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Local Polynomial Convexity of Certain Graphs in C^2

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1. Introduction

Let K be a compact subset of $\mathbf{C}^n$, and denote by $\hat{K}$ the polynomial convex hull of K:

$$\hat{K} = \{z \in \mathbf{C}^n : |p(z)| \le \|p\|_K \text{ for every polynomial } p \text{ in } \mathbf{C}^n\}.$$

We say that K is *polynomially convex* if $\hat{K} = K$. A compact K is called *locally polynomially convex* at $a \in K$ if there exists a closed ball $\bar{B}(a,r)$, centered at a and with radius $r > 0$, such that $\bar{B}(a,r) \cap K$ is polynomially convex. A compact $K \subset \mathbf{C}$ is polynomially convex if $\mathbf{C} \setminus K$ is connected. In higher dimensions there is no such topological characterization of polynomially convex sets, and it is usually hard to determine whether a given compact set is polynomially convex. By a well-known result of Wermer ([We]; see also [AWe, Thm. 17.1]), every totally real manifold is locally polynomially convex. Recall that a $\mathcal{C}^1$ smooth real manifold M is called *totally real* at $p \in M$ if the real tangent space $T_p M$ contains no complex line. In this paper, we are concerned with local polynomial convexity at the origin of the graph Γ_f of a $\mathcal{C}^2$ smooth function f near $0 \in \mathbf{C}$ such that $f(0) = 0$.

By the theorem of Wermer just cited, we know that if $\frac{\partial f}{\partial \bar{z}}(0) \neq 0$ then Γ_f is locally polynomially convex at the origin. Thus it remains to consider the case where $\frac{\partial f}{\partial \bar{z}}(0) = 0$. The work associated with this direction of research is too numerous to list here; instead, the reader is referred to [B1; B2; Wi] and the references given therein. The general scheme of studying local polynomially convexity of Γ_f is by pulling back Γ_f under a proper holomorphic map. Then the inverse of Γ_f is a finite union $X_1 \cup \dots \cup X_k$ of totally real graphs. Using Wermer's theorem, we conclude that the inverse of Γ_f is a finite union of *locally* polynomially convex compact sets meeting only at the origin.

Next, under some appropriate assumptions on f, one can show that there is a polynomial map $p\colon \mathbf{C}^2 \to \mathbf{C}$ such that the sets $p(X_k)$ are contained in disjoint open sectors with vertex at the origin. Then a lemma of Kallin ([Ka]; see also [Pa]) implies that $X_1 \cup \dots \cup X_k$ is locally polynomially convex at the origin. Finally, since polynomial convexity behaves nicely under proper holomorphic transformations, we conclude that Γ_f is locally polynomially convex at the origin.

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This scheme was first carried out in [FS], where it is proved that if $f(z) = |z|^2 + \gamma(z^2 + \bar{z})^2 + o(|z|^2)$ then Γ_f is locally polynomially convex provided $\gamma > 1/2$. Using the same line of argument, Bharali [B1; B2] studies the case where f vanishes to higher order. One drawback of this strategy is that it is not possible to describe *explicitly* a Stein neighborhood basis of Γ_f near the origin when Γ_f is locally polynomially convex at that point.

In this paper, as in [D], we employ the theory of plurisubharmonic functions and plurisubharmonic hulls to attack the problem. More precisely, we construct nonnegative smooth functions vanishing exactly on Γ_f. These functions are, in general, plurisubharmonic only on open sets whose boundaries contain the origin. Under some technical assumptions, we may add small strictly plurisubharmonic functions to obtain plurisubharmonic functions on certain open sets containing the (local) polynomially convex hull of Γ_f. By invoking the classical (and nontrivial) fact about equivalence of plurisubharmonic and polynomial hulls, we conclude that Γ_f is locally polynomially convex at the origin. One advantage of this approach is that it gives a fairly explicit construction of a Stein neighborhood basis of Γ_f when Γ_f is locally polynomially convex at the origin; see the remarks following the proof of Theorem 2.1. We should say that a complete characterization of smooth functions f such that Γ_f is locally polynomially convex at the origin is still open even when f is real-analytic.

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2. Local Polynomial Convexity of Graphs

First we fix the notation that will be used later on. We let $\Delta f(z) := \frac{\partial^2 f}{\partial z \partial \bar{z}}(z)$, where f is a C^2 smooth function. We denote by $B(a,r)$ the open ball with center a and radius r.

Now we come to the main result of this work.

THEOREM 2.1. *Let f be a C^2 smooth function on a neighborhood of $0 \in \mathbf{C}$. Assume that there exist a holomorphic function g defined near $0 \in \mathbf{C}$ and a constant $\lambda \in (0,1)$ satisfying the following conditions*:

(i) $|f|^2 \le \operatorname{Re}(fg)$;
(ii) $g(0) = 0$;
(iii) $|g\Delta\bar{f}| < \lambda\left|\frac{\partial f}{\partial \bar{z}}\right|^2 + \operatorname{Re}(f\Delta\bar{f})$ *for every* $z \neq 0$.

Then Γ_f is locally polynomially convex at the origin in $\mathbf{C}^2$. Furthermore, there exists an $r > 0$ small enough such that a continuous function on $X_r := \Gamma_f \cap \bar{B}(0,r)$ can be approximated uniformly by polynomials.

Observe that (i) and (ii) imply $|f| \leq |g|$ and $f(0) = 0$. Also, it follows from (iii) that the set $\left\{z : \frac{\partial f}{\partial \bar{z}}(z) = 0\right\}$ consists of the origin only.

Before taking up the proof of Theorem 2.1, it is worth mentioning the following simple consequence, which is easier to appreciate. The Γ_f considered here is the same as in Theorem 1.1 in [B1]. Our sufficient condition (i) has the same formal structure as condition (1.1) in [B1], but we shall soon show how our choice of γ in (i) makes it less restrictive than Bharali's condition. See the remarks following the proof of Corollary 2.2.

COROLLARY 2.2. *Let $k \geq 1$ be an integer. Let*

$$f(z) = \alpha z^{2k} + \beta |z|^{2k} + \gamma \bar{z}^{2k} + \mathcal{R}(z)$$

be a $\mathcal{C}^2$ smooth function near $0 \in \mathbf{C}$. Assume that:

(i) $|\gamma| > \alpha|\beta|$, *where α is the largest root of the equation*

$$4x^3 - 10x^2 + 5x - 3 = 0;$$

(ii) $\frac{\partial \mathcal{R}}{\partial \bar{z}}(z) = o(|z|^{2k-1})$ *and* $\Delta\mathcal{R}(z) = o(|z|^{2k-2})$.

Then Γ_f is locally polynomially convex at the origin.

Proof. By considering the biholomorphic transformation $(z, w) \mapsto (z, w - \alpha z^{2k})$, we may assume that $\alpha = 0$. After a rotation, we may establish further that $\gamma > 0$. We rewrite condition (i) as

$$|\beta|(|\beta| + |\gamma|)^2 < (|\gamma| - |\beta|)(4|\gamma|^2 - 5|\gamma\beta| + 2|\beta|^2). \tag{1}$$

According to Theorem 2.1, it suffices to show that the function $g(z) = az^{2k}$ satisfies the conditions of the theorem for appropriately chosen $a > 0$. For this, by (ii) we may write

$$\frac{\partial f}{\partial \bar{z}}(z) = 2k\gamma\bar{z}^{2k-1} + k\beta z^k \bar{z}^{k-1} + o(|z|^{2k-1}),$$
$$\Delta\bar{f}(z) = k^2\bar{\beta}|z|^{2k-2} + o(|z|^{2k-2}).$$

On the one hand, for every $\lambda > 0$ and z small enough we have

$$\begin{aligned}
&\lambda\left|\frac{\partial f}{\partial \bar{z}}(z)\right|^2 + \mathrm{Re}(f(z)\Delta\bar{f}(z)) \\
&\quad = k^2|z|^{2k-2}(\lambda|2\gamma\bar{z}^k + \beta z^k|^2 + |\beta|^2|z|^{2k} + \mathrm{Re}(\bar{\beta}\gamma\bar{z}^{2k})) + o(|z|^{4k-2}) \\
&\quad \geq k^2|z|^{4k-2}(\lambda(4|\gamma|^2 - 4|\gamma\beta| + |\beta|^2) + |\beta|^2 - |\beta\gamma|) + o(|z|^{4k-2}).
\end{aligned}$$

On the other hand,

$$\mathrm{Re}(g(z)\Delta\bar{f}(z)) \leq ak^2|\beta||z|^{4k-2} + o(|z|^{4k-2}).$$

In view of (i), we may choose $a > 0$ and $\lambda \in (0, 1)$ such that

$$a|\beta| < \lambda(4|\gamma|^2 - 4|\gamma\beta| + |\beta|^2) + |\beta|^2 - |\gamma\beta|$$

and

$$(|\beta| + |\gamma|)^2 < a(|\gamma| - |\beta|).$$

It is straightforward to check that the two conditions of Theorem 2.1 are fullfilled with the given choices of g and λ. □

REMARKS. 1. If $k = 1$ then our result is a special case of the Forstneric–Stout theorem mentioned in the Introduction. In [Sl, Prop. 1], the author constructs an explicit Stein neighborhood basis for Γ_f in the case where $k = 1$, $\beta \in [0, 1/2)$, and $\gamma = 1/4$. This method, however, does not seem to cover the case where f vanishes to higher order. In contrast, once we have furnished a proof of Theorem 2.1, it will be fairly easy to construct an explicit Stein neighborhood basis for a large subclass of the graphs Γ_f considered in Theorem 2.1. See the remark following the proof of Theorem 2.1.

2. It is also natural to compare Corollary 2.2 with the main results in [B1; B2] concerning local polynomial convexity of graphs. Roughly speaking, in our context [B1, Thms. 1.1, 1.2] and [B2, Thm. 1.2] state that Γ_f is locally polynomially convex if $|\gamma| > \lambda_k|\beta|$ for $\lambda_k > 0$ satisfying $\lambda_k \uparrow \infty$ when $k \to \infty$. This condition is much more restrictive than our condition (i), at least when k is large enough. However, we must say that a condition like our (ii) is not needed in [B1] or [B2], where less regularity is assumed on f. Nevertheless, in the most interesting case—namely, when f is real-analytic near the origin—condition (ii) is redundant and Corollary 2.2 follows from (i) alone.

For the proof of Theorem 2.1, we first introduce the following auxiliary functions. Consider the function $\theta\colon [0, +\infty) \to \mathbf{R}$ defined by

$$\theta(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t = 0. \end{cases}$$

For $x \geq 0$, set

$$\chi(x) = \int_0^x \theta(t)\,dt.$$

It is easy to check that $\chi(0) = \chi'(0) = \chi''(0) = 0$ and that $\chi'(x) = x^2\chi''(x)$ and $\chi''(x) > 0$ for $x > 0$. Moreover, $\chi''(x)$ decreases to 0 faster than any exponent of x.

The key technical lemma in the proof of Theorem 2.1 is the following.

LEMMA 2.3. *Let h be a C^2 smooth function on an open set $U \subset \mathbf{C}$. Set*

$$F(z, w) = |w - h(z)|^2 \quad \forall (z, w) \in U \times \mathbf{C}.$$

Let Ω be the open subset of $U \times \mathbf{C}$ defined by

$$\Omega := \left\{(z, w) \in U \times \mathbf{C} : \operatorname{Re}((w - h(z))\Delta\bar{h}(z))(1 + F(z, w)) < \left|\tfrac{\partial f}{\partial \bar{z}}(z)\right|^2\right\}.$$

Then $\varphi := \chi \circ F$ is plurisubharmonic on Ω.

Proof. We have

$$\frac{\partial^2\varphi}{\partial w\partial\bar{w}}(z,w) = \frac{\partial^2 F}{\partial w\partial\bar{w}}(z,w)\chi'(F(z,w)) + \left|\frac{\partial F}{\partial w}(z,w)\right|^2 \chi''(F(z,w)),$$
$$\frac{\partial^2\varphi}{\partial z\partial\bar{z}}(z,w) = \frac{\partial^2 F}{\partial z\partial\bar{z}}(z,w)\chi'(F(z,w)) + \left|\frac{\partial F}{\partial z}(z,w)\right|^2 \chi''(F(z,w)),$$
$$\frac{\partial^2\varphi}{\partial z\partial\bar{w}}(z,w) = \frac{\partial^2 F}{\partial z\partial\bar{w}}(z,w)\chi'(F(z,w)) + \frac{\partial F}{\partial z}(z,w)\frac{\partial F}{\partial \bar{w}}(z,w)\chi''(F(z,w)).$$

Now it suffices to check the following inequalities on Ω:

(a) $\frac{\partial^2\varphi}{\partial w\partial\bar{w}} \geq 0$, $\frac{\partial^2\varphi}{\partial z\partial\bar{z}} \geq 0$;
(b) $\frac{\partial^2\varphi}{\partial w\partial\bar{w}}\frac{\partial^2\varphi}{\partial z\partial\bar{z}} \geq \left|\frac{\partial^2\varphi}{\partial z\partial\bar{w}}\right|^2$.

Since $\chi'(F) = F^2\chi''(F)$ and $\frac{\partial^2 F}{\partial w\partial\bar{w}} = 1$, an easy computation gives $\frac{\partial^2\varphi}{\partial w\partial\bar{w}} \geq 0$ on $U \times \mathbf{C}$. Moreover,

$$\frac{\partial^2\varphi}{\partial w\partial\bar{w}}(z,w) = 0 \implies F(z,w) = 0 \implies \frac{\partial^2\varphi}{\partial z\partial\bar{z}}(z,w) = 0.$$

Therefore, (a) is a consequence of (b). It is also immediate to check that (b) is equivalent to the following inequality on Ω:

$$\left(\frac{\partial^2 F}{\partial w\partial\bar{w}}F^2 + \left|\frac{\partial F}{\partial w}\right|^2\right)\left(\frac{\partial^2 F}{\partial z\partial\bar{z}}F^2 + \left|\frac{\partial F}{\partial z}\right|^2\right) \geq \left|\frac{\partial^2 F}{\partial z\partial\bar{w}}F^2 + \frac{\partial F}{\partial z}\frac{\partial F}{\partial\bar{w}}\right|^2,$$

which (after rearranging) is a consequence of the following inequality on Ω:

$$F^2\frac{\partial^2 F}{\partial w\partial\bar{w}}\frac{\partial^2 F}{\partial z\partial\bar{z}} + \left|\frac{\partial F}{\partial w}\right|^2\frac{\partial^2 F}{\partial z\partial\bar{z}} + \frac{\partial^2 F}{\partial w\partial\bar{w}}\left|\frac{\partial F}{\partial z}\right|^2 \geq F^2\left|\frac{\partial^2 F}{\partial z\partial\bar{w}}\right|^2 + 2\,\mathrm{Re}\left(\frac{\partial^2 F}{\partial w\partial\bar{z}}\frac{\partial F}{\partial z}\frac{\partial F}{\partial\bar{w}}\right). \quad (2)$$

We now apply the special form of F to obtain, for every $(z,w) \in U \times \mathbf{C}$, the expressions

$$\frac{\partial F}{\partial z}(z,w) = -(\bar{w} - \overline{h(z)})\frac{\partial h}{\partial z}(z) - (w - h(z))\frac{\partial\bar{h}}{\partial z},$$
$$\frac{\partial F}{\partial w}(z,w) = \bar{w} - \overline{h(z)}, \qquad \frac{\partial^2 F}{\partial z\partial\bar{w}}(z,w) = -\frac{\partial h}{\partial z}(z),$$
$$\frac{\partial^2 F}{\partial z\partial\bar{z}}(z,w) = \left|\frac{\partial h}{\partial z}(z)\right|^2 + \left|\frac{\partial h}{\partial\bar{z}}(z)\right|^2 + 2\,\mathrm{Re}((w - h(z))\Delta\bar{h}(z)).$$

Inserting these relations into both sides of (2) and then applying the definition of Ω, we obtain the desired conclusion. □

LEMMA 2.4. *Let h be a C^2 smooth function near $0 \in \mathbf{C}$ such that $h(0) = 0$. Then, for all $\alpha \in (0,1)$, there is an $r > 0$ sufficiently small that the function $\varphi(z,w) = \chi(|w - h(z)|^2)$ is plurisubharmonic on the open set*

$$\Omega_r = \left\{(z,w) \in B(0,r) : \mathrm{Re}((w - h(z))\Delta\bar{h}(z)) < \alpha\left|\tfrac{\partial h}{\partial \bar{z}}(z)\right|^2\right\}.$$

Proof. The result follows directly from Lemma 2.3 and the continuity of both h and Δh. □

We also need the following classical facts.

PROPOSITION 2.5. *Let $K \subset \mathbf{C}^n$ be compact.*

(i) *If $K = \hat{K}$ then there exists a continuous plurisubharmonic function u on $\mathbf{C}^n$ such that $u = 0$ on K and $u > 0$ on $\mathbf{C}^n \setminus K$.*

(ii) *$z \in \hat{K}$ if only if $u(z) \le \sup_K u$ for every plurisubharmonic function u on $\mathbf{C}^n$.*

The first assertion, due to Catlin (see [Si]), is a variation of [H, Thm. 2.6.11]. The other is a consequence of the solution of the Levi problem in $\mathbf{C}^n$ (see [H, Thm. 4.3.4]).

The following auxiliary fact is of independent interest.

LEMMA 2.6. *Let $K \subset \mathbf{C}^n$ be compact and let $U \subset \mathbf{C}^n$ be an open neighborhood of $\hat{K}$. Assume that there is a plurisubharmonic function u on U such that $u \le 0$ on K. Then $u \le 0$ on $\hat{K}$.*

Proof. After shrinking U and convolving u with suitable smoothing kernels, we may assume that u is bounded from below. By Proposition 2.5(i), there exists a continuous plurisubharmonic function v on $\mathbf{C}^n$ such that $v = 0$ on $\hat{K}$ whereas $v > 0$ on $\mathbf{C}^n \setminus \hat{K}$. By [Po, Lemma 4.1] we can find a plurisubharmonic function u' on $\mathbf{C}^n$ such that $u' = u$ on $\hat{K}$. Applying Proposition 2.5(ii) yields

$$\sup_{\hat{K}} u = \sup_{\hat{K}} u' = \sup_{K} u' = \sup_{K} u \le 0.$$

The lemma is proved. □

Proof of Theorem 2.1. We may assume that $g \not\equiv 0$, since the case $g \equiv 0$ is trivial. Thus g vanishes near the origin only at that point. For $r > 0$, put $X_r = \Gamma_f \cap \bar{B}(0,r)$. We claim that, for $r > 0$ small enough,

$$\hat{X}_r \subset K_r := \{(z,w) \in \bar{B}(0,r) : |w| \le |g(z)|\}. \tag{3}$$

Indeed, consider the function

$$\psi(z,w) = |w|^2 - \mathrm{Re}(wg(z)).$$

Clearly ψ is plurisubharmonic near the origin. Moreover, by part (i) of the theorem we have $\psi \le 0$ on X_r for $r > 0$ small enough. Thus by Lemma 2.6 we have $\psi \le 0$ on $\hat{X}_r$. This proves our claim. Let χ be the function defined before Lemma 2.3. Using Lemma 2.4, we can find $r > 0$ small enough such that the function $\chi(|w - f(z)|^2)$ is plurisubharmonic on the open set

$$\Omega_r := \left\{(z,w) \in B(0,r) : \mathrm{Re}((w - f(z))\Delta\bar{f}(z)) < \lambda\left|\frac{\partial f}{\partial \bar{z}}\right|^2\right\}.$$

Next, for $(z,w) \in K_r$ with $z \neq 0$, by (iii) we have

$$\begin{aligned}\mathrm{Re}((w - f(z))\Delta\bar{f}(z)) &\leq |w\Delta\bar{f}(z)| - \mathrm{Re}(f(z)\Delta\bar{f}(z)) \\ &\leq |g(z)\Delta\bar{f}(z)| - \mathrm{Re}(f(z)\Delta\bar{f}(z)) \\ &< \lambda\left|\frac{\partial f}{\partial \bar{z}}\right|^2.\end{aligned}$$

It follows from the previous estimates, from (3), and from assumption (ii) that, for every $\varepsilon > 0$, there exists a $\delta_\varepsilon \in (0,r)$ (independent of r) such that the function

$$\varphi_\varepsilon(z,w) := \chi(|w - f(z)|^2) + \varepsilon(|z|^2 + |w|^2) \tag{4}$$

is plurisubharmonic on $\Omega_r \cup B(0,\delta_\varepsilon)$, an open neighborhood of $\hat{X}_r$. Observe that $\varphi_\varepsilon \leq \varepsilon r^2$ on $\hat{X}_r$. Therefore, applying Lemma 2.6 yields $\varphi_\varepsilon \leq \varepsilon r^2$ on $\hat{X}_r$. By letting $\varepsilon \to 0$ we infer that $\hat{X}_r = X_r$. Finally, we note that $\Gamma_f \setminus \{(0,0)\}$ is locally contained in totally real surfaces. Using the main theorem in [OPW], we conclude that continuous functions on X_r are uniformly approximable by polynomials. The proof is thereby completed. □

REMARKS. 1. Under the additional assumption that there exists a $\lambda' \in (0,1)$ satisfying

$$|f|^2 \leq \lambda' \,\mathrm{Re}(fg), \tag{5}$$

we will construct an explicit Stein neighborhood basis for Γ_f near the origin in $\mathbf{C}^2$. The case $g \equiv 0$ is trivial, so consider the case $g \not\equiv 0$. We first determine δ_ε such that the function φ_ε defined in (4) is plurisubharmonic on the ball $B(0,\delta_\varepsilon)$. Since $g(0) = 0$ and g is holomorphic, there exists an integer $k \geq 1$ and $0 < a_1 < a_2$ such that, for z near the origin,

$$a_1|z|^k \leq |g(z)| \leq a_2|z|^k. \tag{6}$$

It follows from (5) that $|f| \leq \lambda'|g|$. Thus we can find a constant $a_3 > 0$ such that, in the closed ball $\bar{B}(0,\delta_\varepsilon)$ with $\delta_\varepsilon > 0$ small enough, the following estimate holds:

$$F(z,w) \leq 2(|w|^2 + |f(z)|^2) < a_3\delta_\varepsilon^2.$$

By the choice of χ, there exists a constant $a_4 > 0$ such that the following estimates hold in a small neighborhood of the origin in $\mathbf{C}^2$:

$$\left|\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right| \leq a_4F^k, \quad \left|\frac{\partial^2\varphi}{\partial z\partial\bar{w}}\right| \leq a_4F^k, \quad \left|\frac{\partial^2\varphi}{\partial w\partial\bar{w}}\right| \leq a_4F^k.$$

Choose

$$\delta_\varepsilon = \left(\frac{\varepsilon}{2a_4a_3^k}\right)^{1/2k}. \tag{7}$$

It is easy to check that φ_ε is plurisubharmonic in $B(0,\delta_\varepsilon)$. Next, we let

$$\psi_\varepsilon(z,w) = \varphi_\varepsilon(z,w) + \tilde{\chi}(|w|^2 - \lambda'\,\mathrm{Re}(g(z)w)),$$

where $\tilde{\chi}(x)$ is equal to 0 when $x \le 0$ and to Ax when $x > 0$; here $A > 0$ is a constant to be chosen later. For $r > 0$ we define

$$U_r := \{(z,w) \in B(0,r) : |w| < |g(z)|\}.$$

By the proof of Theorem 2.1, there exists an $r_0 > 0$ such that φ_ε is plurisubharmonic on U_{r_0}; this implies that ψ_ε is plurisubharmonic on $U_{r_0} \cup B(0,\delta_\varepsilon)$ for every $0 < \varepsilon < r_0$ small enough. Now we fix $r \in (0, r_0)$. For $s \in (r, r_0)$ and $\varepsilon > 0$ small enough, define the open sets

$$V_{s,\varepsilon} := \{(z,w) \in U_s \cup B(0,\delta_\varepsilon) : \psi_\varepsilon(z,w) < \varepsilon s^2\}.$$

Clearly

$$\bigcap_{s>r,\,\varepsilon>0} V_{s,\varepsilon} = X_r.$$

It remains to check that $V_{s,\varepsilon}$ is pseudoconvex. Toward this end, since U_s and $B(0,\delta_\varepsilon)$ are pseudoconvex, it suffices to check that

$$\partial V_{s,\varepsilon} \cap \partial B(0,\delta_\varepsilon) \cap \partial U_s = \emptyset.$$

Assume the contrary; then we can find a point $(z_0, w_0) \in \partial B(0,\delta_\varepsilon) \cap \partial U_s$ such that $\psi_\varepsilon(z_0,w_0) \le \varepsilon s^2$. On the one hand, from (6) we obtain

$$\begin{aligned}|w_0|^2 - \lambda' \operatorname{Re}(g(z_0)w_0) &\ge |w_0|^2 - \lambda'|w_0||g(z_0)| \\ &\ge (1-\lambda')|g(z_0)|^2 \ge (1-\lambda')a_1^2|z_0|^{2k}. \qquad (8)\end{aligned}$$

On the other hand, for ε small enough, (6) also implies

$$\delta_\varepsilon^2 = |z_0|^2 + |w_0|^2 = |z_0|^2 + |g(z_0)|^2 \le 4|z_0|^2. \qquad (9)$$

Combining (7), (8), and (9) yields

$$|w_0|^2 - \lambda' \operatorname{Re}(g(z_0)w_0) \ge a_5\varepsilon,$$

where a_5 is a positive constant that is independent of ε. Now we choose $A = r^2/a_5$. It follows that

$$\psi_\varepsilon(z_0,w_0) > \tilde{\chi}(|w_0|^2 - \lambda' \operatorname{Re}(g(z_0)w_0)) > \varepsilon r^2.$$

This is a contradiction, so we are done.

2. The function f in Corollary 2.2 satisfies condition (5) with g as defined in the proof of Corollary 2.2.

The next result shows that local polynomial convexity of totally real graphs is well-behaved under nondegenerate holomorphic mappings.

PROPOSITION 2.7. *Let f be a $\mathcal{C}^1$ smooth function near $0 \in \mathbf{C}$ such that $f(0) = 0$. Let p be a holomorphic function defined near the origin in $\mathbf{C}^2$ such that $p(0,0) = 0$. Assume that*:

(i) *Γ_g is locally polynomially convex at the origin, where $g(z) = p(z, f(z))$*;
(ii) *Γ_g contains no complex variety of dimension 1*; *and*

(iii) *$\Gamma_f \cap Q$ is contained in a union of smooth arcs passing through the origin, where*

$$Q := \left\{(z, w) : \tfrac{\partial p}{\partial w}(z, w) = 0\right\}.$$

Then Γ_f is locally polynomially convex at the origin.

Proof. Set $\Phi(z, w) = (z, p(z, w))$. Clearly Φ is a local biholomorphism outside Q. For $r > 0$ we set

$$X_r := \Gamma_f \cap \bar{B}(0, r) \quad \text{and} \quad Y_r := \Gamma_g \cap \bar{B}(0, r).$$

By part (i) of the proposition, we can choose $r > 0$ such that Y_r is polynomially convex and p is holomorphic on an open neighborhood of $\bar{B}(0, r)$. Pick $r' > 0$ small enough such that $\Phi(X_{r'}) \subset Y_r$. Then

$$\hat{X}_{r'} \subset \Phi^{-1}(Y_r).$$

We claim that $\hat{X}_{r'} \backslash X_{r'} \subset Q$. So assume there exists a point $p \in \hat{X}_{r'} \backslash (X_{r'} \cup Q)$. Because Φ is a local biholomorphism near p, we can find a $\delta > 0$ such that

$$Z := B(p, \delta) \cap (\hat{X}_{r'} \backslash X_{r'})$$

has finite two-dimensional Hausdorff measure and $\Phi \colon Z \to \Phi(Z)$ is bijective. According to a theorem of Alexander and Sibony (see [AWe, Thm. 21.9]), Z is a one-dimensional complex subvariety of $B(p, \delta)$. This contradicts (ii) and so the claim follows. Finally, we assume that there exists a point $q \in Q \cap (\hat{X}_{r'} \backslash X_{r'})$. Then, by the Rossi local maximum principle (see [AWe, Thm. 9.1]), $q \in (X_{r'} \cap Q)\hat{}$. Observe that, by (iii) and the Stolzenberg–Alexander theorem on polynomially convex hulls of finite union of smooth curves (see [AWe, Thm. 12.1]), the set $X_{r'} \cap Q$ is polynomially convex. This is absurd, so we are done. □

Finally, we have the following by-product of Corollary 2.2 and Proposition 2.7.

COROLLARY 2.8. *Let $k \geq 2$ be an integer and let β, γ be numbers satisfying condition* (i) *of Corollary 2.2. Define*

$$f(z) = \begin{cases} \beta \bar{z}^k + \gamma \bar{z}^{2k}/z^k, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Then Γ_f is locally polynomially convex at the origin.

Proof. It suffices to apply Proposition 2.7 with $g(z, w) = z^k w$; then the desired conclusion follows from Corollary 2.2. □

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