Intersection Numbers and Automorphisms of Stable Curves

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1. Introduction

Denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus-g complex algebraic curves. We have the morphism that forgets the last marked point

$$\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \ldots, \sigma_n$ the canonical sections of π and by $D_1, \ldots, D_n$ the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf. Then we have the following tautological classes on moduli spaces of curves:

$$\psi_i = c_1(\sigma_i^*(\omega_\pi));$$

$$\kappa_i = \pi_*\left(c_1\left(\omega_\pi\left(\sum D_i\right)\right)^{i+1}\right);$$

$$\lambda_l = c_l(\pi_*(\omega_\pi)), \quad 1 \leq l \leq g.$$

The classes κ_i were first introduced by Mumford [22] on $\overline{\mathcal{M}}_g$; their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello–Cornalba [1]. For background materials about the intersection theory of moduli spaces of curves, we refer to the book [19] and the survey paper [24].

Hodge integrals are intersection numbers of the form

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g},$$

which are rational numbers because the moduli spaces of curves are orbifolds. They are nonzero only when $\sum_{i=1}^{n} d_i + \sum_{i=1}^{m} a_i + \sum_{i=1}^{g} i k_i = 3g - 3 + n$.

Hodge integrals arise naturally in the localization computation of Gromov–Witten invariants. They have been extensively studied by mathematicians and physicists. Hodge integrals involving only ψ classes can be computed recursively by the the celebrated Witten–Kontsevich theorem [18; 26], which can be equivalently formulated by the DVV recursion relation [5]

$$\langle \tau_{k+1} \tau_{d_1} \cdots \tau_{d_n} \rangle_g$$

$$= \frac{1}{(2k+3)!!} \Bigg[\sum_{j=1}^{n} \frac{(2k+2d_j+1)!!}{(2d_j-1)!!} \langle \tau_{d_1} \cdots \tau_{d_j+k} \cdots \tau_{d_n} \rangle_g$$

$$+ \frac{1}{2} \sum_{r+s=k-1} (2r+1)!! \, (2s+1)!! \, \langle \tau_r \tau_s \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{r+s=k-1} (2r+1)!! \, (2s+1)!! \sum_{\underline{n}=I \coprod J} \Big\langle \tau_r \prod_{i\in I} \tau_{d_i} \Big\rangle_{g'} \Big\langle \tau_s \prod_{i\in J} \tau_{d_i} \Big\rangle_{g-g'} \Bigg],$$

$$\tag{1}$$

where $\underline{n} = \{1, 2, \ldots, n\}$.

Now there are several new proofs of Witten's conjecture; see [3; 14; 15; 16; 21; 23].

Let $\mathrm{denom}(r)$ denote the denominator of a rational number r in reduced form (coprime numerator and denominator, positive denominator). For $2g - 2 + n \geq 1$ we define

$$D_{g,n} = \mathrm{lcm} \bigg\{ \mathrm{denom} \bigg(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \bigg) \, \bigg| \, \sum_{i=1}^{n} d_i = 3g - 3 + n \bigg\},$$

and for $g \geq 2$ we define

$$\mathcal{D}_g = \mathrm{lcm} \bigg\{ \mathrm{denom} \bigg(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \bigg) \, \bigg| \, \sum_{i=1}^{n} d_i = 3g - 3 + n, \, d_i \geq 2, \, n \geq 1 \bigg\},$$

$$\widetilde{\mathcal{D}}_g = \mathrm{lcm} \bigg\{ \mathrm{denom} \bigg(\int_{\overline{\mathcal{M}}_g} \kappa_{a_1} \cdots \kappa_{a_m} \bigg) \, \bigg| \, \sum_{i=1}^{m} a_m = 3g - 3 \bigg\},$$

where lcm denotes the least common multiple. Note that $\mathcal{D}_g$ was previously defined by Itzykson and Zuber [12].

We know that a neighborhood of $\Sigma \in \overline{\mathcal{M}}_{g,n}$ is of the form $U/\mathrm{Aut}(\Sigma)$, where U is an open subset of $\mathbb{C}^{3g-3+n}$. This gives the orbifold structure for $\overline{\mathcal{M}}_{g,n}$. Since denominators of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ all come from these orbifold quotient singularities, the divisibility properties of $D_{g,n}$ and $\mathcal{D}_g$ should reflect the overall behavior of singularities.

In Section 2, we study basic relations between $D_{g,n}$, $\mathcal{D}_g$, and $\widetilde{\mathcal{D}}_g$. In Section 3, we briefly discuss automorphism groups of Riemann surfaces and stable curves. In Section 4, we study prime factors of $\mathcal{D}_g$ and prove a strong form of a conjecture of Itzykson and Zuber [12] concerning denominators of intersection numbers. In Section 5, we present a conjectural multinomial-type property for intersection numbers and verify it in low genera.

2. Basic Properties of $\mathcal{D}_g$

If we take $k = -1$ and $k = 0$ respectively in DVV formula (1), we get the string equation

$$\left\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_g = \sum_{j=1}^{n} \left\langle \tau_{k_j-1} \prod_{i \neq j} \tau_{k_i} \right\rangle_g$$

and the dilaton equation

$$\left\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_g = (2g - 2 + n) \left\langle \prod_{i=1}^{n} \tau_{k_i} \right\rangle_g.$$

Their proof may be found in [19].

LEMMA 2.1.   *If $n \geq 1$, then*

(i) $D_{0,n} = 1$,
(ii) $D_{1,n} = 24$,
(iii) $D_{g,1} = 24^g \cdot g!$.

Proof. The lemma follows from the string equation, the dilaton equation, and the following well-known formulas:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \binom{n-3}{d_1 \cdots d_n} = \frac{(n-3)!}{d_1! \cdots d_n!};$$

$$\langle \tau_1 \rangle_1 = \frac{1}{24}, \qquad \langle \tau_{3g-2} \rangle_g = \frac{1}{24^g g!}.$$

Their proofs can be found in [19; 26].   □

Note that $D_{0,n} = 1$ is expected since $\overline{\mathcal{M}}_{0,n}$ is a smooth manifold.

THEOREM 2.2.   *We have*
$$D_{g,n} \mid D_{g,n+1}.$$

Proof. Let $q^s \mid D_{g,n}$, where q is a prime number and $q^{s+1} \nmid D_{g,n}$.

We sort $\{\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \mid \sum_{i=1}^{n} d_i = 3g - 3 + n, 0 \leq d_1 \leq \cdots \leq d_n\}$ in lexicographical order, and we say $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g \prec \langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g$ if there is some i such that $k_j = m_j$, $j < i$, and $k_i < m_i$. Let $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$ be the minimal element with respect to the lexicographical order such that its denominator is divisible by q^s.

There exist integers c, d, a_i, b_i where $i = 1, \ldots, n - 1$ such that

$$\langle \tau_0 \tau_{k_1} \cdots \tau_{k_n+1} \rangle_g = \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g + \sum_{i=1}^{n-1} \langle \tau_{k_1} \cdots \tau_{k_i-1} \cdots \tau_{k_{n-1}} \tau_{k_n+1} \rangle_g$$

$$= \frac{c}{q^s d} + \sum_{i=1}^{n-1} \frac{b_i}{a_i}.$$

We require $q \nmid c$, $q \nmid d$, and $(a_i, b_i) = 1$.

For $i = 1, \ldots, n-1$ we have $\langle \tau_{k_1} \cdots \tau_{k_i-1} \cdots \tau_{k_{n-1}} \tau_{k_n+1} \rangle_g \prec \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$ and so $a_i = q^{s_i} e_i$, where $s_i < l$ and $q \nmid e_i$. We now have

$$\langle \tau_0 \tau_{k_1} \cdots \tau_{k_n+1} \rangle_g = \frac{c \prod_{i=1}^{n-1} e_i + qd \left(\sum_{j=1}^{n-1} q^{s-s_j-1} \prod_{i \neq j} e_i \right)}{q^s d \prod_{i=1}^{n-1} e_i};$$

we see that q cannot divide the numerator, so we have proved $q^s \mid D_{g,n+1}$. Since q is arbitrary, this proves the theorem. □

THEOREM 2.3. *We have $D_{g,n} \mid \widetilde{D}_g$ for all $g \geq 2$ and $n \geq 1$. Moreover, $\widetilde{D}_g = D_{g,3g-3}$.*

Proof. Let

$$\pi_n \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$$

be the morphism that forgets the last marked point. Then (see [1]) we have

$$(\pi_1 \cdots \pi_n)_* (\psi_1^{a_1+1} \cdots \psi_n^{a_n+1}) = \sum_{\sigma \in S_n} \kappa_\sigma, \tag{2}$$

where κ_σ is defined as follows. Write the permutation σ as a product of $\nu(\sigma)$ disjoint cycles, including 1-cycles: $\sigma = \beta_1 \cdots \beta_{\nu(\sigma)}$, where we think of the symmetric group S_n as acting on the n-tuple $(a_1, \ldots, a_n)$. Denote by $|\beta|$ the sum of the elements of a cycle β. Then

$$\kappa_\sigma = \kappa_{|\beta_1|} \kappa_{|\beta_2|} \cdots \kappa_{|\beta_{\nu(\sigma)}|}.$$

From equation (2), we get

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1+1} \cdots \psi_n^{a_n+1} = \sum_{\sigma \in S_n} \int_{\overline{\mathcal{M}}_g} \kappa_\sigma,$$

so we have proved $D_{g,n} \mid \widetilde{D}_g$.

On the other hand, any $\int_{\overline{\mathcal{M}}_g} \kappa_{a_1} \cdots \kappa_{a_m}$ can be written as a sum of $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$'s. This can be seen by induction on the number of kappa classes. For integrals with only one kappa class, we have $\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1} \psi_1^{d_1} \cdots \psi_n^{d_n}$; we also have

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \cdots \kappa_{a_m} \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n+m}} \psi_{n+1}^{a_1+1} \cdots \psi_{n+m}^{a_m+1} \psi_1^{d_1} \cdots \psi_n^{d_n}$$
$$- \{\text{integrals with at most } m-1 \; \kappa \text{ classes}\},$$

thus finishing the induction argument. So we have proved $\widetilde{D}_g = D_{g,3g-3}$. □

COROLLARY 2.4. *For $g \geq 2$, we have $\mathcal{D}_g = \widetilde{D}_g$.*

We have computed $\mathcal{D}_g$ for $g \leq 20$ using the DVV formula (1) and observed the following conjectural exact values of $\mathcal{D}_g$ (see also [20]).

CONJECTURE 2.5. *Let p be a prime number and $g \geq 2$. Let $\mathrm{ord}(p, n)$ denote the maximum integer such that $p^{\mathrm{ord}(p,n)} \mid n$. Then:*

  (i) $\mathrm{ord}(2, \mathcal{D}_g) = 3g + \mathrm{ord}(2, g!)$;
  (ii) $\mathrm{ord}(3, \mathcal{D}_g) = g + \mathrm{ord}(3, g!)$;
  (iii) $\mathrm{ord}(p, \mathcal{D}_g) = \left\lfloor \frac{2g}{p-1} \right\rfloor$ *for $p \geq 5$, where $\lfloor x \rfloor$ denotes the maximum integer that is not larger than x.*

On the other hand, we may obtain explicit expressions for multiples of $\mathcal{D}_g$ by applying either Kazarian–Lando's formula [15] expressing intersection indices by Hurwitz numbers or Proposition 4.4.

3. Automorphism Groups of Stable Curves

First we recall some facts about automorphisms of compact Riemann surfaces, following [8].

Let X be a compact Riemann surface of genus g and let $\mathrm{Aut}(X)$ be the group of conformal automorphisms of X. It's a classical theorem of Hurwitz that if $g \geq 2$ then $|\mathrm{Aut}(X)| \leq 84(g-1)$.

Let $G \subset \mathrm{Aut}(X)$ be a group of automorphisms of X, and consider the natural map

$$\pi : X \to X/G.$$

We know that π has degree $|G|$ and that X/G is a compact Riemann surface of genus g_0.

The mapping π is branched only at the fixed points of G, and the branching order

$$b(P) = \mathrm{ord}\, G_P - 1,$$

where G_P is the isotropy group at $P \in X$ that is known to be cyclic.

Let $P_1, \ldots, P_r$ be a maximal set of inequivalent fixed points of elements of $G \setminus \{1\}$ (i.e., $P_i \neq h(P_j)$ for all $h \in G$ and all $i \neq j$).

Let $n_i = \mathrm{ord}\, G_{P_i}$. Then the total branch number of π is given by

$$B = \sum_{i=1}^{r} \frac{|G|}{n_i}(n_i - 1) = |G| \sum_{i=1}^{r} \left(1 - \frac{1}{n_i}\right).$$

The Riemann–Hurwitz formula now reads

$$2g - 2 = |G|\left[2g_0 - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{n_i}\right)\right],$$

so we have

$$|G| \mid (2g - 2) \cdot \mathrm{lcm}(n_1, \ldots, n_r). \tag{3}$$

This fact is crucial in the study of automorphism groups of compact Riemann surfaces.

The following is a special case of a theorem due to W. Harvey [11, Thm. 6].

PROPOSITION 3.1 [11]. *The minimum genus g of a compact Riemann surface that admits an automorphism of order p^r (p prime) is given by*

$$g = \max\left\{2, \frac{p-1}{2}p^{r-1}\right\}.$$

In (3), we have $n_i = \text{ord } G_{P_i}$ and G_{P_i} is cyclic, so Proposition 3.1 implies the following result.

COROLLARY 3.2. *Let X be a compact Riemann surface of genus $g \geq 2$ and let $G = |\text{Aut}(X)|$. Then*

$$\text{ord}(p, |G|) \leq \left\lfloor \log_p \frac{2pg}{p-1} \right\rfloor + \text{ord}(p, 2(g-1)).$$

In particular, $p \nmid |G|$ if $p > 2g + 1$.

DEFINITION 3.3. A *node* on a curve is a point that is locally analytically isomorphic to a neighborhood of the orgin of $xy = 0$ in the complex plane $\mathbb{C}^2$.

If Σ is a nodal curve, define its *normalization* $\tilde{\Sigma}$ to be the Riemann surface obtained by "ungluing" its nodes. Let $p: \tilde{\Sigma} \to \Sigma$ denote the canonical normalization map. The preimages in $\tilde{\Sigma}$ of the nodes of Σ are called *node branches*.

A *stable curve* is a connected and compact nodal curve, which means that its singular points are nodes and satisfy the stability conditions: (i) each genus-0 component has at least three node branches; (ii) each genus-1 component has at least one node branch.

Stability is equivalent to the finiteness of the automorphism group. Suppose Σ is a stable curve of arithmetic genus g such that its normalization has m components $\Sigma_1, \ldots, \Sigma_m$ of genus $g_1, \ldots, g_m$.

DEFINITION 3.4. An automorphism φ of the dual graph Γ of Σ will be called *geometric* if it is induced by an automorphism of the corresponding stable curve Σ. All geometric automorphisms of Γ form a group $G\text{Aut}(\Gamma)$, which is a subgroup of $\text{Aut}(\Gamma)$.

The notion of geometric automorphism is introduced by Opstall and Veliche [25] in their study of sharp bounds for the automorphism group of stable curves of a given genus.

THEOREM 3.5. *Let $\widetilde{\text{Aut}}(\Sigma_i)$ be the group of automorphisms of Σ_i fixing node branches on Σ_i. Then*

$$|\text{Aut}(\Sigma)| = |G\text{Aut}(\Gamma)| \cdot \prod_{i=1}^{m} |\widetilde{\text{Aut}}(\Sigma_i)|.$$

Proof. First note the following fact. If $f(x)$ and $g(y)$ are two holomorphic functions defined near the origin of $\mathbb{C}^1$ and satisfy $f(0) = g(0)$, then $F(x, y) =$

$f(x) + g(y) - f(0)$ is a holomorphic function near the origin of $\mathbb{C}^2$ satisfying $F(x, 0) = f(x)$ and $F(0, y) = g(y)$. So to check whether a function on a nodal curve is analytic, we need only check whether it is analytic restricting to each connected component.

There is a natural map $p \colon \mathrm{Aut}(\Sigma) \to G \mathrm{Aut}(\Gamma)$ mapping an automorphism of Σ to the induced automorphism on its dual graph Γ.

For each $b \in G \mathrm{Aut}(\Gamma)$, fix a $T_b \in \mathrm{Aut}(\Sigma)$ such that $p(T_b) = b$. If $f_i \in \widetilde{\mathrm{Aut}}(\Sigma_i)$ for $i = 1, \ldots, m$, we denote by $(f_1, \ldots, f_m) \in \mathrm{Aut}(\Sigma)$ the gluing morphism. We define the map

$$G \mathrm{Aut}(\Gamma) \times \prod_{i=1}^{m} \widetilde{\mathrm{Aut}}(\Sigma_i) \longrightarrow \mathrm{Aut}(\Sigma),$$

$$(b, f_1, \ldots, f_m) \longmapsto T_b \circ (f_1, \ldots, f_m).$$

It's not difficult to see that this map is, in fact, bijective. Its converse is

$$\mathrm{Aut}(\Sigma) \longrightarrow G \mathrm{Aut}(\Gamma) \times \prod_{i=1}^{m} \widetilde{\mathrm{Aut}}(\Sigma_i),$$

$$T \longmapsto (p(T), (T_{p(T)}^{-1} \circ T)|_{\Sigma_1}, \ldots, (T_{p(T)}^{-1} \circ T)|_{\Sigma_m}).$$

Thus we have proved the theorem. $\qquad\square$

PROPOSITION 3.6. *Let Σ be a stable curve of arithmetic genus $g \geq 2$. If a prime number p divides $|\mathrm{Aut}(\Sigma)|$, then $p \leq 2g + 1$.*

Proof. Let's assume that there are δ nodes on Σ and δ_i node branches on each Σ_i. Then we have the following relations:

$$g = \sum_{i=1}^{m} (g_i - 1) + \delta + 1, \tag{4}$$

$$2g_i + \delta_i - 2 \geq 1, \tag{5}$$

$$2\delta = \sum_{i=1}^{m} \delta_i. \tag{6}$$

Summing up (5) for $i = 1, \ldots, n$ and substituting (4) and (6) into (5), we get

$$m \leq 2g - 2.$$

Now let e_{ij} denote the number of edges between Σ_i and Σ_j in the dual graph of Σ. Then it's obvious that $e_{ij} \leq g + 1$.

Since $|\mathrm{Aut}(\Gamma)|$ divides $m! \prod_{(i,j)} (e_{ij}!)$, which is not divisible by prime numbers greater than $2g + 1$, and since $g_i \leq g$, the proposition follows from Theorem 3.5 and Corollary 3.2. $\qquad\square$

We remark that Proposition 3.6 may not hold for nonstable nodal curves.

4. Prime Factors of $\mathcal{D}_g$

DEFINITION 4.1.   In [7], the generating function

$$F(x_1, \ldots, x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_i = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} x_i^{d_i}$$

is called the *n-point function*.

In particular, the 2-point function has a simple explicit form due to Dijkgraaf (see [7]):

$$F(x_1, x_2) = \frac{1}{x_1 + x_2} \exp\left(\frac{x_1^3}{24} + \frac{x_2^3}{24}\right) \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \left(\frac{1}{2}x_1 x_2(x_1 + x_2)\right)^k.$$

LEMMA 4.2.   *Let p be a prime number and let $g \geq 2$. Then the following statements hold*:

(i) *if $p > 2g + 1$, then $p \nmid D_{g,2}$;*

(ii) *if $g + 1 \leq p \leq 2g + 1$, then*

$$p \mid \text{denom}\langle \tau_{(p-1)/2} \tau_{3g-1-(p-1)/2} \rangle_g;$$

(iii) *if $2g+1$ is prime, then $(2g+1) \mid \text{denom}\langle \tau_d \tau_{3g-1-d} \rangle_g$ if and only if $g \leq d \leq 2g - 1$;*

(iv) *if $2g+1$ is prime, then $\text{ord}(2g+1, D_{g,2}) = 1$.*

Proof. From the 2-point function, we get

$$\langle \tau_d \tau_{3g-1-d} \rangle_g = \sum_{i=0}^{g} \sum_{k} \binom{g-k}{i} \binom{k-1}{d-3i-k} \frac{k!}{(g-k)! \, 24^{g-k}(2k+1)! \, 2^k}$$

$$+ \frac{(-1)^{d \bmod 3}}{g! \, 24^g} \binom{g-1}{\lfloor \frac{d}{3} \rfloor},$$

where the summation range of k is $\max\left(\frac{d_1-3i+1}{2}, 1\right) \leq k \leq \min(g-i, d_1 - 3i)$. Then the lemma follows easily. $\square$

THEOREM 4.3.   *Let p be a prime number, let $g \geq 2$, and let $\text{ord}(p, q)$ denote the maximum integer such that $p^{\text{ord}(p,q)} \mid q$. Then*:

(i) *if $p > 2g + 1$, then $p \nmid \mathcal{D}_g$;*

(ii) *for any prime $p \leq 2g + 1$, we have $p \mid \mathcal{D}_g$;*

(iii) *if $2g+1$ is prime, then $\text{ord}(2g+1, \mathcal{D}_g) = 1$;*

(iv) *$\text{ord}(2, \mathcal{D}_g) = 3g + \text{ord}(2, g!)$.*

Proof. For part (i), we use induction on the pair of genus and the number of marked points (g, n) to prove that denominators of all ψ-class intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$ are not divisible by prime numbers greater than $2g + 1$. If $p > 2g + 1$, then $p \nmid D_{g,2}$ has been proved in Lemma 4.2(i). Also $\mathcal{D}_2 = 2^7 \cdot 3^2 \cdot 5$ is

not divisible by $p > 5$. Hence we may assume that $g \geq 3$ and $n \geq 3$. We rewrite the DVV formula as

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$$

$$= \frac{1}{(2d_1+1)!!} \left[\sum_{j=2}^{n} \frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!} \langle \tau_{d_2} \cdots \tau_{d_j+d_1-1} \cdots \tau_{d_n} \rangle_g \right.$$

$$+ \frac{1}{2} \sum_{r+s=d_1-1} (2r+1)!! \, (2s+1)!! \, \langle \tau_r \tau_s \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1}$$

$$\left. + \frac{1}{2} \sum_{r+s=d_1-1} (2r+1)!! \, (2s+1)!! \sum_{\{2,\dots,n\}=I \coprod J} \left\langle \tau_r \prod_{i \in I} \tau_{d_i} \right\rangle_{g'} \left\langle \tau_s \prod_{i \in J} \tau_{d_i} \right\rangle_{g-g'} \right].$$

For $n \geq 3$ marked points we may take $d_1 \leq g$; then, by induction on (g,n), it's easy to see that the denominator of the right-hand side is not divisible by prime numbers greater than $2g+1$. Part (ii) follows from Lemma 2.1(iii), Theorem 2.3, and Lemma 4.2(ii).

For part (iii), we again use induction on (g,n) as in the proof of part (i). We may assume that $n \geq 3$. In view of Lemma 4.2(iii)–(iv), we need only prove $\mathrm{ord}(2g+1, D_{g,n}) \leq 1$. If $n > 3$, then we may take $d_1 < g$ in $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$, whose denominator is not divisible by $(2g+1)^2$. This is easily seen by induction on the right-hand side of the DVV formula. So there is only left to prove that the denominator of $\langle \tau_g \tau_g \tau_g \rangle_g$ is not divisible by $(2g+1)^2$. We have

$$\langle \tau_g \tau_g \tau_g \rangle_g = \frac{1}{(2g+1)!!} \left[\frac{2(4g-1)!!}{(2g-1)!!} \langle \tau_g \tau_{2g-1} \rangle_g + \{\text{lower genus terms}\} \right].$$

Since the factor $2g+1$ in the denominator of $\langle \tau_g \tau_{2g-1} \rangle$ will be cancelled by $(4g-1)!!$, we have proved (iii) by induction.

For part (iv), since $\langle \tau_{3g-2} \rangle_g = 1/24^g g!$ we have $\mathrm{ord}(2, \mathcal{D}_g) \geq 3g + \mathrm{ord}(2, g!)$. The reverse inequality can be seen from the DVV formula by induction on (g,n) and noting that

$$\frac{1}{2} \sum_{r+s=k-1} (2r+1)!! \, (2s+1)!! \sum_{n=I \coprod J} \left\langle \tau_r \prod_{i \in I} \tau_{d_i} \right\rangle_{g'} \left\langle \tau_s \prod_{i \in J} \tau_{d_i} \right\rangle_{g-g'}$$

$$= \sum_{r+s=k-1} (2r+1)!! \, (2s+1)!! \sum_{\{2,\dots,n\}=I \coprod J} \left\langle \tau_r \tau_{d_1} \prod_{i \in I} \tau_{d_i} \right\rangle_{g'} \left\langle \tau_s \prod_{i \in J} \tau_{d_i} \right\rangle_{g-g'}. \quad \square$$

For a single intersection number, we have the following estimate on its denominator.

PROPOSITION 4.4. *If $p \geq 3$ is a prime number, then*

$$\mathrm{ord}(p, \mathrm{denom}\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g) \leq \mathrm{ord}\left(p, \prod_{j=1}^{n} (2d_j+1)!! \right).$$

Proof. Following Dijkgraaf's notation [4], let

$$\langle \tilde{\tau}_{d_1} \cdots \tilde{\tau}_{d_n} \rangle_g = \prod_{j=1}^{n} (2d_j + 1)!! \cdot \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g.$$

Then the DVV formula can be written as

$$\left\langle \tilde{\tau}_k \prod_{j=1}^{n} \tilde{\tau}_{d_j} \right\rangle_g = \sum_{j=1}^{n} (2d_j + 1) \langle \tilde{\tau}_{d_1} \cdots \tilde{\tau}_{d_j+k-1} \cdots \tilde{\tau}_{d_n} \rangle_g + \frac{1}{2} \sum_{r+s=k-2} \left\langle \tilde{\tau}_r \tilde{\tau}_s \prod_{j=1}^{n} \tilde{\tau}_{d_j} \right\rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{r+s=k-2 \\ \underline{n}=I \coprod J}} \left\langle \tilde{\tau}_r \prod_{i \in I} \tilde{\tau}_{d_i} \right\rangle_{g'} \left\langle \tilde{\tau}_s \prod_{i \in J} \tilde{\tau}_{d_i} \right\rangle_{g-g'},$$

where $\underline{n} = \{1, \ldots, n\}$.

Because $\langle \tilde{\tau}_1 \rangle_1 = \frac{1}{8}$, by induction on (g, n) it is easy to show that, for any prime number $p \geq 3$,

$$p \nmid \operatorname{denom} \langle \tilde{\tau}_{d_1} \cdots \tilde{\tau}_{d_n} \rangle_g.$$

The proposition is proved. □

LEMMA 4.5. *Let B_m denote the Bernoulli numbers in the expansion*

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

The denominator of B_{2m} is given by

$$\prod_{(p-1)|2m} p,$$

where the product is taken over the primes p.

Proof. The lemma follows easily from Staudt's theorem (see [10]),

$$-B_{2m} \equiv \sum_{(p-1)|2m} \frac{1}{p} \pmod 1,$$

where the sum is taken over the primes p. □

THEOREM 4.6. *The denominator of intersection numbers of the form*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \tag{7}$$

can only contain prime factors less than or equal to $2g + 1$.

Proof. Mumford [22] proved the following formula for the Chern character of a Hodge bundle:

$$\operatorname{ch}_{2m-1}(\mathbb{E}) = \frac{B_{2m}}{(2m)!} \left[\kappa_{2m-1} - \sum_{i=1}^{n} \psi_i^{2m-1} + \frac{1}{2} \sum_{\xi \in \Delta} l_{\xi_*} \left(\sum_{i=0}^{2m-2} \psi_1^i (-\psi_2)^{2m-2-i} \right) \right].$$

We know that any λ_i can be expressed as a polynomial of $\mathrm{ch}_j(\mathbb{E})$'s:

$$\lambda_i = \sum_{\mu \vdash i} (-1)^{i-l(\mu)} \prod_{r \geq 1} \frac{((r-1)!)^{m_r}}{m_r!} \, \mathrm{ch}_\mu(\mathbb{E}), \quad i \geq 1,$$

where the sum ranges over all partitions μ of i, m_r is the number of r in μ, and $\mathrm{ch}_\mu(\mathbb{E}) = \mathrm{ch}_{\mu_1}(\mathbb{E}) \cdots \mathrm{ch}_{\mu_l}(\mathbb{E})$.

By Faber's algorithm [6], we can reduce any Hodge integral (7) to a sum of integrals with only ψ and κ classes using the Mumford formula. So the extra denominators that may have prime factors larger than $2g+1$ come only from $B_{2m}/(2m)!$.

Note that $\mathrm{ch}_0(\mathbb{E}) = g$, $\mathrm{ch}_{2r}(\mathbb{E}) = 0$ for $r \geq 1$, and $\mathrm{ch}_r(\mathbb{E}) = 0$ for $r \geq 2g$. So the theorem follows from Lemma 4.5 and Theorem 4.3(1). □

In view of Proposition 3.6, Theorem 4.6 should also follow from Mumford's [22] definition of the Chow ring of $\overline{\mathcal{M}}_{g,n}$. We include a proof here because it's conceptually simple and direct.

LEMMA 4.7. *If $2 \leq p \leq g+1$ is a prime number, then* $\mathrm{ord}(p, D_{g,3}) \geq 2$.

Proof. From Lemma 2.1(3) we have $24^g \mid D_{g,3}$, so Lemma 4.7 is obvious for $p = 2$ or 3. We assume $p \geq 5$ hereafter.

The following formulation of the special 3-point function is due to Faber [7]:

$$F_g(x, y, -y)$$

$$= \sum_{b \geq 0} \sum_{j=0}^{2b} (-1)^j \langle \tau_{3g-2b} \tau_j \tau_{2b-j} \rangle_g x^{3g-2b} y^{2b}$$

$$= \sum_{\substack{a+b+c=g \\ b \geq a}} \frac{(a+b)!}{4^{a+b} 24^c (2a+2b+1)!! \, (b-a)! \, (2a+1)! \, c!} x^{3a+3c+b} y^{2b}.$$

If $p > \frac{2g+1}{3}$, then consider the coefficient of $x^{3g-p+1} y^{p-1}$ in $F_g(x, y, -y)$:

$$[F_g(x, y, -y)]_{x^{3g-p+1} y^{p-1}}$$

$$= \sum_{\substack{a+b+c=g \\ a \leq \frac{p-1}{2}}} \frac{(a+b)!}{4^{a+b} 24^c (2a+2b+1)!! \, (b-a)! \, (2a+1)! \, c!},$$

where $b = \frac{p-1}{2}$. We must have $c < p$, so it's not difficult to see that only the term with $a = b = \frac{p-1}{2}$ can contain factor p^2 in the denominator.

If $p \leq \frac{2g+1}{3}$, then

$$[F_g(x, y, -y)]_{x^g y^{2g}} = \frac{1}{4^g (2g+1)!!}$$

and $\mathrm{ord}(p, (2g+1)!!) \geq 2$. Hence we have proved the lemma. □

THEOREM 4.8. *Let X be a compact Riemann surface of genus $g' \geq 2$ and let $g \geq g'$. Then* $|\mathrm{Aut}(X)|$ *divides* $D_{g,3}$.

Proof. We first prove the case $g' = g$.

Let p denote a prime number. By Corollary 3.2, it is sufficient to prove

$$\left\lfloor \log_p \frac{2pg}{p-1} \right\rfloor + \text{ord}(p, 2(g-1)) \leq \text{ord}(p, D_{g,3}) \qquad (8)$$

for all prime $p \leq 2g + 1$.

If $\max(g, 5) \leq p \leq 2g + 1$, then $\lfloor \log_p \frac{2pg}{p-1} \rfloor \leq 1$ and $\text{ord}(p, 2(g-1)) = 0$; therefore, by Theorem 4.3(2), inequality (8) holds. Next we assume $5 \leq p \leq g - 1$. Before addressing $p = 2$ and $p = 3$, we examine three subcases as follows.

Case (i). If $p = g - 1 \geq 5$ is prime, then $(g-1)(g-2) > 2g$. By Lemma 4.7, we have

$$\left\lfloor \log_{g-1} \frac{2g(g-1)}{g-2} \right\rfloor + 1 \leq 2 \leq \text{ord}(g-1, D_{g,3}).$$

Case (ii). Otherwise, if $p \nmid (g-1)$ then, since $\text{ord}(p, 2(g-1)) = 0$, it follows that $g! \mid D_{g,3}$ and $\text{ord}(p, g!) \geq \lfloor \frac{g}{p} \rfloor$; hence, in order to check (8), it's sufficient to prove

$$\left\lfloor \log_p \frac{2pg}{p-1} \right\rfloor \leq \left\lfloor \frac{g}{p} \right\rfloor.$$

Let $g = kp + r$, where $-p \leq r < 0$. Then $\lfloor \frac{g}{p} \rfloor = k - 1$. Since for fixed k the left-hand side takes its maximum value when $g = kp - 1$, we need only prove the above inequality for $g = kp - 1$, which is equivalent, for all $k \geq 2$ and $p \geq 5$, to

$$p^k > \frac{2p(kp-1)}{p-1} \quad \text{(i.e., } p^k - p^{k-1} - 2kp + 2 > 0\text{)},$$

which is not difficult to check.

Case (iii). If $p \mid (g-1)$ and $5 \leq p < g - 1$, let $\text{ord}(p, 2(g-1)) = r$. Then $p^r \mid (g-1)$ and we have

$$\text{ord}(p, D_{g,3}) \geq \text{ord}(p, g!) = \left\lfloor \frac{g}{p} \right\rfloor + \left\lfloor \frac{g}{p^2} \right\rfloor + \left\lfloor \frac{g}{p^3} \right\rfloor + \cdots$$

$$\geq \left\lfloor \frac{g}{p} \right\rfloor + r - 1.$$

So it's sufficient to prove

$$\left\lfloor \log_p \frac{2pg}{p-1} \right\rfloor + 1 \leq \left\lfloor \frac{g}{p} \right\rfloor.$$

Let $g = kp + 1$ and $k \geq 2$. We need to prove

$$p^k > \frac{2p(kp+1)}{p-1} \quad \text{(i.e., } p^k - p^{k-1} - 2kp - 2 > 0\text{)}.$$

The inequality holds except in the case where $p = 5$, $k = 2$, and $g = 11$, which should be treated separately. We have

$$\text{ord}(5, |G|) \leq \left\lfloor \log_5 \frac{110}{4} \right\rfloor + 1 = 3$$

and ord$(5, D_{11,3}) = 3$; in fact,

$$D_{11,3} = 2^{41} \cdot 3^{15} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23.$$

We have finished checking in this case.

Now we consider when $p = 2$ or $p = 3$. Note that $24^g g! \mid D_{g,3}$. If $p = 2$, then it's sufficient to prove that $\log_2 4g \leq 3g - 1$. If $p = 3$, then it's sufficient to prove that $\log_3 3g \leq g$. Both cases are easy to check, so we have concluded the proof of the theorem when $g' = g$.

The proof of the cases $g' < g$ can be proved by exactly the same argument and using Lemma 4.7. $\qquad \square$

There exists a compact Riemann surface X of genus 6 with $|\mathrm{Aut}(X)| = 150$ (see Table 13 in [2]). However, the power of 5 in $D_{6,2} = 2^{22} \cdot 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ is only 1, so $|\mathrm{Aut}(X)| \nmid D_{6,2}$. In this sense, we may say that Theorem 4.8 is optimal.

The following immediate corollary of Theorem 4.8 is a conjecture of Itzykson and Zuber, stated at the end of [12, Sec. 5].

COROLLARY 4.9. *For $1 < g' \leq g$, the order of automorphism group of any compact Riemann surface of genus g' divides $\mathcal{D}_g$.*

We remark that the statement of Corollary 4.9 doesn't hold for stable curves: there exists some stable curve, of genus g, whose automorphism group order does not divide $\mathcal{D}_g$. A counterexample can be constructed as follows. Let $n = \lfloor \frac{2g}{p-1} \rfloor$ Riemann surfaces of genus $\frac{p-1}{2}$ be attached to a sphere at $e^{2\pi i/n}$ for $0 \leq i \leq n - 1$. When $n \geq p$, the order of automorphism group of such a stable curve will have a power of p larger than $\lfloor \frac{2g}{p-1} \rfloor$ (see Conjecture 2.5).

5. A Conjectural Numerical Property of Intersection Numbers

During our work on intersection numbers, we noticed a multinomial-type property for intersection numbers. Although the property is still conjectural, we feel that it would establish interesting constraints on intersection numbers of moduli spaces and so briefly present it here.

From

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \binom{n-3}{d_1 \cdots d_n} = \frac{(n-3)!}{d_1! \cdots d_n!},$$

we see that if $d_1 < d_2$ then

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_0 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_0.$$

Now we prove that the same inequality holds in genus 1.

PROPOSITION 5.1. *For $\sum_{i=1}^n d_i = n$ and $d_1 < d_2$, we have*

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1.$$

Proof. We prove the inequality by induction on n. If $n = 2$, then

$$\langle \tau_0 \tau_2 \rangle_1 = \langle \tau_1 \tau_1 \rangle_1 = \frac{1}{24}.$$

Now assume that the proposition has been proved for $n - 1$. We may also assume that $d_2 - d_1 \geq 2$, for otherwise the result is trivial. So by the symmetry property of intersection numbers, we may assume without loss of generality that $d_n = 0$ or $d_n = 1$.

If $d_n = 1$ then, by the dilaton equation, we have

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 = (n-1) \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_{n-1}} \rangle_1,$$
$$\langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1 = (n-1) \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_{n-1}} \rangle_1.$$

So $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1$ holds in this case by induction.

If $d_n = 0$ then, by the string equation, we have

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 = \langle \tau_{d_1-1} \tau_{d_2} \cdots \tau_{d_{n-1}} \rangle_1 + \langle \tau_{d_1} \tau_{d_2-1} \cdots \tau_{d_{n-1}} \rangle_1$$
$$+ \sum_{i=3}^{n-1} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_i-1} \cdots \tau_{d_{n-1}} \rangle_1,$$
$$\langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1 = \langle \tau_{d_1} \tau_{d_2-1} \cdots \tau_{d_{n-1}} \rangle_1 + \langle \tau_{d_1+1} \tau_{d_2-2} \cdots \tau_{d_{n-1}} \rangle_1$$
$$+ \sum_{i=3}^{n-1} \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_i-1} \cdots \tau_{d_{n-1}} \rangle_1.$$

So $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1$ holds again by induction. $\square$

Now we formulate the following conjecture.

CONJECTURE 5.2. *For* $\sum_{i=1}^{n} d_i = 3g - 3 + n$ *and* $d_1 < d_2$,

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_g.$$

In other words: The more evenly $3g - 3 + n$ is distributed among indices, the larger are the intersection numbers.

By the same argument of Proposition 5.1, we can see that for each g it's enough to check only those intersection numbers with $n \leq 3g - 1$ and $d_3 \geq 2, \ldots, d_n \geq 2$.

We checked Conjecture 5.2 for $g \leq 16$ with the help of Faber's Maple program. Moreover, for $n = 2$ we checked all $g \leq 300$ using Dijkgraaf's 2-point function; and for $n = 3$ we checked all $g \leq 50$ using Zagier's 3-point function.

References

[1] E. Arbarello and M. Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves,* J. Algebraic Geom. 5 (1996), 705–709.

[2] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces,* London Math. Soc. Lecture Note Ser., 280, Cambridge Univ. Press, Cambridge, 2000.

[3]   L. Chen, Y. Li, and K. Liu, *Localization, Hurwitz numbers and the Witten conjecture,* preprint, math. AG/0609263.

[4]   R. Dijkgraaf, *Intersection theory, integrable hierarchies and topological field theory,* New symmetry principles in quantum field theory (Cargse, 1991), NATO Adv. Sci. Inst. Ser. B Phys., 295, pp. 95–158, Plenum, New York, 1992.

[5]   R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Topological strings in $d < 1$,* Nuclear Phys. B 352 (1991), 59–86.

[6]   C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians,* New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., 264, pp. 93–109, Cambridge Univ. Press, Cambridge, 1999.

[7]   ———, *A conjectural description of the tautological ring of the moduli space of curves,* Moduli of curves and abelian varieties, Aspects Math., E33, pp. 109–129, Vieweg, Braunschweig, 1999.

[8]   H. M. Farkas and I. Kra, *Riemann surfaces,* Grad. Texts in Math., 71, Springer-Verlag, New York, 1980.

[9]   E. Getzler and R. Pandharipande, *Virasoro constraints and the Chern classes of the Hodge bundle,* Nuclear Phys. B 530 (1998), 701–714.

[10] G. Hardy and E. Wright, *An introduction to the theory of numbers,* Oxford Univ. Press, New York, 1979.

[11] W. Harvey, *Cyclic groups of automorphisms of a compact Riemann surface,* Quart. J. Math. 17 (1966), 86–97.

[12] C. Itzykson and J.-N. Zuber, *Combinatorics of the modular group II: The Kontsevich integrals,* Internat. J. Modern Phys. A 7 (1992), 5661–5705.

[13] R. Kaufmann, Yu. Manin, and D. Zagier, *Higher Weil–Petersson volumes of moduli spaces of stable n-pointed curves,* Comm. Math. Phys. 181 (1996), 763–787.

[14] M. E. Kazarian, *KP hierarchy for Hodge integrals,* preprint, arXiv:0809.3263.

[15] M. E. Kazarian and S. K. Lando, *An algebro-geometric proof of Witten's conjecture,* J. Amer. Math. Soc. 20 (2007), 1079–1089.

[16] Y.-S. Kim and K. Liu, *A simple proof of Witten conjecture through localization,* preprint, math. AG/0508384.

[17] F. Knudsen, *The projectivity of the moduli space of stable curves. III. The line bundles on $\mathcal{M}_{g,n}$, and a proof of the projectivity of $\overline{\mathcal{M}}_{g,n}$ in characteristic 0,* Math. Scand. 52 (1983), 200–212.

[18] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function,* Comm. Math. Phys. 147 (1992), 1–23.

[19] S. Lando and A. Zvonkin, *Graphs on surfaces and their applications,* Encyclopaedia Math. Sci., 141, Springer-Verlag, Berlin, 2004.

[20] K. Liu and H. Xu, *New properties of the intersection numbers on moduli spaces of curves,* Math. Res. Lett. 14 (2007), 1041–1054.

[21] M. Mirzakhani, *Weil–Petersson volumes and intersection theory on the moduli space of curves,* J. Amer. Math. Soc. 20 (2007), 1–23.

[22] D. Mumford, *Towards an enumerative geometry of the moduli space of curves,* Arithmetic and geometry (M. Artin, J. Tate, eds.), part II, Progr. Math., 36, pp. 271–328, Birkhäuser, Boston, 1983.

[23] A. Okounkov and R. Pandharipande, *Gromov–Witten theory, Hurwitz numbers, and Matrix models, I,* preprint, math. AG/0101147.

[24] R. Vakil, *The moduli space of curves and Gromov–Witten theory,* preprint, math. AG/0602347.

[25] M. van Opstall and R. Veliche, *Maximally symmetric stable curves,* Michigan Math. J. 55 (2007), 513–534.

[26] E. Witten, *Two-dimensional gravity and intersection theory on moduli spaces,* Surveys in differential geometry (Cambridge, MA, 1990), pp. 243–310, Lehigh Univ., Bethlehem, PA, 1991.

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