Notre Dame Journal of Formal Logic
Volume 39, Number 4, Fall 1998

Lattice Ordered *O*-Minimal Structures

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Abstract We propose a notion of *o*-minimality for partially ordered structures. Then we study *o*-minimal partially ordered structures $(A, \leq, \ldots)$ such that $(A, \leq)$ is a Boolean algebra. We prove that they admit prime models over arbitrary subsets and we characterize ω-categoricity in their setting. Finally, we classify *o*-minimal Boolean algebras as well as *o*-minimal measure spaces.

1 Introduction Totally ordered *o*-minimal structures were introduced by Van den Dries in [9] and intensively studied in Pillay and Steinhorn [7] and Knight, Pillay, and Steinhorn [4]. They are unstable according to the Shelah classification theory and yet they enjoy some good model theoretic properties. For instance, it was shown in [7] that if T is a complete theory whose models are totally ordered *o*-minimal structures and Ω denotes a big saturated model of T, then

1. the algebraic closure in Ω satisfies the Steinitz exchange principle;
2. for each subset X of Ω (of smaller cardinality), there is a unique model of T elementarily prime over X.

Definable sets in totally ordered *o*-minimal structures were largely studied also in [4] where, in particular, it was proved that *o*-minimality is preserved by elementary equivalence in this setting. Other motivations and connections are explained in Marker [6]. Of course, one may wonder what happens with respect to *o*-minimality when one replaces total orders with arbitrary partial orders. But the class of partially ordered structures may be too large to allow reasonably general and significant results. So we prefer a more particular starting point, and we deal here with Boolean lattice ordered structures, namely, partially ordered structures $\mathcal{A} = (A, \leq, \ldots)$ such that $(A, \leq)$ is a Boolean algebra. We explore *o*-minimality in this setting.

In particular, in Section 2 we propose a possible definition of *o*-minimality for these structures. Accordingly, we classify *o*-minimal Boolean algebras; we see that they are exactly the Boolean algebras with finitely many atoms. Then we point out that for a complete theory T whose models are Boolean lattice ordered *o*-minimal

September 24, 1997; revised March 4, 1998

structures, the exchange principle may fail but over each subset X of Ω (of smaller cardinality) there does exist a model of T elementarily prime. This is the matter of Section 3.

Section 4 is devoted to classifying the ω-categorical theories whose models are Boolean lattice ordered o-minimal structures. It turns out that they do not exceed the theories of expansions of o-minimal Boolean algebras by finitely many constants.

Finally, we investigate some other possible examples among measure spaces (Section 5). These are not Boolean lattice ordered structures in a literal sense but, of course, they are very near. The analysis again shows that the o-minimal structures in this class are comparatively poor. So Section 4 and Section 5 may suggest that the complete theories whose models are Boolean lattice ordered o-minimal structures should be confined among some trivial expansions of the theories of Boolean algebras with finitely many atoms and raise the question of finding, if possible, *nontrivial* examples.

We refer to Hodges [3] for basic model theory and to Birkhoff [1] and Halmos [2] for Boolean algebra. As already stated, for a given complete theory T, we fix a big saturated model Ω of T and we work inside Ω by assuming that any model of T is an elementary substructure of Ω.

2 ***o*-minimal and quasi-*o*-minimal structures** Let $\mathcal{A} = (A, \leq, \dots)$ be a structure partially ordered by $\leq$.

Definition 2.1 $\mathcal{A}$ is *quasi-o-minimal* if and only if the only subsets of A definable in $\mathcal{A}$ are the finite Boolean combinations of sets defined by formulas $a \leq v$ or $v \leq b$ with a and b in A. $\mathcal{A}$ is *o-minimal* if and only if for every $X \subseteq A$, the only X-definable subsets of A are the finite Boolean combinations of sets defined by formulas $a \leq v$ and $v \leq b$ with a and b in the algebraic closure $acl(X)$ of X.

What happens for totally ordered structures? In this case, quasi-o-minimality implies o-minimality as implicitly observed in [7], so the two notions are equivalent. Furthermore, if $\mathcal{A} = (A, \leq, \dots)$ is a totally ordered o-minimal structure, then any definable subset of A is a finite Boolean combination and even a finite union of points and open intervals (possibly with endpoints $\pm\infty$). The latter are just a neighborhood basis for a topology of A (the *intrinsic topology*).

When $\mathcal{A} = (A, \leq, \dots)$ is a lattice ordered structure with a least element and a greatest element (in particular, when $\mathcal{A}$ is Boolean lattice ordered), then a similar topology can be defined on A by taking the convex subsets $[a, b] = \{x \in A : a \leq x \leq b\}$ as a sub-basis of closed sets (see [1], 10.12). This is the so-called interval topology. So when $\mathcal{A}$ is quasi-o-minimal, the definable subsets are just the finite Boolean combinations of closed sets in the sub-basis and hence are constructible sets in the interval topology. Notice also that, for arbitrary partially ordered structures $\mathcal{A} = (A, \leq, \dots)$, o-minimality implies quasi o-minimality, but the converse is not always true: there do exist partially ordered quasi-o-minimal structures which are not o-minimal. The following is an example.

Example 2.2 Let $\mathcal{A} = (A, \leq)$, where A is the disjoint union of a copy $\mathbf{Q}$ of the rationals, a copy $\mathbf{Z}$ of the integers, and two additional elements $0_{\mathcal{A}}$ and $1_{\mathcal{A}}$. $\mathbf{Q}$ and $\mathbf{Z}$

are ordered in the usual way; an element of **Q** and an element of **Z** are not comparable with each other. $0_{\mathcal{A}}$ is the least element of A and $1_{\mathcal{A}}$ is the greatest one. Notice that $acl(\emptyset) = \{0_{\mathcal{A}}, 1_{\mathcal{A}}\}$; for any two elements in **Q** or in **Z** are in the same orbit of $Aut(\mathcal{A})$, so any $\emptyset$-definable set overlapping **Q** or **Z** must include **Q**, **Z**, respectively, and hence is infinite. Furthermore, **Q** is $\emptyset$-definable by the following formula.

$$\forall w_1 \forall w_2 \exists u_1 \exists u_2 (w_1 < v < w_2 \rightarrow w_1 < u_1 < v < u_2 < w_2) \\ \wedge \exists w_1 \exists w_2 (w_1 < v < w_2).$$

But **Q** cannot be obtained as a finite Boolean combination of formulas $a \leq v$ or $v \leq b$ with a and b in $acl(\emptyset)$. Consequently, $\mathcal{A}$ is not *o*-minimal. Nevertheless, $\mathcal{A}$ is quasi-*o*-minimal. In fact, fix $r \in \mathbf{Q}$. Then **Q** is defined in $\mathcal{A}$ by

$$\neg(v \leq 0_{\mathcal{A}}) \wedge \neg(v \geq 1_{\mathcal{A}}) \wedge (v \leq r \vee r \leq v).$$

Similarly, **Z** is definable in $\mathcal{A}$. Now take any definable subset D of A. We claim that D is a finite Boolean combination of sets defined by formulas $a \leq v$ or $v \leq b$ with a and b in A. This is trivial for $D \cap \{0_{\mathcal{A}}, 1_{\mathcal{A}}\}$. So owing to the previous remarks on **Q** and **Z**, it suffices to show our claim for $D \cap \mathbf{Q}$ and $D \cap \mathbf{Z}$, and to assume correspondingly, $D \subseteq \mathbf{Q}$ or $D \subseteq \mathbf{Z}$. Then let $D \subseteq \mathbf{Q}$, $D = \varphi(\mathcal{A}, \vec{r}, \vec{s})$ with $\vec{r}$ in **Q** and $\vec{s}$ in **Z**. More precisely, put $\vec{r} = (r_0, r_1, \ldots, r_n)$ where with no loss of generality, $r_0 < r_1 < \cdots < r_n$. Look at the open intervals

$$]0_{\mathcal{A}}, r_0[, \quad]r_0, r_1[, \quad \ldots, \quad]r_n, 1_{\mathcal{A}}[$$

in **Q**. It is easy to see that each of these intervals, when overlapping D, is included in D. Hence, D is defined by

$$(v \leq r_0 \vee v \geq r_0) \wedge \neg(v \leq 0_{\mathcal{A}}) \wedge \neg(v \geq 1_{\mathcal{A}}) \wedge \varphi^{\star}(v, \vec{r}),$$

where $\varphi^{\star}(v, \vec{r})$ defines the open intervals

$$]0_{\mathcal{A}}, r_0[, \quad]r_0, r_1[, \quad \ldots, \quad]r_n, 1_{\mathcal{A}}[$$

and the endpoints $r_0, \ldots, r_n$ contained in D. A similar (slightly more complicated) procedure works when $D \subseteq \mathbf{Z}$.

However, the equivalence

$$\mathcal{A} \text{ } o\text{-minimal} \iff \mathcal{A} \text{ quasi-}o\text{-minimal}$$

holds when $\mathcal{A}$ is a *pure* Boolean algebra. But before proving this result, let us fix some notation. Recall that a Boolean algebra $\mathcal{A}$ can be regarded as a structure in a language with a unique 2-ary relation symbol for $\leq$. Let $\sqcap$, $\sqcup$, and $'$ denote, respectively, the meet, join, and complement operations in $\mathcal{A}$, and $0_{\mathcal{A}}$ and $1_{\mathcal{A}}$ denote the least and the greatest elements in $\mathcal{A}$. All of them are $\emptyset$-definable in $\{\leq\}$. So we can freely add the corresponding symbols to our language because it is easy to see that no result below depends on this additional alphabet. So when considering Boolean algebras, we work in this extended language L_0. Furthermore, if $\mathcal{A}$ is a Boolean algebra and a is an element of A, $\mathcal{A}|a$ denotes the Boolean algebra having domain $\{x \in A : x \leq a\}$ and the obvious structure.

Theorem 2.3 *Let $\mathcal{A}$ be an infinite Boolean algebra. The following propositions are equivalent:*

(i) $\mathcal{A}$ *is o-minimal;*
(ii) $\mathcal{A}$ *is quasi-o-minimal;*
(iii) $\mathcal{A}$ *has only finitely many atoms.*

Proof: (i) $\Longrightarrow$ (ii) The proof is trivial. (ii) $\Longrightarrow$ (iii) Let $At(\mathcal{A})$ denote the set of atoms in $\mathcal{A}$. Obviously $At(\mathcal{A})$ is $\emptyset$-definable in $\mathcal{A}$. If (ii) holds, then $At(\mathcal{A})$ can be expressed as a finite union of sets D defined by a conjunction of formulas,

1. $a_0 \leq v, \ldots, a_s \leq v$,
2. $v \leq b_0, \ldots, v \leq b_t$,
3. $c_0 \not\leq v, \ldots, c_n \not\leq v$,
4. $v \not\leq d_0, \ldots, v \not\leq d_m$,

with parameters from A. Assume $At(\mathcal{A})$ is infinite. Hence at least one set D is infinite too. Fix such a D and the corresponding conjunction of formulas. By putting $a_0 = 0_{\mathcal{A}}$, we can assume that at least one formula occurs in (1). By replacing $a_0, \ldots, a_s$ with their join a, we can suppose that (1) contains exactly one formula $a \leq v$. In a similar way, (2) can be restricted to a unique formula $v \leq b$. Clearly $a \leq b$. Furthermore, we can assume

$$c_i \sqcap a = 0_{\mathcal{A}}, \quad c_i \leq b \quad \forall i \leq n,$$

$$a \leq d_j \leq b \quad \forall j \leq m.$$

In fact, fix $i \leq n$. For $x \in A$ and $x \geq a$,

$$x \geq c_i \Longleftrightarrow x \geq c_i \sqcap a'.$$

So we can replace c_i with $c_i \sqcap a'$. Moreover, if $c_i \not\leq b$, then no element $x \leq b$ in A satisfies $x \geq c_i$ and $v \not\geq c_i$ can be taken out of (3). Proceed similarly for d_j, $j \leq m$.

We can also assume $c_i \neq 0_{\mathcal{A}}$ for all $i \leq n$ and $d_j \neq b$, in fact, ${d_j}' \sqcap b \neq 0_{\mathcal{A}}$, for all $j \leq m$. By replacing $c_0, \ldots, c_n, {d_0}' \sqcap b, \ldots, {d_m}' \sqcap b$ with the atoms of the subalgebra $\mathcal{A}_0$ they generate, we can arrange that

$$c_0, \ldots, c_n, {d_0}' \sqcap b, \ldots, {d_m}' \sqcap b$$

are atoms in $\mathcal{A}_0$. In particular, $c_0, \ldots, c_n$ are pairwise disjoint and ${d_0}' \sqcap b, \ldots, {d_m}' \sqcap b$ are pairwise disjoint (where disjointedness means that the meet is $0_{\mathcal{A}}$).

In conclusion, the atoms of A lying in D are just the elements $x \in A$ satisfying $a \leq x \leq b$ and

$$x' \sqcap c_i \neq 0_{\mathcal{A}} \quad \forall i \leq n,$$

$$x \sqcap ({d_j}' \sqcap b) \neq 0_{\mathcal{A}} \quad \forall j \leq m.$$

Notice that this reduction does not use the hypotheses $D \subseteq At(\mathcal{A})$ and D infinite. But these further assumptions force $a = 0_{\mathcal{A}}$, otherwise $v \geq a$ is satisfied by at most one atom. Moreover, $m \leq 0$ because no atom x in A can overlap two disjoint elements $\neq 0_{\mathcal{A}}$. But $m = 0$ implies that the only element in D is ${d_0}' \sqcap b$ and this contradicts the fact that D is infinite. So no formula occurs in (4).

At this point $b \sqcap (\sqcup c_i)'$ is in D, hence $b \sqcap (\sqcup c_i)'$ is an atom. If every c_i is a finite union of atoms, then the same holds for b, and D is not infinite. So there is $i \leq n$ such that c_i is not a finite union of atoms. Take x and y in A such that $x, y \neq 0_{\mathcal{A}}$, $x \sqcap y = 0_{\mathcal{A}}$, and $x \sqcup y < c_i$. Then $x \sqcup y$ is not an atom, but $x \sqcup y$ is in D. This is a contradiction. So no D is infinite and $At(\mathcal{A})$ is finite.

$(iii) \Longrightarrow (i)$ Assume $At(\mathcal{A})$ is finite, for instance $At(\mathcal{A}) = \{x_0, \ldots, x_m\}$ where $x_0 \neq \cdots \neq x_m$. Put $x = \sqcup_{j \leq m} x_j$. Notice that each element $a \in A$ decomposes uniquely as

$$(a \sqcap x) \sqcup (a \sqcap x'),$$

where either $a \sqcap x$ is $0_{\mathcal{A}}$ or any element $c \neq 0_{\mathcal{A}}$, $c \leq a \sqcap x$, contains some atoms and $a \sqcap x'$ does not contain any atoms. Hence $\mathcal{A}$ is separable according to the terminology of [5]. We know (for instance from [5]) that the theory of separable Boolean algebras is quantifier eliminable in the language $L_1 = L_0 \cup \{R, R_n : n \in \mathbf{N}, n > 0\}$ where R, R_n are 1-ary relation symbols to be interpreted within an arbitrary Boolean algebra $\mathcal{B}$ as follows. For every $b \in B$,

1. $b \in R^{\mathcal{B}}$ if and only if for every element $c \in B$ satisfying $0_{\mathcal{B}} < c \leq b$, there is some atom d in B such that $d \leq b$,
2. $b \in R_n^{\mathcal{B}}$ if and only if there are at least n atoms below b.

In particular, $R^{\mathcal{B}}$ and $R_n^{\mathcal{B}}$ are $\emptyset$-definable in L_0. At this point let us come back to our algebra $\mathcal{A}$. Let $\varphi(v, \vec{y})$ be a formula of L_0 with parameters $\vec{y}$ from A. Hence $\varphi(v, \vec{y})$ is L_1-equivalent in the theory of $\mathcal{A}$ to a suitable finite Boolean combination of formulas

$$p(v, \vec{y}) \leq q(v, \vec{y}), \quad R(p(v, \vec{y})), \quad R_n(p(v, \vec{y})),$$

where $p(v, \vec{y})$ and $q(v, \vec{y})$ are Boolean polynomials (namely, L_1-terms) in v, $\vec{y}$ and n ranges over the positive integers. In the theory of $\mathcal{A}$,

1. $p(v, \vec{y}) \leq q(v, \vec{y})$ is equivalent to $p(v, \vec{y}) \sqcap (q(v, \vec{y}))' = 0_{\mathcal{A}}$,
2. $R(p(v, \vec{y}))$ to $p(v, \vec{y}) \sqcap x' = 0_{\mathcal{A}}$,
3. $R_n(p(v, \vec{y}))$ to either

$$\bigvee_{0 \leq i_1 < \cdots < i_n \leq m} \quad \bigwedge_{1 \leq j \leq n} ((p(v, \vec{y}))' \sqcap x_{i_j} = 0_{\mathcal{A}})$$

(when $n \leq m + 1$) or $v \sqcup 1_{\mathcal{A}} = 0_{\mathcal{A}}$ (otherwise).

To sum up, $\varphi(v, \vec{y})$ is equivalent in the theory of $\mathcal{A}$ to a Boolean combination of formulas of the kind

$$r(v, \vec{y}, \vec{x}) = 0_{\mathcal{A}}$$

where r is a Boolean polynomial in v, $\vec{y}$ and $\vec{x}$ and $\vec{x} = (x_0, \ldots, x_m)$ is in $acl(\emptyset)$ because $At(\mathcal{A})$ is $\emptyset$-definable and finite. Hence $(\vec{y}, \vec{x})$ is in $acl(\vec{y})$. We can assume that $r(v, \vec{y}, \vec{x})$ is a finite join of finite meets of elements among v, $\vec{y}$, $\vec{x}$ and their complements. Thus $r(v, \vec{y}, \vec{x}) = 0_{\mathcal{A}}$ implies that each meet is $0_{\mathcal{A}}$ (and conversely). But in the theory of $\mathcal{A}$, for a given $a \in A$,

1. $v \sqcap a = 0_{\mathcal{A}}$ is equivalent to $v \leq a'$,
2. $v' \sqcap a = 0_{\mathcal{A}}$ to $v \geq a$,

3. $v \sqcap v' \sqcap a = 0_{\mathcal{A}}$ is always satisfied.

In conclusion, $\varphi(v, \vec{y})$ is equivalent in the theory of $\mathcal{A}$ to a finite Boolean combination of formulas $a \leq v$, $v \leq b$ with a and b in $acl(\vec{y})$. Hence $\mathcal{A}$ is o-minimal. □

Actually, the previous proofs of $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$ work for arbitrary expansions of Boolean algebras and only $(iii) \Longrightarrow (i)$ needs the purity assumption. Accordingly one may ask the following.

Problem 2.4 Does

$$\mathcal{A} \text{ quasi-}o\text{-minimal} \Longrightarrow \mathcal{A}\ o\text{-minimal}$$

hold for any Boolean lattice ordered structure $\mathcal{A} = (A, \leq, \ldots)$?

Corollary 2.5 *Infinite o-minimal Boolean algebras $\mathcal{A}$ do not satisfy the exchange principle: there are a, b, and c in A such that $a \in acl(b, c) - acl(c)$ but $b \notin acl(a, c)$.*

Proof: Decompose $\mathcal{A}$ as $\mathcal{A}_0 \times \mathcal{A}_1$ where $\mathcal{A}_1$ is the (finite) subalgebra generated by the atoms of $\mathcal{A}$ and $\mathcal{A}_0$ is atomless. Fix $0_{\mathcal{A}_0} < c < a < 1_{\mathcal{A}_0}$ in A_0 and take $b \in A_0$, $a \sqcap c' < b < a$. Then $a = b \sqcup c \in acl(b, c)$ but $a \notin acl(c)$ and $b \notin acl(a, c)$. □

3 Prime models Let T be a complete theory and Ω still denote a big saturated model of T. When X is a subset of Ω (of smaller cardinality), a model $\mathcal{A}$ of T whose domain contains X is called *(elementarily) prime* over X if and only if for every model $\mathcal{B}$ of T such that $X \subseteq B$, there is an elementary embedding of $\mathcal{A}$ in $\mathcal{B}$ acting identically on X. The existence and the uniqueness (up to X-isomorphism) of prime models over arbitrary subsets X is guaranteed for complete theories of totally ordered o-minimal structures [7] as well as for the theories of Boolean algebras with finitely many atoms (see [10]). This section is devoted to partly extending the existence theorem for theories of Boolean lattice ordered o-minimal structures.

Theorem 3.1 *Let T be a complete theory whose models are Boolean lattice ordered o-minimal structures, X be a subset of Ω (of smaller cardinality). Then there exists a model of T (elementarily) prime over X.*

Proof: It is sufficient to show that, for every subset X of Ω (of smaller cardinality), the isolated 1-types over X are dense in $S_1(X)$. We can restrict our analysis to algebraically closed sets X (so we shall assume $X = acl(X)$). Hence take a formula $\varphi(v)$ in the language $L(X)$ obtained from the language L of T by adding a constant symbol for every element in X. Assume $\varphi(\Omega)$ to be nonempty. By o-minimality, $\varphi(v)$ is equivalent to a disjunction of (consistent) conjunctions of formulas,

1. $a \leq v$,
2. $v \leq b$,
3. $c_0 \not\leq v, \ldots, c_n \not\leq v$,
4. $v \not\leq d_0, \ldots, v \not\leq d_m$,

with parameters from $X = acl(X)$. We need to find a complete formula $\theta(v)$ of $L(X)$ such that $\theta(\Omega)$ is not empty and $\theta(v)$ implies in Ω at least one of the conjunctions in $\varphi(v)$. Hence we can assume that $\varphi(v)$ is just the conjunction (1) – (4), possibly with

$a = 0_\Omega$ and $b = 1_\Omega$. Here are some further permissible assumptions on $c_0, \ldots, c_n$ and $d_0, \ldots, d_m$ (see Theorem 2.3):

(i) $\forall i \leq n$, $c_i \neq 0_\Omega$ and $c_i \sqcap a = 0_\Omega$;
(ii) $\forall j \leq m$, $a \leq d_j < b$;
(iii) $c_0, \ldots, c_n, {d_0}' \sqcap b, \ldots, {d_m}' \sqcap b$ are atoms in the subalgebra of Ω they generate (notice that all these atoms still belong to $X = acl(X)$); in particular, $c_0, \ldots, c_n$ are pairwise disjoint and ${d_0}' \sqcap b, \ldots, {d_m}' \sqcap b$ are pairwise disjoint.

Finally, we can also suppose

(iv) $\forall i \leq n$, c_i is not an atom of Ω (otherwise, for every $s \in \Omega$, $s \not\geq c_i$ if and only if $s \leq {c_i}'$, and by replacing b with $b \sqcap {c_i}'$, we can eliminate $v \not\geq c_i$ in (3));
(v) $\forall j \leq m$, ${d_j}' \sqcap b$ is not an atom of Ω (otherwise, for every $s \leq b$ in Ω, $s \not\leq d_j$ if and only if $s \geq {d_j}' \sqcap b$, and by replacing a with $a \sqcup ({d_j}' \sqcap b)$, we can eliminate $v \not\leq d_j$ in (4)).

Now choose $y_0, \ldots, y_n, z_0, \ldots, z_m \in \Omega$ as follows.

(a) Let $i \leq n$, $c_i \neq {d_j}' \sqcap b$ for every $j \leq m$. Then take $y_i \in \Omega$ satisfying $0_\Omega < y_i < c_i$ (this is possible owing to (iv)); pick $y_i \in X$ when X overlaps $]0_\Omega, c_i[$.
(b) Let $j \leq m$, ${d_j}' \sqcap b \neq c_i$ for every $i \leq n$. Then choose $z_j \in \Omega$ satisfying $0_\Omega < z_j < {d_j}' \sqcap b$ (as allowed by (v)); again pick $z_j \in X$ if possible.
(c) Finally let $i \leq n$, $j \leq m$ satisfy $c_i = {d_j}' \sqcap b$. Accordingly take $y_i, z_j \in \Omega$ such that $0_\Omega < y_i < c_i$, $z_j = {y_i}' \sqcap c_i$; also in this case choose y_i (hence z_j) in X if possible.

We emphasize that $y_0, \ldots, y_n, z_0, \ldots, z_m$ are $\neq 0_\Omega$ and pairwise disjoint. Put

$$y = \sqcup_{i \leq n} y_i, \quad z = \sqcup_{j \leq m} z_j.$$

Notice that, for every $j \leq m$, $z_j \leq {d_j}' \sqcap b \leq {d_j}' \leq a'$. Now let

$$t = (a \sqcup z) \sqcap y'.$$

Then $t \in \varphi(\Omega)$, in fact

1. $a \leq t$ ((i) implies $a \leq {c_i}'$ for all $i \leq n$; hence $a \leq {y_i}'$ for all $i \leq n$ and, consequently, $a \leq y'$; clearly $a \leq a \sqcup z$);
2. $t \leq b$ (for every $j \leq m$, $z_j \leq {d_j}' \sqcap b \leq b$; so $z \leq b$ and $a \sqcup z \leq b$; this forces $t \leq b$);
3. for all $i \leq n$, $c_i \not\leq t$ (it suffices to notice that $c_i \sqcap t' = c_i \sqcap (y \sqcup (a' \sqcap z')) \geq c_i \sqcap y = y_i > 0_\Omega$);
4. for all $j \leq m$, $t \not\leq d_j$ (in fact, $t \sqcap {d_j}' = y' \sqcap (a \sqcup z) \sqcap {d_j}' = y' \sqcap z \sqcap {d_j}'$ because $a \sqcap {d_j}' = 0_\Omega$; hence $t \sqcap {d_j}' \geq y' \sqcap z_j = z_j > 0_\Omega$).

To sum up, we have shown what follows. Fix, if possible,

$$x_i \in X \cap]0_\Omega, c_i[\text{ for } i \leq n,$$

$$t_j \in X \cap]0_\Omega, {d_j}' \sqcap b[\text{ for } j \leq m,$$

$${x_i}' \sqcap c_i = t_j \text{ when } c_i = {d_j}' \sqcap b;$$

put in $L(X)$

$$\begin{aligned}
\theta(v) \quad : \quad & \exists y_0, \ldots, \exists y_n \exists z_0, \ldots, \exists z_m \Big(\bigwedge_{i \le n} 0_\Omega < y_i < c_i \\
\wedge \quad & \bigwedge_{j \le m} 0_\Omega < z_j < d_j{}' \sqcap b \wedge \bigwedge_{i \le n,\, X \cap]0_\Omega, c_i[\neq \varnothing} y_i = x_i \\
\wedge \quad & \bigwedge_{j \le m,\, X \cap]0_\Omega, d_j{}' \sqcap b[\neq \varnothing} z_j = t_j \wedge \bigwedge_{i \le n,\, j \le m} y_i \sqcap z_j = 0_\Omega \\
\wedge \quad & v = (a \sqcup \sqcup_{j \le m} z_j) \sqcap \sqcap_{i \le n} y_i{}' \Big);
\end{aligned}$$

then $\theta(\Omega) \subseteq \varphi(\Omega)$.

At this point it is sufficient to show that $\theta(v)$ is complete in $L(X)$, and hence that two elements t and s in $\theta(\Omega)$ have the same type over X in the language L of T. But the 1-type of t (or s) over X is uniquely determined by its formulas

$$v \le x, \quad v \ge x$$

(or negations) when x ranges over X. For Ω is o-minimal. Accordingly, it is enough to show that two elements $t, s \in \theta(\Omega)$ satisfy the same type over X in $\{\le\}$ (or also in L_0). So let

$$y_0, \ldots, y_n, z_0, \ldots, z_m \in \Omega,$$

$$u_0, \ldots, u_n, w_0, \ldots, w_m \in \Omega$$

witness $t \in \theta(\Omega)$, $s \in \theta(\Omega)$, respectively. Recall the following.

(a) $\forall i \le n$, $0_\Omega < y_i, u_i < c_i$ and $y_i = u_i = x_i \in X$ if $X \cap]0_\Omega, c_i[\neq \varnothing$;

(b) $\forall j \le m$, $0_\Omega < z_j, w_j < d_j{}' \sqcap b$ and $z_j = w_j = t_j \in X$ if $X \cap]0_\Omega, d_j{}' \sqcap b[\neq \varnothing$;

(c) For $i \le n$, $j \le m$ and $c_i = d_j{}' \sqcap b$, $y_i \sqcap z_j = 0_\Omega$ and $y_i \sqcup z_j = c_i$, $u_i \sqcap w_j = 0_\Omega$ and $u_i \sqcup w_j = c_i$.

Take $i \le n$, $X \cap]0_\Omega, c_i[= \varnothing$. Then all the elements $\neq 0_\Omega, c_i$ in $\Omega | c_i$ satisfy the same type over $\varnothing$ in the language L_0 (in $\Omega | c_i$). In fact, Ω is o-minimal also as a Boolean algebra and hence admits only finitely many atoms; each of them is in $acl(\varnothing)$, hence in X, because $At(\Omega)$ is $\varnothing$-definable. So no atom is in $\Omega | c_i$. As $c_i \neq 0_\Omega$ and c_i is not an atom in Ω, $\Omega | c_i$ is an infinite atomless Boolean algebra. Hence all the elements $\neq 0_\Omega, c_i$ in $\Omega | c_i$—in particular, y_i, u_i, or also z_j, w_j when $c_i = d_j{}' \sqcap b$ for some $j \le m$—satisfy the same type over $\varnothing$ in L_0. The same holds when $X \cap]0_\Omega, d_j{}' \sqcap b[= \varnothing$ with $j \le m$ and $d_j{}' \sqcap b \neq c_i$ for every $i \le n$. Consequently there exist some L_0-automorphisms of $\Omega | c_i$ (for $i \le n$), $\Omega | d_j{}' \sqcap b$ (for $j \le m$) mapping, respectively,

y_i into u_i	Case (a),
z_j into w_j	Case (b),
y_i into u_i, z_j into w_j	Case (c);

of course, these automorphisms act just identically when $y_i = u_i \in X$ or $z_j = w_j \in X$. Recalling the properties of $c_0, \ldots, c_n$ and $d_0{}' \sqcap b, \ldots, d_m{}' \sqcap b$, we can glue these L_0-automorphisms to build an automorphism of Ω (in L_0) fixing X pointwise and mapping y_i in u_i for $i \leq n$ and z_j in w_j for $j \leq m$, hence t in s. So t and s have the same type over X in L_0 or also in the language L of T, and we are done. $\square$

Problem 3.2 Let T be a complete theory whose models are Boolean lattice ordered o-minimal structures, Ω be a big saturated model of T, X be a subset of Ω (of smaller cardinality). Is the model of T elementarily prime over X unique (up to X-isomorphism)?

4 The ω-categorical case Let T be a complete theory whose models are Boolean lattice ordered o-minimal structures. As before, we work inside a big saturated model Ω of T. L_0 denotes the language for Boolean algebras and L is the language of T. Our aim is to find the conditions ensuring that T is ω-categorical. For this purpose, it may be worth recalling that, in particular, every o-minimal Boolean algebra (every Boolean algebra with only finitely many atoms) has an ω-categorical complete theory.

Proposition 4.1 *Let T be as before. The following propositions are equivalent:*

(i) *T is ω-categorical;*
(ii) *for every finite subset X of Ω, $acl(X)$ is finite;*
(iii) *there is a function f from $\mathbf{N}$ in $\mathbf{N}$ such that, for every finite $X \subseteq \Omega$ of power n, $|acl(X)| \leq f(n)$.*

Proof: The implication $(i) \Longrightarrow (iii)$ follows from the Ryll-Nardzewski Theorem and $(iii) \Longrightarrow (ii)$ is immediate. But $(ii) \Longrightarrow (i)$, and even $(iii) \Longrightarrow (i)$, may fail in the general setting.

However we claim that, under our assumptions on T, $(ii) \Longrightarrow (i)$ holds. First of all, it suffices to show that for every algebraically closed finite subset X of Ω, $S_1(X)$ is finite. In fact, suppose that this is true. Notice that, consequently, for every finite $X \subseteq \Omega$, $S_1(X)$ is finite because (ii) implies that $acl(X)$ is finite and $S_1(acl(X))$ projects onto $S_1(X)$ by the restriction map. At this point an induction argument shows that $S_n(\varnothing)$ is finite for every positive integer n (and consequently that T is ω-categorical).

So pick $X \subset \Omega$, X finite, X algebraically closed. In particular, X is closed under joint, meet, and complement and contains both 0_Ω and 1_Ω; in other words, X is a finite Boolean subalgebra of Ω. Let $x_0, \ldots, x_k$ denote its atoms. Since Ω is o-minimal even as a Boolean algebra, Ω contains only finitely many atoms. So, for every $j \leq k$, either x_j is an atom of Ω or $\Omega|x_j$ is an infinite atomless Boolean algebra. Furthermore, $1_\Omega = \sqcup_{j \leq k} x_j$ and hence Ω decomposes up to L_0-isomorphism (as a Boolean algebra) in the following way:

$$\Omega \simeq \prod_{j \leq k} \Omega | x_j.$$

So every element $a \in \Omega$ can be expressed as $a = \sqcup_{j \leq k} a_j$ where a_j abbreviates $a \sqcap x_j$ for all $j \leq k$. Notice also that all the elements in $\Omega|x_j$, excepting 0_Ω and x_j, have the same type over $\varnothing$ in $\Omega|x_j$ (as a Boolean algebra). Now observe that, for every $a \in \Omega$,

> the L-type of a over X is fully determined by the ordered sequence of the L_0-types of the a_j's ($j \leq k$) over $\emptyset$ in $\Omega|x_j$,

(so there are only finitely many 1-types over X in L and the claim is proved). In fact, let $b \in \Omega$, put $b_j = b \sqcap x_j$ for every $j \leq k$, and suppose $tp(a_j/\emptyset) = tp(b_j/\emptyset)$ in $\Omega|x_j$ (as a structure of L_0). So, for each $j \leq k$, there is an automorphism of $\Omega|x_j$ in L_0 mapping a_j in b_j. Glue these automorphisms for $j \leq k$. One gets an automorphism of Ω in L_0 fixing each x_j, hence X pointwise, and mapping a in b. So a and b have the same type over X in L_0 and, by o-minimality, in L. □

So if T is a theory satisfying the assumptions at the beginning of this section, and T is ω-categorical, then $acl(\emptyset)$ is a finite Boolean subalgebra of Ω (viewed as a structure of L_0). Let $x_0, \ldots, x_k$ denote its atoms, then $x_0, \ldots, x_k$ generate $acl(\emptyset)$ as a Boolean algebra. As before, as Ω has only finitely many atoms (say n atoms), for every $j \leq k$, either x_j is an atom of Ω or $\Omega|x_j$ is an infinite atomless Boolean algebra.

Lemma 4.2 *Let T be an ω-categorical theory satisfying the assumptions at the beginning of Section 4. Then, for every finite subset X of Ω, $acl(X)$ is the Boolean subalgebra generated by $X \cup \{x_0, \ldots, x_k\}$.*

Actually, we already know that $acl(X)$ is finite and is even a subalgebra: this depends on the ω-categoricity of T and does not use o-minimality. What we want to emphasize here is that the o-minimality of Ω implies that $acl(X)$ is just the subalgebra generated by $X \cup \{x_0, \ldots, x_k\}$ in L_0.

Proof: We proceed by induction on the power of X. The case X empty is clear. Assume $X = \{a\}$ with a in Ω. As before, decompose $a = \sqcup_{j\leq k} a_j$ where, for every $j \leq k$, $a_j = a \sqcap x_j$. Then $a_0, \ldots, a_k \in acl(a)$ and $acl(a) = acl(a_0, \ldots, a_k)$. Let $b \in acl(a)$, decompose $b = \sqcup_{j\leq k} b_j$, where $b_j = b \sqcap x_j$ for every $j \leq k$; hence $b_j \in acl(a)$. Let $0_\Omega < b_j < a_j$. By arguing as in Proposition 4.1, $a_0 \sqcup \cdots \sqcup b_j \sqcup \cdots \sqcup a_k$ has the same type as a over the empty set. Hence there is $c_j \in \Omega$ such that $0_\Omega < c_j < b_j$ and $c_j \in acl(a_0, \ldots, b_j, \ldots, a_k) \subseteq acl(a)$. Repeating this procedure, one builds an infinite strictly decreasing sequence of elements in $acl(a)$ and this contradicts T ω-categorical. In a similar way, one excludes $a_j < b_j < x_j$. Hence either b_j is among $0_\Omega, a_j, x_j$, or b_j, a_j are not comparable. In the latter case, the previous remarks force $b_j \sqcap a_j = 0_\Omega$ and $b_j \sqcup a_j = x_j$ and so $b_j = a_j{}' \sqcap x_j$. In both cases b_j is in the Boolean subalgebra generated by a_j and x_j. Hence $b_0, \ldots, b_k$ are in the Boolean subalgebra generated by a, $x_0, \ldots, x_k$ and then the same is true for b. Therefore $acl(a)$ is contained in this subalgebra and consequently equals it.

Now let us deal with the general case. For every finite set $X \subseteq \Omega$ and for every $b \in \Omega - acl(X)$, $acl(X \cup \{b\})$ equals the algebraic closure of b in the theory of Ω_X (so after adding the elements of X as parameters). Both ω-categoricity and o-minimality are preserved under expanding the language by finitely many constants. So the algebraic closure of b in the theory of Ω_X is just the Boolean subalgebra generated by the union of b and the algebraic closure of $\emptyset$ in Ω_X (namely, the algebraic closure of X in Ω); but, by induction, this is just the subalgebra generated by $X \cup \{x_0, \ldots, x_k\}$. Hence the algebraic closure of $X \cup \{b\}$ in Ω is the Boolean subalgebra generated by $X \cup \{b, x_0, \ldots, x_k\}$. □

Now we want to show that, by adding new constants for $x_0, \ldots, x_k$ to L (this affects neither ω-categoricity nor o-minimality), an ω-categorical theory T satisfying our assumptions is, more or less, the theory of infinite Boolean algebras with n atoms (recall $n = |At(\Omega)|$). Here is the precise statement.

Theorem 4.3 *Let T be an ω-categorical theory satisfying the assumptions at the beginning of Section 4 and let $x_0, \ldots, x_k$ list, as above, the atoms in Ω (so $x_0, \ldots, x_k$ generate $acl(\emptyset)$ as a Boolean algebra). For every L-formula $\varphi(\vec{v}, w_0, \ldots, w_k)$, there is a formula $\varphi_0(\vec{v}, w_0, \ldots, w_k)$ of L_0 such that $\varphi(\vec{v}, x_0, \ldots, x_k)$ and $\varphi_0(\vec{v}, x_0, \ldots, x_k)$ are equivalent in Ω.*

Proof: With no loss of generality, we can assume that L contains some constant symbols for $x_0, \ldots, x_k$, hence $w_0, \ldots, w_k$ do not occur in φ and φ_0. As T is complete, our claim is trivial when φ is a sentence (just choose $\varphi_0 : \forall v(v = v)$ when $\varphi \in T$ and $\varphi_0 : \neg(\forall v(v = v))$ otherwise).

When $\varphi = \varphi(v)$ contains a unique free variable, then by o-minimality, $\varphi(v)$ is equivalent in T to a Boolean combination of formulas $v \geq p$ or $v \leq p$ where $p \in acl(\emptyset)$ and hence p can be expressed as a Boolean polynomial in $x_0, \ldots, x_k$. This provides the required formula $\varphi_0(v)$. Finally, consider the case $\varphi = \varphi(v, \vec{v})$, where $\vec{v} = (v_1, \ldots, v_m)$ is a sequence of variables of length $m > 0$. For every $\vec{y} \in \Omega^m$, $\varphi(v, \vec{y})$ is equivalent in Ω to a Boolean combination of formulas $v \geq p$ or $v \leq p$ where $p \in acl(\vec{y})$, and hence, owing to Lemma 4.2, is a Boolean polynomial in $\vec{y}, x_0, \ldots, x_k$. The decomposition of $\varphi(v, \vec{y})$ as a Boolean combination of formulas $v \geq p$ or $v \leq p$, as well as the decomposition of each p as a Boolean polynomial in $\vec{y}, x_0, \ldots, x_k$, do not depend directly on $\vec{y}$ but only on its type over $\emptyset$, and so are preserved under replacing $\vec{y}$ with another sequence in Ω^m having the same type over $\emptyset$.

Since T is ω-categorical, there are only finitely many (say $s + 1$) m-types over $\emptyset$. Moreover, the type of a given sequence $\vec{y} \in \Omega^m$ is fully determined by

$$(tp(y_{i+1}/y_1, \ldots, y_i) : 0 \leq i < m)$$

in the following sense. Let $\vec{z} \in \Omega^m$ and assume that

1. z_1 has the same type as y_1 over $\emptyset$ (so there is an automorphism f_1 of Ω mapping y_1 into z_1),
2. z_2 has the same type as $f_1(y_2)$ over $z_1 = f_1(y_1)$ (so there is an automorphism f_2 of Ω mapping y_1, y_2 into z_1, z_2 respectively),

and so on; then $tp(\vec{z}/\emptyset) = tp(\vec{y}/\emptyset)$. By ω-categoricity, each type $tp(y_{i+1}/y_1, \ldots, y_i)$ (with $0 \leq i < m$) is isolated. By o-minimality, a formula isolating it can be expressed as a Boolean combination of

$$v_{i+1} \geq q_i, \quad v_{i+1} \leq q_i$$

where q_i is a Boolean polynomial in $x_0, \ldots, x_k, y_1, \ldots, y_i$. Then $tp(\vec{y}/\emptyset)$ is isolated by a Boolean combination of formulas

$$v_{i+1} \geq q_i, \quad v_{i+1} \leq q_i$$

where $0 \leq i < m$ and $q_i = q_i(v_1, \ldots, v_i, x_0, \ldots, x_k)$ is a Boolean polynomial in $v_1, \ldots, v_i, x_0, \ldots, x_k$ (recall that all automorphisms of Ω fix $x_0, \ldots, x_k$). Let

$\theta_0(\vec{v}), \ldots, \theta_s(\vec{v})$ list the formulas defined in this way to isolate the $s+1$ m-types over $\emptyset$. As seen before, for every $j \leq s$ and every $\vec{y} \in \Omega^m, \varphi(v, \vec{y})$ is equivalent to a Boolean combination $\eta_j(v, \vec{y})$ of formulas $v \geq p$ or $v \leq p$ where $p = p(\vec{y}, x_0, \ldots, x_k)$ is a Boolean polynomial in $\vec{y}, x_0, \ldots, x_k$. Altogether, $\varphi(v, \vec{v})$ is equivalent to

$$\varphi_0(v, \vec{v}) \quad : \quad \bigvee_{j \leq s} (\theta_j(\vec{v}) \wedge \eta_j(v, \vec{v})),$$

which is a formula of L_0 and even a Boolean combination of

$$v_{i+1} \geq q_i(v_1, \ldots, v_i, x_0, \ldots, x_k),$$

$$v_{i+1} \leq q_i(v_1, \ldots, v_i, x_0, \ldots, x_k)$$

(with $0 \leq i < m$) and

$$v \geq p(\vec{v}, x_0, \ldots, x_k),$$

$$v \leq p(\vec{v}, x_0, \ldots, x_k).$$

$\square$

5 ***o-minimal measure spaces*** A *measure space* is a triple $(\mathcal{A}, F, m)$ where $\mathcal{A}$ is a(n infinite) Boolean algebra, F is an ordered field, and m (the *measure*) is a function of A in F satisfying

1. $m(0_{\mathcal{A}}) = 0_F$ and for all a in A $m(a) \geq 0_F$,
2. for every a and b in A such that $a \sqcap b = 0_{\mathcal{A}}$, $m(a \sqcup b) = m(a) + m(b)$.

$+$, 0_F denote here the addition in F and its zero element. So measure spaces can be viewed as first order 2-sorted structures in a suitable language L_m extending L_0 by a 1-ary operation symbol for m, two new constants for 0_F and the multiplicative identity 1_F of F, three operation symbols for $+$, $\cdot$ and $-$ in F; of course, we interpret $\leq$ into a relation extending the Boolean order on A and the linear order on F, and we assume that an element of A and an element of F are not comparable with each other with respect to this relation. In particular, every measure space can be considered a partially ordered structure with respect to (the interpretation of) $\leq$; moreover, A and F are $\emptyset$-definable in $(\mathcal{A}, F, m)$: A is the set of elements satisfying $v \geq 0_{\mathcal{A}}$ and F is its complement.

We want to characterize the (quasi)-o-minimal measure spaces. Here is the classification theorem.

Theorem 5.1 *Let $(\mathcal{A}, F, m)$ be a measure space. The following propositions are equivalent:*

(i) $(\mathcal{A}, F, m)$ *is o-minimal;*
(ii) $(\mathcal{A}, F, m)$ *is quasi-o-minimal;*
(iii) *F is a real closed field; $\mathcal{A} \simeq \mathcal{A}_0 \times \mathcal{A}_1$ is the direct product of an infinite atomless Boolean algebra $\mathcal{A}_0$ and a finite algebra $\mathcal{A}_1$; $m(A_0) = 0_F$.*

Before beginning the proof, some comments. In this case (as well as in the previous section), it turns out that o-minimal examples are comparatively trivial; in fact, the measure functions m can take only finitely many values. Consequently no existentially closed measure space with a nonzero measure

as well as

no existentially closed measure space with an effective measure

is *o*-minimal (recall that m is said to be *effective* when $0_{\mathcal{A}}$ is the only element whose measure is 0_F; existentially closed measure spaces in the quoted classes are classified in [8]). This remark suggests the following question (already raised in Section 1): are there some significant examples of *o*-minimal Boolean lattice ordered structures besides Boolean algebras with finitely many atoms, or trivial variations?

But now let us give the proof of Theorem 5.1. We know that $(i) \Longrightarrow (ii)$ is clear.

Proof of $(ii) \Longrightarrow (iii)$*:* Let $(\mathcal{A}, F, m)$ be a quasi-*o*-minimal measure space. Then $\mathcal{A}$—as a structure of L_m—is quasi-*o*-minimal. For, let X be a subset of A definable in L_m inside $\mathcal{A}$; X is definable also in $(\mathcal{A}, F, m)$ (just conjunct '$v \geq 0_{\mathcal{A}}$') and hence is a Boolean combination of sets defined by formulas

$$v \geq a, \quad v \leq b$$

with $a, b \in A \cup F$. But actually no parameter in F is necessary because no element in A is comparable with F. Then $\mathcal{A}$ is quasi-*o*-minimal in L_m. In particular $\mathcal{A}$ is quasi-*o*-minimal, hence *o*-minimal, also as a Boolean algebra. Accordingly, $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1$ where $\mathcal{A}_1$ is a finite algebra and $\mathcal{A}_0$ is an infinite atomless algebra; if s denotes the join of the atoms in $\mathcal{A}$, then $\mathcal{A}_1$ is just isomorphic to $\mathcal{A}|s$, and $\mathcal{A}_0$ to $\mathcal{A}|s'$. As $s \in dcl(\emptyset)$, both $\mathcal{A}_0$ and $\mathcal{A}_1$ are $\emptyset$-definable. In the same way, one sees that F (as a structure of L_m or also as an ordered field) is quasi-*o*-minimal. By Theorem 2.3 in [7], F is real closed.

So what we have still to check is that $m(A_0) = 0_F$. As $\mathcal{A}_0$ is $\emptyset$-definable in $\mathcal{A}$, we can assume with no loss of generality $A_0 = A$, hence $\mathcal{A}$ atomless (and infinite). Suppose toward a contradiction $m(A) \neq 0_F$. In particular, we can normalize m and fix $m(1_{\mathcal{A}}) = 1_F$.

Lemma 5.2 *Let $\mathcal{A} = (A, \leq, \ldots)$ be an infinite Boolean lattice ordered quasi-o-minimal structure such that the underlying Boolean algebra is atomless. Then every filter (ideal) definable in $\mathcal{A}$ is principal.*

Proof: By duality, we can limit our analysis to filters. Let U be a filter of $(A, \leq)$ definable in $\mathcal{A}$. By quasi *o*-minimality, U is a finite union of sets defined by formulas

1. $v \geq a$,
2. $v \leq b$,
3. $v \not\geq c_0, \ldots, v \not\geq c_n$,
4. $v \not\leq d_0, \ldots, v \not\leq d_m$

with parameters in A. As before, we can assume that

(i) $\forall i \leq n$, $c_i \neq 0_{\mathcal{A}}$, $c_i \sqcap a = 0_{\mathcal{A}}$ and $c_i \leq b$;

(ii) $\forall j \leq m$, $a \leq d_j < b$;

(iii) $c_0, \ldots, c_n, {d_0}' \sqcap b, \ldots, {d_m}' \sqcap b$ are atoms in the subalgebra they generate; in particular, $c_0, \ldots, c_n$ are pairwise disjoint and ${d_0}' \sqcap b, \ldots, {d_m}' \sqcap b$ are pairwise disjoint.

Let $x \in A$ satisfy $a \le x \le b$, $x \not\ge c_0, \dots, c_n$, $x \not\le d_1, \dots, d_m$ but $x \le d_0$, namely, $x \sqcap {d_0}' \sqcap b = 0_{\mathcal{A}}$. As ${d_0}' \sqcap b \ne 0_{\mathcal{A}}$ and $\mathcal{A}$ is atomless, there is some y in A, $0_{\mathcal{A}} < y < {d_0}' \sqcap b$; put $z_0 = x \sqcup y$, $z_1 = x \sqcup (y' \sqcap {d_0}' \sqcap b)$, so $z_0, z_1 \in U$ because both z_0 and z_1 realize (1)–(4). However $x = z_0 \sqcap z_1$, hence also x is in U. In other words, we can eliminate the condition $v \not\le d_0$ in (4). By repeating the procedure, we can conclude that (4) is unnecessary. Accordingly U is a finite union of sets S such that each of them is defined by a conjunction of formulas in (1)–(3). So, for every S, the corresponding a is in $S \subseteq U$; consequently every element $x \ge a$ in A is in U. Since the meet of all possible a's is in U, it follows that U is the principal filter generated by some a. □

Let us come back now to our measure space $(\mathcal{A}, F, m)$. Recall that we assume $\mathcal{A}$ atomless and $m(1_{\mathcal{A}}) = 1_F$. As a consequence of Lemma 5.2, we can obtain the following corollary.

Corollary 5.3 *There exists $d \in A$ such that for all $x \in A$, $m(x_F) = 0_F$ if and only if $x \le d$.*

Proof: Let I be the set of the elements in A whose measure is 0_F. Then I is a (proper) ideal of $\mathcal{A}$; furthermore, I is $\emptyset$-definable in $\mathcal{A}$ (as a structure of L_m). By Lemma 5.2, I is principal. Let $d \in A$ generate I. Then d is the required element. □

Clearly, d is unique, and owing to our assumptions, $d \ne 1_{\mathcal{A}}$. Decompose $\mathcal{A}$ (up to L_0-isomorphism) as $\mathcal{A}|d \times \mathcal{A}|d'$. $d' \ne 0_{\mathcal{A}}$ implies that $\mathcal{A}|d'$ is an infinite atomless Boolean algebra. Furthermore, m is effective in $\mathcal{A}|d'$; in other words, no element $x \ne 0_{\mathcal{A}}$ in $\mathcal{A}|d'$ satisfies $m(x) = 0_F$. Without loss of generality, we can replace $\mathcal{A}$ with $\mathcal{A}|d'$ and assume that m is effective in $\mathcal{A}$. Notice that if $x \in A$ and $x \ne 0_{\mathcal{A}}$, then there is $y \in A$ such that $0_{\mathcal{A}} < y < x$; consequently,

$$0_F < m(y), m(y'), \quad m(x) = m(y) + m(y'),$$

hence either $m(y)$ or $m(y')$ is $\le \frac{1}{2} m(x)$.

Lemma 5.4 *Let $b \in A$, $b \ne 0_{\mathcal{A}}$. Then for every $\epsilon \in F$ with $0_F < \epsilon < m(b)$ there is $c \in A$ such that $0_{\mathcal{A}} < c < b$ and $m(c) \le \epsilon$.*

Proof: Let S be the set of the elements $s \in F$ satisfying the following conditions:

1. $0_F < s < m(b)$,
2. for all $c \in A$ with $0_{\mathcal{A}} < c < b$, $m(c) > s$.

S is definable in F (as a structure of L_m). So S (if nonempty) is a finite disjoint union of points and intervals in $F^{>0_F}$. Furthermore, S is downward closed in $F^{>0_F}$ and upperly bounded (by $m(b)$). Hence S contains a maximal interval of the form $]0_F, r[$ or $]0_F, r]$ with $r \in F$, $r > 0_F$. In the former case, there is $y \in A$ such that $y \le b$ and $m(y) = r$. For, if $r = m(b)$, then $y = b$, of course; otherwise $r < m(b)$, and there is $y \in A$ such that $0_{\mathcal{A}} < y < b$ and $m(y) \le r$; owing to the choice of r, $m(y) = r$. Then there is $c \in A$ such that $0_{\mathcal{A}} < c < y \le b$ and $m(c) \le \frac{r}{2} < r$ and this is impossible because $\frac{r}{2} \in S$. In the latter case, take $r_1 \in F$ with $r < r_1 < 2r$, $r_1 \notin S$, $r_1 < m(b)$. For some $y \in A$ with $0_{\mathcal{A}} < y < b$, $m(y) \le r_1$. Then there is $c \in A$ such that $0_{\mathcal{A}} < c < y$ and $m(c) \le \frac{r_1}{2} < r$. Again we get a contradiction. So S must be empty. □

Consider now $X = \{x \in A : m(x) < \frac{1}{2}\}$. X is definable and nonempty, and hence is a finite union of sets $X_0, \ldots, X_k$ such that, for every $t \leq k$, X_t is defined by a conjunction of formulas:

1. $v \geq a_t$,
2. $v \leq b_t$,
3. $v \not\geq c_{t0}, \ldots, v \not\geq c_{tn_t}$,
4. $v \not\leq d_{t0}, \ldots, v \not\leq d_{tm_t}$

where the parameters $a_t, b_t, c_{t0}, \ldots, c_{tn_t}, d_{t0}, \ldots, d_{tm_t}$ satisfy the usual assumptions (i), (ii), and (iii). Notice that

$$b_0 \sqcup \cdots \sqcup b_k = 1_{\mathcal{A}};$$

otherwise $b = (b_0 \sqcup \cdots \sqcup b_k)'$ has measure $> 0_F$ and there is $c < b$ in A such that $0_F < m(c) < \frac{1}{2}$ (Lemma 5.4); but $c \not\leq b_t$ for any $t \leq k$, hence $c \notin X$.

Now we claim that for every $t \leq k$, $m(b_t) \leq \frac{1}{2}$. Assume not. Then $m(b_t) > \frac{1}{2}$ for some $t \leq k$. There is $x \in A$ for which $0_{\mathcal{A}} < x < b_t$ and $m(x) < m(b_t) - \frac{1}{2}$ (Lemma 5.4). Without loss of generality, we can assume $x \sqcap a_t = 0_{\mathcal{A}}$ and

$$0_{\mathcal{A}} < x \sqcap c_{it} < c_{it} \quad \forall i \leq n_t,$$

$$0_{\mathcal{A}} < x \sqcap d_{jt}{}' \sqcap b_t < d_{jt}{}' \sqcap b_t \quad \forall j \leq m_t$$

(use again Lemma 5.4). Hence $x' \sqcap b_t \in X_t$ because $a_t \leq x' \sqcap b_t \leq b_t$, $0_{\mathcal{A}} < x' \sqcap b_t \sqcap c_{it} < c_{it}$ for all $i \leq n_t$ and $0_{\mathcal{A}} < x' \sqcap b_t \sqcap d_{jt}{}' < d_{jt}{}' \sqcap b_t$ for all $j \leq m_t$; consequently, $m(x' \sqcap b_t) < \frac{1}{2}$. It follows that

$$m(b_t) = m(x) + m(x' \sqcap b_t) < m(b_t) - \frac{1}{2} + \frac{1}{2} = m(b_t),$$

and this is a contradiction.

Therefore, $m(b_t) \leq \frac{1}{2}$ for all $t \leq k$. Notice that, if $m(b_t) < \frac{1}{2}$, then every element $x \leq b_t$ in A satisfies $m(x) < \frac{1}{2}$, whereas, if $m(b_t) = \frac{1}{2}$, then, as m is effective, the elements $x \in A$ satisfying $m(x) < \frac{1}{2}$ and $x \leq b_t$ are just those $< b_t$. In conclusion, for all $a \in A$,

$$x \in X$$

if and only if

$$x \leq b_t \text{ for some } t \leq k, \text{ and } x < b_t \text{ when } m(b_t) = \frac{1}{2}.$$

Now suppose $m(b_t) < \frac{1}{2}$ and put $m_t = \frac{1}{2} - m(b_t)$. So $m(b_t{}') = m_t + \frac{1}{2} > \frac{1}{2}$ and one can find $c_t \in A$ such that $0_{\mathcal{A}} < c_t < b_t{}'$ and $m(c_t) < m_t$. Form $b_t \sqcup c_t$ and notice

$$m(b_t \sqcup c_t) = m(b_t) + m(c_t) < \frac{1}{2}.$$

Hence there is $s \leq k$ such that $b_t \sqcup c_t \leq b_s$. Clearly, $t \neq s$ because $0_{\mathcal{A}} < c_t < b_t{}'$. So $b_t < b_s$ for some $s \leq k$, $s \neq t$. Consequently, all the indices $t \leq k$ for which $m(b_t) < \frac{1}{2}$ can be eliminated and we can assume $m(b_t) = \frac{1}{2}$ for all $t \leq k$. Notice that this preserves $\sqcup_{t \leq k} b_t = 1_{\mathcal{A}}$. Consequently, for all $x \in A$,

$$x \in X \text{ if and only if } x < b_t \text{ for some } t \leq k.$$

Let k be minimal. If $k = 0$, then $b_0 = 1_{\mathcal{A}}$, so $m(x) < \frac{1}{2}$ for every $x \neq 1_{\mathcal{A}}$ in A and this is obviously false. Hence $k > 0$. Owing to the minimality of k, for all $t \leq k$, there is $x_t \in A$ such that $0_{\mathcal{A}} < x_t < b_t$ (and $m(x_t) < \frac{1}{2}$) but $x_t \sqcap b_s{}' \neq 0_{\mathcal{A}}$ for every $s \leq k$, $s \neq t$. By Lemma 5.4, we can choose x_t such that

$$m(x_t) < \frac{1}{2(k+1)}$$

(keeping the assumption $x_t \sqcap b_s{}' \neq 0_{\mathcal{A}}$ for $s \leq k$ and $s \neq t$). Build $x = \sqcup_{t \leq k} x_t$, then

$$m(x) \leq \sum_{t \leq k} m(x_t) < (k+1)\frac{1}{2(k+1)} = \frac{1}{2},$$

so $x \in X$ and $x \leq b_t$ for some $t \leq k$. Choose $s \leq k$, $s \neq t$, then $x_s \leq x \leq b_t$. This is a contradiction.

In conclusion $m(A) = 0_F$. This completes the proof of $(ii) \Longrightarrow (iii)$. □

Proof of $(iii) \Longrightarrow (i)$*:* Let $(\mathcal{A}, F, m)$ be a measure space such that F is a real closed field, $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1$ where $\mathcal{A}_1$ is finite, $\mathcal{A}_0$ is infinite and atomless, and $m(A_0) = 0_F$.

Fix a and b in A, $X \subseteq A$, X algebraically closed (in $\mathcal{A}$ with respect to L_0). We claim that

$$\text{if } tp(a/X) = tp(b/X) \text{ in } \mathcal{A}, \text{ then } tp(a/X \cup F) = tp(b/X \cup F) \text{ in } (\mathcal{A}, F, m).$$

For, let $(\overline{\mathcal{A}}, \overline{F}, \overline{m})$ be a (big) saturated elementary extension of $(\mathcal{A}, F, m)$; in particular, $\overline{\mathcal{A}}$ is a saturated elementary extension of $\mathcal{A}$, hence there is an automorphism f of $\overline{\mathcal{A}}$ in L_0 fixing X pointwise and mapping a in b; f fixes every atom in $\overline{\mathcal{A}}$, hence acts identically on $\overline{\mathcal{A}_0}$. Therefore, we can extend f to an automorphism g of $(\overline{\mathcal{A}}, \overline{F}, \overline{m})$ fixing $\overline{F}$ pointwise; in fact, if s denotes the join of atoms in $\mathcal{A}$ (and in $\overline{\mathcal{A}}$), then for all $x \in \overline{A}$,

$$\overline{m}(x) = \overline{m}(s \sqcap x) + \overline{m}(s' \sqcap x) = \overline{m}(s \sqcap x)$$

and

$$f(x) = f((s \sqcap x) \sqcup (s' \sqcap x)) = (s \sqcap x) \sqcup (s' \sqcap f(x)),$$

so that

$$g(\overline{m}(x)) = \overline{m}(x) = \overline{m}(s \sqcap x) = \overline{m}(f(x)) = \overline{m}(g(x)).$$

In particular, a and b have the same type over $X \cup F$ in $(\mathcal{A}, F, m)$.

Now take a formula $\varphi(v)$ of L_m having parameters in $X \cup F$ (and implying $v \geq 0_{\mathcal{A}}$, namely, "$v \in A$"). For every 1-type p over $X \cup F$ in $(\mathcal{A}, F, m)$ containing $\varphi(v)$, there is a formula $\varphi_p(v) \in p$ in L_0 with parameters in X such that $\varphi_p(v)$ implies $\varphi(v)$. So $\varphi(v)$ is equivalent to the (possibly infinite) disjunction $\vee_p \varphi_p(v)$. By compactness, $\varphi(v)$ is equivalent to a finite disjunction $\varphi^\star(v)$ of formulas $\varphi_p(v)$. As $\mathcal{A}$ is o-minimal, $\varphi^\star(v)$ is in its turn equivalent to a Boolean combination of formulas $a \leq v$, $v \leq b$ with a and b in X (recall $X = acl(X)$ in $\mathcal{A}$).

Now let a, b in F, $Y \subseteq F$. Notice that if

$$tp(a/Y \cup m(A)) = tp(b/Y \cup m(A)) \text{ in } F,$$

then

$$tp(a/A \cup Y) = tp(b/A \cup Y) \text{ in } (\mathcal{A}, F, m).$$

For, let $(\overline{\mathcal{A}}, \overline{F}, \overline{m})$ be as before. Then there is an automorphism f of $\overline{F}$ fixing $Y \cup m(A)$ pointwise and mapping a in b; glue f and the identity of $\overline{\mathcal{A}}$ and get an automorphism g of $(\overline{\mathcal{A}}, \overline{F}, \overline{m})$ (recall that f fixes $m(A)$ pointwise). So a and b have the same type over $A \cup Y$ in $(\mathcal{A}, F, m)$.

At this point, given a formula $\varphi(v)$ of L_m having parameters from $A \cup Y$ (and including '$v \in F$'), proceed as before to build an equivalent formula $\varphi^\star(v)$ in the language of ordered fields, with parameters in $Y \cup m(A)$. As F is real closed, hence o-minimal, $\varphi^\star(v)$ is in its turn equivalent to a Boolean combination of formulas $a \leq v$, $v \leq b$ where a and b are in the definable closure of $Y \cup m(A)$ in F. As $m(A)$ is included in the algebraic closure of $\emptyset$ in $(\mathcal{A}, F, m)$, $a, b \in acl(Y)$ in $(\mathcal{A}, F, m)$.

In conclusion $(\mathcal{A}, F, m)$ is o-minimal. □

Acknowledgments I thank the referee for suggesting several improvements.

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