 Introduction In [4], Shelah settled the Main Gap for countable theories. Roughly speaking, this result shows that for a countable first order theory T either there is a reasonable set of cardinal invariants which describe the isomorphism type of models of T or in some technical sense the class of models of T is chaotic. For countable unidimensional theories this dichotomy is explained by two properties, *prime models over pairs* (PMOP) and the *omitting types order property* (OTOP) both of which we proceed to define.

A stable theory T is said to have PMOP if whenever $M_0 \prec M_i \prec N \models T$ for $i = 1, 2$ and M_1 is independent from M_2 over M_0 then there is a prime model over $M_1 \cup M_2$. If a unidimensional theory T of cardinality λ has PMOP then the class of models of T is well behaved. For instance, every model of T is determined up to isomorphism by its cardinality and a choice of relatively $|T|^+$-saturated submodel. This implies that $I(\kappa, T) \leq 2^{2^{|T|}}$ for all $\kappa > |T|$.

A theory T has the OTOP if there is a type $p(\bar{x}, \bar{y}, \bar{z})$ so that for every λ there is M, a model of T, and a sequence $\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ in M so that $p(\bar{x}, \bar{a}_\alpha, \bar{a}_\beta)$ is realized in M if and only if $\alpha < \beta$. This is a particular instance of the more general concept of *having an $L_{\infty,\omega}$-definable order*. T has an $L_{\infty,\omega}$-definable order if there is an $L_{\infty,\omega}$-formula $\varphi(x, y)$ so that for every λ there is M, a model of T, and a sequence $\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ in M so that $M \models \varphi(\bar{a}_\alpha, \bar{a}_\beta)$ if and only if $\alpha < \beta$.

Theories which have $L_{\infty,\omega}$-definable orders were first studied in Shelah [3]. A theory T which has an $L_{\infty,\omega}$-definable order has a chaotic class of models and is *unclassifiable*. A theory T is said to be unclassifiable if for any regular cardinal $\lambda > |T|$ there are two nonisomorphic models of T which can be forced to be isomorphic by a forcing notion which preserves cardinals (and adds no new small subsets of λ). A λ-closed forcing is an example of such a forcing. The class of models of such a theory

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cannot have a reasonable set of cardinal invariants which determine isomorphism. It should be remarked that any unidimensional theory with PMOP is not unclassifiable (is classifiable).

A countable unidimensional theory which does not have the OTOP has PMOP (see [4], ch. 12). In this paper we show that this is not true for uncountable unidimensional theories. Precisely we give an example of a unidimensional theory with no $L_{\infty,\omega}$-definable order which nevertheless is unclassifiable.

We begin in this section by describing a particular theory, T, which has cardinality $2^{\aleph_0}$ and is unidimensional with $R^{\infty}(x = x) = 2$. In Section 2 we prove that T has quantifier elimination and that one cannot code arbitrarily long $L_{\infty,\omega}$-definable orders in models of T. In the third section we prove that T is unclassifiable. In the final section we consider a family of examples similar to T and give a characterization of those which are classifiable and those which are not.

Let us first describe the particular theory T somewhat informally. The language for T will be multisorted. One sort, V, will contain a vector space over the two element field. There will be predicates to pick out a descending chain of subspaces U_n where $[U_n : U_{n+1}] = 2$. It will be convenient to introduce predicates for all the cosets of U_n. There will be continuum many other sorts S_η for $\eta \in 2^\omega$. Each of these can be thought of as a cover of V and the fiber in S_η above any $v \in V$ is acted on regularly by V. For any fixed η, there is no interaction between the fibers in S_η above different elements from V. However, for every $v \in V$, there is interaction between the fiber above v in S_η and S_μ for $\eta, \mu \in 2^\omega$, $\eta \neq \mu$. Suppose that $\eta|_n = \mu|_n$ but $\eta(n) \neq \mu(n)$. U_n, the subspace of V, induces an equivalence relation with 2^n many classes on the fiber above v in both S_η and S_μ. We will introduce relations between S_η and S_μ which will give a bijection between these sets of equivalence classes.

Now we give a formal description of T. We begin with a definition.

Definition 1.1 Given $\mu \neq \eta \in 2^\omega$, $i(\mu, \eta)$ = the greatest $n < \omega$ for which $\mu|_n = \eta|_n$.

The language L is defined as follows.

1. L has sorts $\{V\} \cup \{S_\eta : \eta \in 2^\omega\}$.
2. In the sort V there is a constant 0, a binary function $+$, and unary predicates $\{U_n^j : n < \omega,\ j < 2^n\}$.
3. For each $\eta \in 2^\omega$, there are functions $f_\eta : S_\eta \to V$, $g_\eta : S_\eta \times S_\eta \to V$ and $h_\eta : V \times S_\eta \to S_\eta$.
4. For each pair $\mu \neq \eta \in 2^\omega$, there are binary relations $R^j_{(\mu,\eta)} \subset S_\mu \times S_\eta$ for $j < 2^{i(\mu,\eta)}$.

Now let M be the following L-structure.

1. V is a countable abelian group in which every element other than the identity has order 2. $+$ and 0 are interpreted by the group operation and the identity element of V, respectively. For each $n < \omega$, U_n^0 is a subgroup of V such that $U_{n+1}^0 \subset U_n^0$ and $[U_n^0 : U_{n+1}^0] = 2$. Finally, for each $n < \omega$, $\{U_n^j : j < 2^n\}$ is a list of the cosets of U_n^0 in V.
2. For each $\eta \in 2^\omega$, $S_\eta = V \times V$ and the functions are interpreted as follows.

    (a) $f_\eta(x, y) = y$.

(b) $g_\eta((x,y),(x',y')) = x - x'$ provided that $y = y'$ and is undefined otherwise.

(c) $h_\eta(u,(x,y)) = (u+x, y)$.

3. Given $\mu \neq \eta \in 2^\omega$, $R^j_{(\mu,\eta)}((x,y),(x',y'))$ holds just if $y = y'$ and $x - x' \in U^j_{i(\mu,\eta)}$.

We define T to be the theory of M in the language L.

Throughout the rest of this paper we will drop the subscripts in the functions f_η, g_η, h_η whenever there is no confusion as to their domains. We end this section with an easy, if somewhat inelegant, lemma which will be of use later.

Lemma 1.2 *Let N be any model of T and μ, η, τ distinct members of 2^ω. Suppose that $a, a' \in S_\mu$, $b, b' \in S_\tau$, and $c, c' \in S_\eta$.*

(i) If $N \models R^j_{(\mu,\tau)}(a,b)$ for some $j < 2^{i(\mu,\tau)}$ and if $u, u' \in V$ with $u - u' \in U^0_{i(\mu,\tau)}$, then $N \models R^j_{(\mu,\tau)}(h(u,a), h(u',b))$.

(ii) If $N \models R^j_{(\mu,\tau)}(a,b) \wedge R^j_{(\mu,\tau)}(a',b')$ for some $j < 2^{i(\mu,\tau)}$ and if $u, u' \in V$ with $u - u' \in U^0_{i(\mu,\tau)}$, then for any $k < 2^{i(\mu,\tau)}$

$$N \models R^k_{(\mu,\tau)}(a, h(u,b)) \longleftrightarrow R^k_{(\mu,\tau)}(a', h(u',b')).$$

(iii) Let $n_0 = i(\mu,\eta)$, $n_1 = i(\mu,\tau)$, and $n_2 = i(\tau,\eta)$ and suppose that $n_2 \leq n_0$. Now choose $j_l < 2^{n_l}$ for $l = 0,1,2$ and suppose that

$$N \models R^{j_0}_{(\mu,\eta)}(a,c) \wedge R^{j_1}_{(\mu,\tau)}(a,b) \wedge R^{j_2}_{(\tau,\eta)}(b,c).$$

Then

$$N \models R^{j_0}_{(\mu,\eta)}(a',c') \wedge R^{j_1}_{(\mu,\tau)}(a',b') \quad\Longrightarrow\quad N \models R^{j_2}_{(\tau,\eta)}(b',c').$$

Proof: (i) and (ii) are immediate consequences of the definition of T. (iii): It suffices to show that (iii) holds when N is our original model M. First note that $\mu|_{n_2} = \eta|_{n_2} = \tau|_{n_2}$ — since $i(\tau,\eta) = n_2 \leq i(\mu,\eta)$ — and we conclude that $n_2 \leq n_1$. Since we are working in M we may assume that $a, a', b, b', c, c' \in V(M)$ and our hypotheses imply that

$$(a-c), (a'-c') \in U^{j_0}_{n_0},\ (a-b), (a'-b') \in U^{j_1}_{n_1} \quad\text{and}\quad b - c \in U^{j_2}_{n_2} \tag{1}$$

and it remains to prove that $b' - c' \in U^{j_2}_{n_2}$. Now

$$(b-c) - (b'-c') = ((a'-b') - (a-b)) + ((a-c) - (a'-c'))$$

and so the required result follows from (1) and the fact that $n_2 \leq \min(n_1, n_0)$. □

2 T does not have OTOP

Proposition 2.1 *Let N be any model of T. Then the set of partial isomorphisms $\{\rho : D \to N : D \subset N$ is finite$\}$ has the back and forth property.*

Proof: Suppose $\overline{a}, \overline{b} \subset N$ satisfy the same atomic formulas (we will write $\overline{a} \equiv_0 \overline{b}$) and fix $c \in N$. We must find $d \in N$ so that $\overline{a}c \equiv_0 \overline{b}d$. Denote by A, B the substructures of N generated by $\overline{a}, \overline{b}$, respectively. Then A is finite and $A \equiv_0 B$ in the obvious way. We may clearly assume that $c \notin A$. Now there are various possibilities.

Case 1: $c \in V(N)$. Since $V(A)$ is finite, we can find $d \in V(N) \backslash V(B)$ so that

$$\text{for all } n < \omega, \quad c - d \in U_n^0(N). \tag{2}$$

We claim that this d works. It is enough to verify that the following holds.

(i) For any $u \in V(A)$, any $n < \omega$ and $j < 2^n$,

$$u + c \in U_n^j(N) \quad \text{if and only if} \quad u' + d \in U_n^j(N)$$

where u' corresponds to u in $V(B)$.

(ii) For any $\eta, \mu \in 2^\omega$, $v \in S_\eta(A)$, $w \in S_\mu(A)$ and $j < 2^{i(\mu,\eta)}$,

$$N \models R^j_{(\mu,\eta)}(w, h(c, v)) \quad \text{if and only if} \quad N \models R^j_{(\mu,\eta)}(w', h(d, v'))$$

where v' corresponds to v in $S_\eta(B)$ and w' corresponds to w in $S_\mu(B)$.

This is an easy consequence of (2) above and Lemma 1.2(ii).

Case 2: For some $\eta \in 2^\omega$, $c \in S_\eta(N)$.

Subcase 1: For all $e \in A \backslash V(A)$, $f(e) \neq f(c)$. In this case, let $u = f(c)$ and choose $u' \in V(N)$, as in Case 1, so that $Au \equiv_0 Bu'$. It is now trivial to check that any $d \in f_\eta^{-1}(u')$ works.

Subcase 2: There is $e \in S_\eta(A)$ with $f(e) = f(c)$. Let $e' \in B$ correspond to e. Now let $u = g(c, e)$ and choose $u' \in V(N)$, as in Case 1, so that $Au \equiv_0 Bu'$. Then since $c = h(u, e)$, taking d to be $h(u', e')$ works. All that remains is

Subcase 3: The set $Q = \{\tau \in 2^\omega : \text{there is } e \in S_\tau(A) \text{with } f(e) = f(c)\}$ is nonempty and does not contain η. In this case, let $n_0 = \max\{i(\tau, \eta) : \tau \in Q\}$ and choose $\mu \in Q$ with $i(\mu, \eta) = n_0$. Further, let $e \in S_\mu(A)$ satisfy $f(e) = f(c)$. Now for some $j_0 < 2^{n_0}$ we have $N \models R^{j_0}_{(\mu,\eta)}(e, c)$. Let e' correspond to e in B and choose $d \in S_\eta(N)$ so that $N \models R^{j_0}_{(\mu,\eta)}(e', d)$. We claim that this d works.

By virtue of our hypotheses, it suffices to check that if $\tau \in 2^\omega$, $j_2 < 2^{i(\tau,\eta)}$, $z \in S_\tau(A)$ and $u \in V(A)$, then

$$N \models R^{j_2}_{(\tau,\eta)}(z, h(u, c)) \quad \Longrightarrow \quad N \models R^{j_2}_{(\tau,\eta)}(z', h(u', d))$$

where z', u' correspond to z, u respectively in B.

So suppose that $N \models R^{j_2}_{(\tau,\eta)}(z, h(u, c))$. By Lemma 1.2(ii) we may assume that $u = u' = 0$. Again by Lemma 1.2(ii), we may assume that $\tau \neq \mu$ (note that if τ were equal to μ, we would have $g(e, z) \equiv_0 g(e', z')$). Now observe that $f(z) = f(c) = f(e)$ (hence $\tau \in Q$) and so for some $j_1 < 2^{i(\mu,\tau)}$,

$$N \models R^{j_1}_{(\mu,\tau)}(e, z) \wedge R^{j_1}_{(\mu,\tau)}(e', z').$$

Now since $\tau \in Q$, it follows that $i(\tau, \eta) \leq n_0$ and thus Lemma 1.2(iii) implies that $N \models R^{j_2}_{(\tau,\eta)}(z', d)$, as required. This completes the proof of the proposition. $\square$

The next corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2 *T admits quantifier elimination.*

The following is now immediate from the quantifier elimination.

Corollary 2.3 *T is superstable, unidimensional, and $R^{\infty}(S_\eta) = 2$ for all $\eta < 2^\omega$.*

Corollary 2.4 *T does not have an $L_{\infty,\omega}$-definable order. In particular, T does not have OTOP.*

Proof: Suppose that there is an $L_{\infty,\omega}$-formula $\varphi(\bar{x}, \bar{y})$ so that for every λ there is a model $M \models T$ and a sequence $\langle a_\alpha : \alpha < \lambda\rangle \subset M^n$ for some $n < \omega$ such that

$$N \models \chi(\overline{a}_\alpha, \overline{a}_\beta) \quad \text{if and only if} \quad \alpha < \beta.$$

By standard arguments then there is a model $N \models T$, an indiscernible sequence $\langle \overline{a}_\alpha : \alpha < \omega\rangle \subset N^n$ for some $n < \omega$ such that

$$N \models \chi(\overline{a}_\alpha, \overline{a}_\beta) \quad \text{if and only if} \quad \alpha < \beta.$$

By indiscernibility the pairs $(\overline{a}_0, \overline{a}_1)$ and $(\overline{a}_1, \overline{a}_0)$ have the same L-type in N. It then follows from Proposition 2.1 that they have the same $L_{\infty,\omega}$-type in N, a contradiction. $\square$

3 *T is unclassifiable*

Definition 3.1 Let L^* be any language and N_1, N_2 L^*-structures. Then a set, $\Pi \neq \varnothing$, of partial isomorphisms $\pi : N_1 \to N_2$, is called a λ-back-and-forth system for N_1, N_2 if

(i) for all $\pi \in \Pi$, $|\text{dom}(\pi)| < \lambda$, and
(ii) given $\pi \in \Pi$ and $C \subset N_1$, $D \subset N_2$ with $|C|, |D| < \lambda$, there are $\pi_1, \pi_2 \in \Pi$ extending π such that $C \subset \text{dom}(\pi_1)$ and $D \subset \text{ran}(\pi_2)$.

If there is a λ-back-and-forth system Π between two models which in addition is closed under unions of chains of length less than λ, then we say that the models are strongly equivalent. Such a Π is λ-closed as a forcing notion (see Kunen [2]) and hence satisfies the forcing condition listed above. Hyttinen and Shelah [1] show that for any unsuperstable theory T and any regular cardinal $\lambda > |T|$ there are models M and N of cardinality λ which are strongly equivalent and nonisomorphic.

We will prove that T is unclassifiable by coding certain graphs into models of T. Given a graph G let I_G, E_G denote the vertex and edge sets, respectively, of G. We will be interested in those graphs which are symmetric and irreflexive, contain no triangles, and satisfy: $|I_G| \geq 2^{\aleph_0}$ and every vertex has valence at least 2. Let $\mathbf{G}$ be the class of all such graphs, and let L' be the language of graphs.

We will show that for every $G \in \mathbf{G}$, there is a corresponding model $N_G \models T$ with $|N_G| = |I_G|$, so that if $G, H \in \mathbf{G}$, then

(i) $N_G \cong N_H$ implies $G \cong H$, and
(ii) if $|I_G| = |I_H| = \lambda > 2^{\aleph_0}$ and G, H are strongly equivalent, then N_G and N_H are strongly equivalent.

It easily follows that T is unclassifiable.

We now fix a large saturated model $\mathbf{C} \models T$, and from here on we will work inside $\mathbf{C}$. Thus V, S_η, and so on, will mean $V(\mathbf{C})$, $S_\eta(\mathbf{C})$, and so on. We also denote U_n^0 simply by U_n for all $n < \omega$ and we denote $\bigcap_{n<\omega} U_n$ by U. Clearly U has $2^{\aleph_0}$ many cosets in V. Notice that by the definition of T, V is a vector space over the field of two elements. With this in mind we will sometimes refer to the linear dependence (independence) of subsets of V, and given any $A \subset V$ we will denote by $\overline{A}$ the subspace generated by A.

We may assume that $M \prec \mathbf{C}$ where M is our original model. Denote $V(M) \subset V$ by V^*. We will assume that $U \cap V^*$ is infinite. Now since M is countable, the set of cosets of U which intersect V^* is countable; let us list this set as $\{W_k : k < \omega\}$.

Definition 3.2 The model $N \prec \mathbf{C}$ is said to be small, if for any coset, W of U,

$$W \cap V(N) \neq \varnothing \quad \text{if and only if} \quad W = W_k \text{ for some } k < \omega.$$

We now describe a general method for constructing small models. First let us adopt the following notation: for $v \in V$, a *sequence above* v is a sequence $\overline{a}_v = \langle a_{v,\eta} : \eta \in 2^\omega \rangle$ where for each $\eta \in 2^\omega$, $a_{v,\eta} \in S_\eta$ and $f(a_{v,\eta}) = v$.

Now let Y be any subset of U and let $V_Y = \overline{Y \cup V^*}$. For each $v \in V_Y$, choose a sequence over v, say $\overline{a}_v$. Now let N be the substructure of $\mathbf{C}$ generated by $V_Y \cup \{\overline{a}_v : v \in V_Y\}$. Then it is easily seen that N is a model of T (simply check that the set of partial isomorphisms, $\rho : N \to M$, has the back and forth property, precisely as in the proof of Proposition 2.1). It follows from Corollary 2.2 that $N \prec \mathbf{C}$ and we also note that $|N| = |Y| + 2^{\aleph_0}$. Moreover, $V(N) = V_Y$ and thus, since $Y \subset U$, it follows that N is small.

Given a graph $G \in \mathbf{G}$ we will construct the model N_G as described above by making an appropriate choice for the set Y and for the sequences $\overline{a}_v$. In order to code G, we will need to be able to recognize a given element of $V(N_G)$ as being one of two types according to which sequence was chosen over that element. The next lemmas demonstrate that this is possible.

Lemma 3.3 *Let $v \in V$. Then there are sequences $\overline{a}_v = \langle a_{v,\eta} : \eta \in 2^\omega \rangle$, $\overline{b}_v = \langle b_{v,\eta} : \eta \in 2^\omega \rangle$ over v such that*

(i) for all $\eta \neq \mu \in 2^\omega$, $\models R^0_{(\mu,\eta)}(a_{v,\mu}, a_{v,\eta})$;

(ii) for all $\eta \neq \mu \in 2^\omega$, $\models \neg R^0_{(\mu,\eta)}(b_{v,\mu}, b_{v,\eta})$.

Moreover, the sequences $\overline{a}_v, \overline{b}_v$ can be chosen for all $v \in V$ so that whenever $v_1, v_2 \in V$ satisfy $v_1 \equiv_0 v_2$, then $\overline{a}_{v_1} \equiv_0 \overline{a}_{v_2}$ and $\overline{b}_{v_1} \equiv_0 \overline{b}_{v_2}$.

Proof: (i) is like (ii) except easier and the moreover clause is routine. (ii): Let $\eta_0, \ldots, \eta_m \in 2^\omega$. By compactness it suffices to show that in our original model M, there are $a_0, \ldots, a_m \in V(M)$ such that

$$\text{for all } j \neq k \leq m, \quad a_j - a_k \in U_{i(\eta_j,\eta_k)-1} \setminus U_{i(\eta_j,\eta_k)}. \tag{$\dagger$}$$

The proof is by induction on m, the case $m = 1$ being trivial. So let us assume that $m > 1$ and that ($\dagger$) holds for all smaller m.

Let $n = \max\{i(\eta_j, \eta_k) : j \neq k \leq m\}$ and assume without loss of generality that $n = i(\eta_{m-1}, \eta_m)$. By the inductive hypothesis, choose $a_0, \ldots, a_{m-1} \in V(M)$ so that (†) holds for these elements. Now choose $a_m \in V(M)$ so that $a_m - a_{m-1} \in U_{n-1} \setminus U_n$. We claim that $\{a_0, \ldots, a_m\}$ satisfies (†).

Fix $j < m-1$ and let $n_1 = i(\eta_j, \eta_m), n_2 = i(\eta_j, \eta_{m-1})$. It suffices to show that $a_j - a_m \in U_{n_1-1} \setminus U_{n_1}$. Now since $a_j - a_{m-1} \in U_{n_2-1} \setminus U_{n_2}$ and $a_m - a_{m-1} \in U_{n-1} \setminus U_n$, it is enough to show that $n > n_1$ and $n_1 = n_2$. But this is an immediate consequence of the maximality of n and Definition 1.1. Thus the lemma is proved. □

We call any sequence satisfying (i) in the lemma *a sequence of Type I* over v, and any sequence satisfying (ii) *a sequence of Type II* over v.

Lemma 3.4 *Suppose that $N \prec \mathbf{C}$ is small and $v \in V(N)$. Then N does not contain both a Type I and a Type II sequence over v.*

Proof: Suppose for a contradiction that $\overline{a} = \langle a_\eta : \eta \in 2^\omega\rangle \subset N$ is a Type I sequence over v and $\overline{b} = \langle b_\eta : \eta \in 2^\omega\rangle \subset N$ is a Type II sequence over v. Since N is small, there are $\mu \neq \eta \in 2^\omega$ and $k < \omega$ such that both $g(a_\eta, b_\eta)$ and $g(a_\mu, b_\mu)$ belong to W_k. Since $a_\eta = h(g(a_\eta, b_\eta), b_\eta)$ and $a_\mu = h(g(a_\mu, b_\mu), b_\mu)$, we obtain a contradiction from Lemma 1.2(i). □

We are now able to construct the models N_G for $G \in \mathbf{G}$. Begin by fixing for each $v \in V$, a Type I sequence $\overline{a}_v$ and a Type II sequence $\overline{b}_v$ over v as in the moreover clause of Lemma 3.3. Further fix a basis X for $V^* \cap U$ and a subspace $Z \subset V^*$ such that $V^* = (V^* \cap U) \oplus Z$ $(= \overline{X} \oplus Z)$.

Now fix $G \in \mathbf{G}$ and let Y_G be a linearly independent subset of U which contains X and such that $|Y_G| = |I_G|$. Index Y_G as $\{y_p : p \in I_G\}$ and let $V_G = \overline{V^* Y_G}$. Observe that by our choice of Y_G we have

$$V_G = \overline{Y_G} \oplus Z. \tag{♮}$$

Now for each $v \in V_G$ choose one of the sequences $\overline{a}_v$ or $\overline{b}_v$ over v as follows.

1. If $v = y_p$ for some $p \in I_G$, then choose the sequence $\overline{a}_v$.
2. If $v = y_p + y_q$ for some $p, q \in I_g$ such that $(p, q) \in E_G$, then choose the sequence $\overline{a}_v$.
3. Otherwise choose the sequence $\overline{b}_v$.

For $v \in V_G$, let *seq(v)* denote the sequence chosen above v (that is, seq(v) is either $\bar{a}_v$ of $\bar{b}_v$) and now let N_G be the substructure generated by $V_G \cup \{\text{seq}(v) : v \in V_G\}$. By our earlier discussion, N_G is a small (elementary) submodel of $\mathbf{C}$ with $|N_G| = |I_G|$. Let us call $v \in V_G (= V(N_G))$ of Type I (Type II) in N_G just if seq(v) is of Type I (Type II). Given distinct elements v_0, v_1, v_2 of V_G, call the set $\{v_0, v_1, v_2\}$ a *notable triple* if each v_i is of Type I in N_G and $v_0 + v_1 + v_2 = 0$. We leave to the reader to verify the following consequence of our assumptions on $\mathbf{G}$ and the linear independence of Y_G.

Lemma 3.5

(i) For any $v \in V_G$, $v = y_p$ for some $p \in I_G$ if and only if v belongs to at least two notable triples.

(ii) For any $p, q \in I_G$, $(p, q) \in E_G$ if and only if $y_p + y_q$ is of Type I in N_G.

Combining Lemmas 3.4 and 3.5 yields

Corollary 3.6 *If $G, H \in \mathbf{G}$ and $N_G \cong N_H$, then $G \cong H$.*

Corollary 3.7 *$I(\lambda, T) = 2^\lambda$ for all $\lambda \geq 2^{\aleph_0}$.*

It remains to prove that if $G, H \in \mathbf{G}$ have cardinality $\lambda > 2^{\aleph_0}$ and are strongly equivalent then N_G and N_H are strongly equivalent.

It is convenient at this juncture to expand the languge L to a new language L^1 by adding a new unary predicate Q to the sort V. For any $G \in \mathbf{G}$, N_G is made into an L^1-structure by interpreting Q as follows:

$$N_G \models Q(v) \text{ just if } v \text{ has Type I in } N_G.$$

Now let us fix $G, H \in \mathbf{G}$ so that $|I_G| = |I_H| = \lambda > 2^{\aleph_0}$ and G, H are strongly equivalent. We also fix a λ-back-and-forth system Φ for G, H which is closed under the union of chains of length less than λ. It certainly suffices to find a λ-back-and-forth system for N_G, N_H with respect to the enriched language L^1 which is also closed under the union of short chains. Call a substructure $A \subset N_G$ *full* if for all $v \in V(A)$, $\text{seq}(v) \subset A$, similarly for substructures of N_H. Given the definition of the sequences $\overline{a}_v, \overline{b}_v$, it is straightforward to verify the following.

Fact 3.8 *Suppose that $A \subset N_G$, $B \subset N_H$ are full substructures. Then the following are equivalent.*

1. *There is an onto L^1-partial isomorphism $\theta : V(A) \to V(B)$.*
2. *There is an onto L^1-partial isomorphism $\overline{\theta} : A \to B$ such that $\theta \subset \overline{\theta}$ and for all $v \in V(A)$, $\overline{\theta}(seq(v)) = seq(\theta(v))$.*

Let L^2 be the restriction of L^1 to the sort V. Then V_G, V_H are naturally L^2-structures, and we conclude from the above fact that it suffices to find a λ-back-and-forth system for the L^2-structures V_G, V_H. We now proceed to define just such a system. Recall that we fixed a λ-back-and-forth system Φ for the graphs G, H. Given $\varphi \in \Phi$, we define a corresponding L^2-partial isomorphism, $\theta_\varphi : V_G \to V_H$, as follows. Suppose that $\text{dom}(\varphi) = J \subset I_G$ and $\text{ran}(\varphi) = K \subset I_H$. Then θ_φ is the unique linear map satisfying

1. $\text{dom}(\theta_\varphi) = \overline{\{y_p : p \in J\}} \oplus Z$ and $\text{ran}(\theta_\varphi) = \overline{\{y_r : r \in K\}} \oplus Z$,
2. $\theta_\varphi(y_p) = y_{\varphi(p)}$, and
3. $\theta_\varphi|_Z = \text{id}|_Z$.

That θ_φ is a L-partial isomorphism follows from the fact that $Y_G, Y_H \subset U$. To see that θ_φ is in fact a L^2-partial isomorphism, we need only check that θ_φ preserves the predicate Q, but this is immediate since φ is an L'-partial isomorphism. Note also that since $|J| < \lambda$, we have $|\text{dom}(\theta_\varphi)| < \lambda$ also. Now let

$$\Theta = \{\theta_\varphi : \varphi \in \Phi\}.$$

Lemma 3.9 *Θ is a λ-back-and-forth system closed under the union of chains of length less than λ.*

Proof: Let $\theta \in \Theta$ and suppose that $C \subset V_G$ satisfies $|C| < \lambda$. We must find $\theta' \in \Theta$ so that $\theta \subset \theta'$ and $C \subset \mathrm{dom}(\theta')$. By definition, there is $\varphi \in \Phi$ such that $\theta = \theta_\varphi$. Now by $(\natural)$ above, there is $J' \subset I_G$ such that $|J'| < \lambda$ and

$$C \quad \subset \quad \overline{\{y_p : p \in J'\}} \oplus Z.$$

Since Φ is a λ-back-and-forth system, there is $\varphi' \in \Phi$ which extends φ and satisfies $J' \subset \mathrm{dom}(\varphi')$. Clearly taking $\theta' = \theta_{\varphi'}$ satisfies our requirements. The closure under unions of short chains is straightforward since Φ is closed under such unions, and hence the lemma is proved. $\square$

We conclude from our discussions above that N_G and N_H are indeed strongly equivalent and therefore that T is unclassifiable.

4 Other examples In this section we will introduce a family of examples of unidimensional theories that explores the ways that PMOP can fail. For each limit ordinal $\delta < \omega_1$, let $\eta_\delta = \langle \delta_n : n \in \omega \rangle$ be an increasing sequence whose limit is δ. Let $S = \omega_1^{<\omega} \cup \{\eta_\delta : \delta \text{ a limit ordinal, } \delta < \omega_1\}$. Fix $X \subseteq S$ and define T_X as in Definition 1.1 except that η now ranges over X and not just 2^ω. So in particular there is a sort S_η for every $\eta \in X$, and all corresponding functions and relations are also defined. Let $\overline{X} = \{\delta : \eta_\delta \in X\}$. We can repeat the proofs in Section 2 and get

Proposition 4.1 *For X as above, T_X is unidimensional and does not have an $L_{\infty,\omega}$-definable order.*

One way that prime models can be built is by inevitable construction.

Definition 4.2 Fix a complete theory T.

1. A type $p \in S(A)$ is called inevitable if p is realized in any model M of T which contains A.
2. Suppose M is a model of T and $A \subseteq M$. We say that M is inevitably constructed over A if M can be enumerated as $\langle a_\alpha : \alpha < \beta \rangle$ for some β so that for all $\alpha < \beta$, $\mathrm{tp}(a_\alpha / \{a_\gamma : \gamma < \alpha\} \cup A)$ is inevitable.

If M is inevitably constructed over A then it is prime over A. A strong way for a theory T to have PMOP would be to have inevitably constructed models over pairs of models independent over a common submodel, ICMOP.

We remind the reader that ω_1^ω is topologized by the initial segment topology: that is, basic open sets are of the form $\{\eta \in \omega_1^\omega : \nu \subseteq \eta\}$ where $\nu \in \omega_1^n$ for some $n \in \omega$. We will prove

Proposition 4.3 *For X as above, T_X has ICMOP if and only if*

1. *there is no uncountable $C \subseteq X$ which can be homeomorphically embedded into 2^ω, and*
2. *$\overline{X}$ is not stationary.*

Proof: Fix a monster model of T_X. We leave it to the reader to verify that T_X has ICMOP if and only if there is an enumeration of all the sorts of T_X, $\langle S_{\eta_i} : i < \alpha \rangle$ so that for some (equivalently any) $a \in V$ and sequence $\langle a_i : i < \alpha \rangle$ with $a_i \in S_{\eta_i}$ and $f_{\eta_i}(a_i) = a$, for each $j < \alpha$, $tp(a_j/\{a_i : i < j\})$ is isolated. In fact, this type will be isolated if and only if η_j is not in the closure of $\{\eta_i : i < j\}$ (in the topology induced by $\omega_1^{<\omega}$ on X). Hence T_X has ICMOP if and only if X can be enumerated so that every initial segment of the enumeration is closed; call such an enumeration a good enumeration.

Let us begin the proof of the proposition by proving that if (1) and (2) hold then T_X has ICMOP. Let $X_\alpha = X \cap \alpha^\omega$; note that X_α is closed. For any $\alpha < \omega_1$, α^ω can be homeomorphically embedded into 2^ω. It follows that X_α can also be so embedded. If for any α, X_α is uncountable this would contradict condition 1. So each X_α is countable. Since $\overline{X}$ is not stationary, there is a cub $C \subseteq \omega_1$ so that $C \cap \overline{X} = \varnothing$. Suppose that $C = \langle \alpha_i : i < \omega_1 \rangle$ and this enumeration is in increasing order. Since C and $\overline{X}$ have empty intersection, for any limit ordinal δ, $X_{\alpha_\delta} = \cup_{i<\delta} X_{\alpha_i}$. Hence, if one enumerates $X_{\alpha_{i+1}} \setminus X_{\alpha_i}$ in order type ω and places these enumerations in increasing order, one obtains an enumeration of X in order type ω_1 which is good.

For the other direction we will show first that if condition 1 fails then X does not have a good enumeration. So suppose that $C \subseteq X$ is an uncountable set which can be homeomorphically embedded into 2^ω. Take any enumeration and concentrate on the first ω_1 many elements of C which are listed. There must be a countable subset of these which is dense. Hence this enumeration cannot be good.

It follows then that if condition 1 does not fail then all the sets X_α introduced above are countable. So X is of size ω_1. We will prove by induction on order type that for all α and all such X, if $\overline{X}$ is stationary then X cannot have a good enumeration of order type α.

The case where $\alpha = \beta + 1$ is straightforward. If $\mathrm{cf}(\alpha) = \omega$ then suppose we have an enumeration of the required type of order type α. Suppose $\alpha = \cup_{n\in\omega} \alpha_n$ and X is enumerated $\langle \eta_i : i < \alpha \rangle$. Let $X_n = \{\eta_i : i < \alpha_n\}$. Since $\overline{X}$ is stationary, for some n, $\overline{X}_n$ is stationary, and since it can be enumerated in order type α_n this is a contradiction.

Finally, suppose that $\mathrm{cf}(\alpha) = \omega_1$. Let $\langle \alpha_i : i < \omega_1 \rangle$ be a continuous increasing chain of ordinals whose union is α. Define a function $g : \omega_1 \to \omega_1$ so that $g(\beta) =$ the least i such that X_γ is enumerated before α_i for all $\gamma < \beta$. g is continuous and cofinal so there is a cub C of limit ordinals so that for every $\beta \in C$, $g(\beta) = \beta$. (Note that $g(\beta)$ is not necessarily greater than or equal to β but by Fodor's lemma, the set of β for which $g(\beta) < \beta$ is not stationary.) □

Claim 4.4 *Let $\overline{Y}$ be the set of $\delta \in \overline{X}$ for which η_δ is in the closure of $\cup_{\beta<\delta} X_\beta$. Then $\overline{Y}$ is stationary.*

Proof: Suppose not. Then since $\overline{X}$ is stationary, there is an $n \in \omega$ with the following property: there is a stationary set $\overline{W} \subseteq \overline{X}$ such that for any $\delta \in \overline{W}$, $\eta_\delta \restriction n \neq \zeta \restriction n$ for any $\zeta \in \cup_{\beta<\delta} X_\beta$. But now by n applications of Fodor's lemma, we can assume that $\eta_\delta \restriction n$ is constant for all $\delta \in \overline{W}$ which is clearly a contradiction to the choice of n.

Define a function $f : C \cap \overline{Y} \to \omega_1$ as follows: for $\delta \in C \cap \overline{Y}$, η_δ is in the closure of $\cup_{\beta<\delta} X_\beta$. So since we are assuming we have a good enumeration, η_δ must be enumerated before α_δ. Let $f(\alpha) = i$ where i is the least such that η_δ is enumerated before α_i. f is regressive so there is a set $Z \subseteq X$ so that $\overline{Z} \subseteq \overline{Y} \cap C$ is stationary and $f(\overline{Z})$ is constant, say ζ. But then all η_α for $\alpha \in \overline{Z}$ are enumerated before α_ζ which contradicts the inductive assumption. □

Proposition 4.5 *If $X \subseteq S$ and $C \subseteq X$ is uncountable and can be homeomorphically embedded into 2^ω, then T_X is unclassifiable.*

Proof: The proof from Section 3 works here as well. □

Remark 4.6 One thing which is not at issue here is whether or not T_X is classifiable or unclassifiable. The definitions of these terms that we have given in this paper seem to leave the possibility that there may be unidimensional theories which are neither. Although that cannot be ruled out now, it can for these theories. To see this, note that what is really at issue in attempting to build a prime model over a pair of small models is what the structure of the collection of fibers over a point in the vector space is. If there is only one way up to isomorphism to define the fibers over a point in the vector space then the theory has PMOP and is classifiable. On the other hand, if there are two ways which are not isomorphic then the proof outlined in Section 3 will show that that theory is unclassifiable.

The point of this investigation was not to show that the dichotomy was satisfied for this class of examples but to isolate the exact properties that gave rise to unclassifiability. We still believe that all examples T_X where $\overline{X}$ is stationary are unclassifiable but at this moment we only know that they do not have inevitably constructed models over pairs.

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