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Triple Products and Cohomological Invariants for Closed 3-Manifolds

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Heegaard Floer homology groups are a powerful tool in low-dimensional topology introduced and studied by Ozsváth and Szabó [7; 8; 9; 10; 11], and they have generated much interest among topologists (see e.g. [1; 5; 6]). The groups are associated to a closed oriented 3-manifold Y together with a choice of spin^c structure $\mathfrak{s}$, and they comprise a number of variations: HF^+, HF^-, $\widehat{HF}$, HF^∞. Of these, HF^∞ is considered to be the least interesting as an invariant—a result of the apparent fact (formulated as a conjecture by Ozsváth and Szabó [8]) that it is determined by the cohomology ring of Y. Although all the evidence supports Ozsváth's and Szabó's conjecture, the structure of HF^∞ can be rather more complicated than a cursory inspection of the cohomology ring of Y might suggest (cf. [2]). Furthermore, in various situations it can be useful for other purposes to understand the behavior of HF^∞; for example, it plays a key role in Ozsváth's and Szabó's [7] proof of Donaldson's diagonalizability theorem for definite 4-manifolds and generalizations.

With these ideas in mind, we introduce here an invariant $HC^\infty_*(Y)$ of the cohomology ring of Y that we call the "cup homology". This invariant is closely related to $HF^\infty(Y,\mathfrak{s})$ for any torsion spin^c structure $\mathfrak{s}$, granted the conjecture mentioned above (more precisely, in this case $HC^\infty_*(Y)$ is the E^∞ term of a spectral sequence converging to $HF^\infty(Y,\mathfrak{s})$, possibly after a grading shift; see Section 4). The cup homology satisfies the following properties.

(i) It is the homology of a free complex $C^\infty_*(Y)$ over $\mathbb{Z}$ whose underlying group is $\Lambda^* H^1(Y;\mathbb{Z})\otimes\mathbb{Z}[U,U^{-1}]$ (where U is a formal variable of degree -2), and whose differential is defined in terms of triple products $\langle a\cup b\cup c,[Y]\rangle$ of elements $a,b,c\in H^1(Y)$.

(ii) Multiplication by U induces an isomorphism between $HC^\infty_k(Y)$ and $HC^\infty_{k-2}(Y)$ for all k, so $HC^\infty_*(Y)$ is determined as a $\mathbb{Z}[U]$-module by its values in two adjacent degrees.

(iii) When $b_1(Y)\geq 1$, the ranks $\mathrm{rk}_{\mathbb{Z}}(HC^\infty_{2k}(Y))$ and $\mathrm{rk}_{\mathbb{Z}}(HC^\infty_{2k+1}(Y))$ are equal.

The reader familiar with Heegaard Floer homology will recognize that $C^\infty_*(Y)$ is identical (at least on the level of groups) with the E_2 term of a spectral sequence that calculates $HF^\infty(Y,\mathfrak{s})$. (The d_2 differential vanishes for trivial reasons, and according to the conjecture of Ozsváth and Szabó, d_3 is given by the boundary operator used here.)

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We define an invariant $h(Y)$ of the cohomology ring of Y by

$$h(Y) = \begin{cases} \operatorname{rk}_{\mathbb{Z}} HC_k^\infty(Y)\ \forall k \in \mathbb{Z} & \text{if } b_1(Y) \geq 1, \\ \frac{1}{2} & \text{if } b_1(Y) = 0. \end{cases}$$

Our goal here is to investigate the possibilities for $HC_*^\infty(Y)$ and, in particular, to study the behavior of $h(Y)$.

Suppose $b_1(Y) \leq 2$. Then there can be no nontrivial triple products of elements in $H^1(Y;\mathbb{Z})$, so the differential on $C_*^\infty(Y)$ vanishes. Therefore, in this case the group $HC_*^\infty(Y)$ is independent of Y and the following statements hold.

$$\begin{aligned} &\text{If } b_1(Y) = 0: \quad HC_{2k}^\infty(Y) = \mathbb{Z} \text{ and } HC_{2k+1}^\infty(Y) = 0. \\ &\text{If } b_1(Y) = 1: \quad HC_k^\infty(Y) = \mathbb{Z}\ \forall k, \text{ so } h(Y) = 1. \\ &\text{If } b_1(Y) = 2: \quad HC_k^\infty(Y) = \mathbb{Z}^2\ \forall k, \text{ so } h(Y) = 2. \end{aligned}$$

In general, we have the following bounds on $h(Y)$.

THEOREM 1. *Fix an integer $b \geq 1$. Then, for any 3-manifold Y with $b_1(Y) = b$,*

$$L(b) \leq h(Y) \leq 2^{b-1}, \tag{1}$$

where

$$L(b) = \begin{cases} 3^{(b-1)/2} & \text{if } b \text{ is odd,} \\ 2 \cdot 3^{b/2-1} & \text{if } b \text{ is even.} \end{cases}$$

Given a fixed value b of $b_1(Y)$, it is natural to ask which values of $h(Y)$ allowed by Theorem 1 can be realized. We can certainly realize the upper bound 2^{b-1} for any $b \geq 1$ by taking Y to be the connected sum of b copies of $S^1 \times S^2$, or any other 3-manifold with $b_1(Y) = b$ having trivial cup products on H^1. In the case $b = 3$, the two possibilities $h = 3$ and $h = 4$ are realized by the 3-torus T^3 and $\#^3 S^1 \times S^2$, respectively. On the other hand, a simple calculation based on the definitions (Lemma 12) shows that, when $b_1(Y) \geq 4$, the value $h = 2^{b_1(Y)-1} - 1$ does not occur. Further examples are discussed in Section 2. In general, one expects that a more complicated cup product structure in the cohomology of Y will result in a smaller value for $h(Y)$, where "complicated" refers roughly to the number of nonvanishing triple products of elements of $H^1(Y;\mathbb{Z})$. It is a result of Sullivan [12] that, in an appropriate sense, all possibilities for triple cup product behavior are realized by closed 3-manifolds Y (see Section 1 for a more precise statement). In principle, this result reduces the determination of which values of h are realized to a purely algebraic combinatorial question.

We have the following general result on the behavior of h under connected sum.

THEOREM 2. *Let Y_1 and Y_2 be closed 3-manifolds as before. Then*

$$h(Y_1 \# Y_2) = 2h(Y_1)h(Y_2).$$

Thus, for example, $h(Y \# S^1 \times S^2) = 2h(Y)$, which gives an easy way to realize values of h recursively. To spell this out, let $\mathcal{H}_b \subset \{L(b), L(b)+1, \ldots, 2^{b-1}\}$ denote the collection of integers n such that $n = h(Y)$ for some 3-manifold Y with $b_1(Y) = b$. We then have our next result.

COROLLARY 3. *For any integer $b \geq 1$, the set $\mathcal{H}_{b+1}$ contains $2\mathcal{H}_b = \{2h \mid h \in \mathcal{H}_b\}$. In fact, the operation $Y \mapsto Y \# S^1 \times S^2$ demonstrates an inclusion*

$$\begin{aligned} \mathcal{H}_b &\to \mathcal{H}_{b+1}, \\ h &\mapsto 2h. \end{aligned}$$

It follows, for example, that if Y is a 3-manifold with $b_1(Y) = b$ realizing the lower bound $h(Y) = L(b)$ for b odd, then $Y \# S^1 \times S^2$ realizes the lower bound $h(Y \# S^1 \times S^2) = L(b+1)$. However, it is clear from Theorem 1 that this naive construction will not help realize small values of $h(Y)$.

COROLLARY 4. *The rank of $HF_k^\infty(T^3 \# S^1 \times S^2; \mathfrak{s}_0)$ (for $\mathfrak{s}_0$ the torsion* spin^c *structure) realizes the smallest possible rank $L(4) = 6$ for HF^∞ in each degree among all 3-manifolds Y having $b_1(Y) = 4$.*

Proof. In Section 4 we will show (Proposition 18) that, whenever $b_1(Y) \leq 4$, the lower bound on $h(Y)$ of Theorem 1 also gives a lower bound on the rank of $HF_k^\infty(Y, \mathfrak{s})$ in any torsion spin^c structure $\mathfrak{s}$. Hence the corollary follows from Theorems 1 and 2. □

Note that the corollary could easily be proved directly in Heegaard Floer homology by using (as in the proof of [8, Lemma 4.8]) a spectral sequence argument.

A more subtle question in regards to the 3-manifold "geography" that we consider here is: Which values of $h(Y)$ can be realized by *irreducible* 3-manifolds? For example, we shall see that if Σ_g is a closed orientable surface of genus $g \geq 1$ then

$$h(\Sigma_g \times S^1) = \binom{2g+1}{g};$$

in particular, $h(\Sigma_2 \times S^1) = 10$. Observe that, since $b_1(\Sigma_2 \times S^1) = 5$, it follows that the bounds supplied by Theorem 1 are $9 \leq h \leq 16$ (the analogue of the latter fact for HF^∞ is part of the content of [8, Lemma 4.8]).

Let us call a 3-manifold Y *rationally irreducible* if, in any connected sum decomposition of Y, at least one of the factors is a rational homology sphere. The connected sum with a rational homology sphere does not change the triple cup product structure of a 3-manifold; therefore, if an integer h is realized as $h(Y)$ for some rationally irreducible 3-manifold Y, then it is also realized by an irreducible 3-manifold.

We have the following immediate corollary of Theorem 2.

COROLLARY 5. *If $h(Y)$ is odd, then Y is rationally irreducible.*

We also have the following result related to irreducibility, which follows from the behavior of h under connected sum.

THEOREM 6. *Let Y_1 and Y_2 be 3-manifolds having first Betti number at least* 1, *and suppose that $b_1(Y_1)$ and $b_1(Y_2)$ are not both odd. Let $Y = Y_1 \# Y_2$ and set $b = b_1(Y) = b_1(Y_1) + b_1(Y_2)$. Then*

$$h(Y) \geq \frac{4}{3} L(b).$$

As a consequence we see that, if one is interested in realizing small values of $h(Y)$ for a fixed odd value of b_1, then the only candidates are (rationally) irreducible.

On the other hand, it follows from the theorem that if Y is a 3-manifold with $b = b_1(Y)$ odd and $h(Y) < \frac{4}{3}L(b) = 4 \cdot 3^{(b-3)/2}$ then Y is rationally irreducible. The inequality in Theorem 6 cannot be strengthened, as shown by the example

$$T^3 \# S^1 \times S^2 \# S^1 \times S^2,$$

for which $h = 12 = \frac{4}{3}L(5)$.

The complex $C_*^\infty(Y)$ can be just as easily be defined using cohomology with coefficients in any commutative ring—for example, using $H^1(Y; \mathbb{Z}_p)$, which gives rise to a mod p cup homology $HC_*^\infty(Y)_p$. We obtain a sequence of invariants $h_p(Y)$ given by the rank in each dimension of $HC_*^\infty(Y)_p$; the arguments used to define $h(Y)$ and to obtain Theorem 1 and the other results listed here are insensitive to this change (where, of course, $b_1(Y)$ is calculated in the appropriate coefficients). These invariants can easily be distinct from each other and from $h(Y)$ (see Section 2), and we can consider the realization problem for each of them.

By the preceding results, if for any $p \geq 1$ we define

$$k_p(Y) = \log_2(2h_p(Y))$$

(where by convention $h_1(Y) = h(Y)$), then $\{k_1(Y), k_2(Y), k_3(Y), \dots\}$ is a sequence of real-valued invariants of 3-manifolds that vanish for rational homology spheres, are additive under connected sum, and satisfy

$$m(b_1(Y)) \leq k_p(Y) \leq b_1(Y);$$

here the lower bound m is a linear function of the first Betti number, which is easily derived from Theorem 1.

In Section 1 we define the chain complex $C_*^\infty(Y)$ and make some simple observations; Section 2 is devoted to a few sample calculations. We prove Theorem 1 in Section 3.1, and in Section 3.2 we prove Theorems 2 and 6. In Section 4 we spell out the conjectural relationship between $HC_*^\infty(Y)$ and $HF^\infty(Y, \mathfrak{s})$. One remark is in order here: there is another version of Floer homology for 3-manifolds (due to Kronheimer and Mrowka [3]) that is based on the Seiberg–Witten equations. This "monopole Floer homology" appears to be isomorphic to Heegaard Floer homology; moreover, the relationship between the cohomology ring and the Seiberg–Witten analogue of HF^∞ (i.e., $\overline{HM}_*$) has been established in that theory. Therefore, according to [3, Chap. IX], our HC_*^∞ is isomorphic to the E_∞ term of a spectral sequence converging to the monopole Floer homology $\overline{HM}_*$ modulo a possible shift in grading. In particular, most of our results here could be rephrased using this monopole homology rather than the more elementary (but somewhat artificial) cup homology.

1. Definitions

Let Y be a closed oriented 3-dimensional manifold and write $H = H^1(Y; \mathbb{Z})$. The cup product structure on Y induces a 3-form on H, written $\mu_Y \in \Lambda^3 H^*$, that is given by $\mu_Y(a, b, c) = \langle a \cup b \cup c, [Y] \rangle$ for $a, b, c \in H$.

REMARK 7. Sullivan [12] has shown that, for any pair (H, μ) with H a finitely generated free abelian group and $\mu \in \Lambda^3 H^*$, there exists a 3-manifold Y with $H^1(Y) = H$ and cup product form $\mu_Y = \mu$.

There is a natural interior product $H^* \otimes \Lambda^k H \to \Lambda^{k-1} H$, written $\omega \otimes \alpha \mapsto \omega \angle \alpha$, induced by the duality between H^* and H and with the property that $\omega \angle (\omega \angle \alpha) = 0$ for $\omega \in H^*$. Hence there is an extension of $\angle$ to the exterior algebra,

$$\Lambda^p H^* \otimes \Lambda^k H \to \Lambda^{k-p} H,$$

that satisfies $(\omega \wedge \eta) \angle \alpha = \omega \angle (\eta \angle \alpha)$.

DEFINITION 8. Let U be a formal variable of degree -2. The *cup complex* of Y is defined to be the chain complex $C_*^\infty = C_*^\infty(Y)$, with chain groups

$$C_*^\infty = \Lambda^* H \otimes \mathbb{Z}[U, U^{-1}]$$

graded in the obvious way and with differential

$$\partial \colon C_k^\infty \to C_{k-1}^\infty,$$
$$\partial(\alpha \otimes U^n) = \mu_Y \angle \alpha \otimes U^{n-1}. \tag{2}$$

Here μ_Y is the 3-form given by cup product defined previously.

Thus, for any k,

$$C_k^\infty \cong \bigoplus_{\ell \equiv k \bmod 2} \Lambda^\ell H \otimes U^{(\ell-k)/2} \tag{3}$$

and

$$\partial \colon \Lambda^\ell H \otimes U^n \to \Lambda^{\ell-3} \otimes U^{n-1}.$$

We have an explicit expression for ∂. Namely, if $a_1, \ldots, a_k \in H^1(Y)$, then

$$\partial(a_1 \cdots a_k) = U^{-1} \cdot \sum_{i_1 < i_2 < i_3} (-1)^{i_1+i_2+i_3} \langle a_{i_1} \cup a_{i_2} \cup a_{i_3}, [Y] \rangle a_1 \cdots \widehat{a_{i_1}} \cdots \widehat{a_{i_2}} \cdots \widehat{a_{i_3}} \cdots a_k,$$

where we use juxtaposition to indicate wedge product.

Since $\mu_Y \angle (\mu_Y \angle \alpha) = (\mu_Y \wedge \mu_Y) \angle \alpha = 0$, we see that $(C_*^\infty(Y), \partial)$ is indeed a chain complex. We write $HC_*^\infty(Y)$ for $H_*(C_*^\infty(Y), \partial)$; clearly, $HC_*^\infty(Y)$ is an invariant of the homotopy type of Y.

It is obvious that $U \colon C_*^\infty \to C_{*-2}^\infty$ is a chain isomorphism, so the homology $HC_*^\infty(Y)$ is determined as a group by its values in two adjacent degrees. Let

$$h_{\mathrm{e}}(Y) = \mathrm{rk}_{\mathbb{Z}}(HC_{2k}^\infty(Y)) \quad \text{and}$$
$$h_{\mathrm{o}}(Y) = \mathrm{rk}_{\mathbb{Z}}(HC_{2k+1}^\infty(Y))$$

for any k. Now observe that

$$HC_{\mathrm{e}}^\infty \cong HC_{2k} \otimes \mathbb{Z}[U, U^{-1}] \quad \text{and}$$
$$HC_{\mathrm{o}}^\infty \cong HC_{2k+1}(Y) \otimes \mathbb{Z}[U, U^{-1}]$$

for any k.

LEMMA 9. *If $b_1(Y) \geq 1$, then $h_e(Y) = h_o(Y)$.*

Proof. As a chain complex over the graded ring $\mathbb{Z}[U, U^{-1}]$, the complex C_*^∞ has even- and odd-graded parts given by $\Lambda^e H \otimes \mathbb{Z}[U, U^{-1}]$ and $\Lambda^o H \otimes \mathbb{Z}[U, U^{-1}]$, respectively. If $b_1(Y) \geq 1$ then the groups $\Lambda^e H$ and $\Lambda^o H$ have the same rank and so C_*^∞ has vanishing Euler characteristic over $\mathbb{Z}[U, U^{-1}]$. Therefore, the same is true of HC_*^∞, so that $0 = \mathrm{rk}_{\mathbb{Z}[U,U^{-1}]}(HC_e^\infty) - \mathrm{rk}_{\mathbb{Z}[U,U^{-1}]}(HC_o^\infty) = h_e - h_o$. □

DEFINITION 10. For Y a closed oriented 3-manifold with $b_1(Y) \geq 1$, define

$$h(Y) = h_e(Y) = h_o(Y).$$

If $b_1(Y) = 0$, set $h(Y) = \frac{1}{2}$.

Thus h is an invariant of the cohomology ring of Y that takes values in $\mathbb{Z}$ when Y is not a rational homology sphere.

LEMMA 11. *For any 3-manifold Y, we have $h(Y) \leq 2^{b_1(Y)-1}$.*

Proof. The lemma is true by definition if $b_1(Y) = 0$, so suppose $b_1(Y) \geq 1$. The rank of $HC_k^\infty(Y)$ is no larger than the rank of C_k^∞, so from (3) we have

$$\mathrm{rk}(HC_k^*(Y)) \leq \sum_{\ell \equiv k \bmod 2} \mathrm{rk}(\Lambda^\ell H) = \begin{cases} \displaystyle\sum_{\ell \text{ even}} \binom{b_1(Y)}{\ell} & \text{for } k \text{ even}, \\ \displaystyle\sum_{\ell \text{ odd}} \binom{b_1(Y)}{\ell} & \text{for } k \text{ odd}. \end{cases}$$

Both sums on the right are equal to $2^{b_1(Y)-1}$, which proves the lemma. □

Following similar terminology in Heegaard Floer homology, we say that a 3-manifold Y has *standard* cup homology if $h(Y) = 2^{b_1(Y)-1}$. Equivalently, the cup homology is standard if and only if the triple product form μ_Y is zero.

Our next lemma is another example of a constraint on h arising from purely algebraic considerations.

LEMMA 12. *If Y is a 3-manifold with $b_1(Y) = b$, $b \geq 4$, and if $h(Y) \neq 2^{b-1}$, then $h(Y) \leq 2^{b-1} - 2$.*

Proof. Suppose the triple product form on Y is given by $\mu_Y = \sum a_{ijk} e_i e_j e_k \in \Lambda^3 H^*$, for a basis $\{e_n\}$ of H^* and a_{ijk} integers, with the sum over $0 < i < j < k \leq b$. Then the differential acts on the top exterior power $\Lambda^b H \otimes U^n$ by

$$\partial \colon e_1 \cdots e_b \otimes U^n \mapsto \sum_{i<j<k} \pm a_{ijk} e_1 \cdots \hat{e}_i \cdots \hat{e}_j \cdots \hat{e}_k \cdots e_b \otimes U^{n-1},$$

which is injective unless $\mu_Y = 0$. Likewise, the component of the differential mapping into $\Lambda^0 H$ is easily seen to be surjective (over the rationals) if and only if $\mu_Y \neq 0$.

Therefore, as soon as $\mu_Y \neq 0$, a 2-dimensional space of cycles disappears from $\Lambda^b H \oplus \Lambda^3 H$ and a 2-dimensional space of boundaries appears in $\Lambda^{b-3} H \oplus \Lambda^0 H$.

This implies that the rank of $HC^\infty_{2k}(Y)$ and $HC^\infty_{2k+1}(Y)$ are each forced to decrease by at least 2 once $b > 3$ and $\mu_Y \neq 0$. □

2. Examples

Here we calculate $h(Y)$ for some sample 3-manifolds Y. As remarked previously, since there can be no nontrivial triple products on $H^1(Y)$ when $b_1(Y) \leq 2$, the differential on $C^\infty_*(Y)$ must vanish in this case. This easily gives the results listed before Theorem 1.

More generally, if all triple cup products of elements of $H^1(Y;\mathbb{Z})$ vanish (i.e., if $\mu_Y = 0$) then the cup homology is standard—that is, $HC^\infty_*(Y) \cong \Lambda^* H^1(Y) \otimes \mathbb{Z}[U, U^{-1}]$. This is the case, for example, for connected sums of copies of $S^1 \times S^2$. For more interesting examples, it was observed by Sullivan [12] that if Y is the link of an isolated algebraic surface singularity then $\mu_Y = 0$. In fact, Sullivan's argument proves the following statement.

PROPOSITION 13. *If Y is a closed 3-manifold bounding an oriented 4-manifold X such that the cup product pairing on $H^2(X, Y)$ is nondegenerate, then $\mu_Y = 0$. Hence $HC^\infty_*(Y)$ is standard for such Y and so $h(Y) = 2^{b_1(Y)-1}$.*

In particular, since the link of a singularity bounds a 4-manifold with negative-definite intersection form on $H_2(X) = H^2(X, Y)$, such 3-manifolds have standard $HC^\infty_*(Y)$.

Proof of Proposition 13. We work over the rationals, since torsion cannot contribute to triple products over $\mathbb{Z}$. Poincaré duality implies that the cup pairing

$$Q_X \colon H^2(X) \otimes H^2(X, Y) \to \mathbb{Q}$$

is nondegenerate; that is, there exists an isomorphism

$$Q_X \colon H^2(X) \to (H^2(X, Y))^*.$$

Let $\tilde{Q}_X \colon H^2(X, Y) \otimes H^2(X, Y) \to \mathbb{Q}$ be the cup product form on $H^2(X, Y)$. Then we have a commutative diagram

$$\begin{array}{ccc} & H^2(X) & \\ {\scriptstyle i}\nearrow & & \searrow^{Q_X}_{\sim} \\ H^2(X, Y) & \xrightarrow[\tilde{Q}_X]{} & (H^2(X, Y))^*. \end{array}$$

Hence $\tilde{Q}_X$ is nondegenerate if and only if i is an isomorphism. In this case, in the sequence

$$\cdots \to H^1(Y) \xrightarrow{j} H^2(X, Y) \xrightarrow{i} H^2(X) \xrightarrow{k} H^2(Y) \to \cdots$$

the homomorphisms j and k vanish. As noted in [12], this means that cup products of elements $a, b \in H^1(Y)$ can be computed by lifting to $H^1(X)$ because j is trivial, multiplying, and restricting to Y, which gives 0 since k is trivial. □

If $b_1(Y) = 3$, then the only possible nontrivial differential in $C^\infty_*(Y)$ is between $\Lambda^3 H \otimes U^n$ and $\Lambda^0 H \otimes U^{n-1}$. If $a \cup b \cup c = 0$ for a basis a, b, c of $H = H^1(Y)$, then $HC^\infty_k(Y) \cong \mathbb{Z}^4$ for each k and so $h(Y) = 4$. Otherwise (i.e., if $a \cup b \cup c \neq 0$), the differential is injective and $HC^\infty_k(Y)$ has rank 3 for all k; hence $h(Y) = 3$. In fact, if $\langle a \cup b \cup c, [Y] \rangle = n \neq 0$ then

$$HC^\infty_{2k}(Y) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/n\mathbb{Z}),$$
$$HC^\infty_{2k+1}(Y) \cong \mathbb{Z}^3.$$

It also follows that, in the notation of the Introduction, $h_p(Y) = 4$ for primes p dividing n and $h_q(Y) = 3$ for other primes q.

A more substantial example is given by $Y = \Sigma_g \times S^1$, where Σ_g is a closed oriented surface of genus $g \geq 1$. In this case it is a simple matter to calculate that $\mu_Y = s \wedge \omega$, where s is the class $[pt \times S^1]$ and $\omega \in \Lambda^2 H_1(\Sigma_g)$ is the symplectic 2-form given by the cup product pairing on the first cohomology of Σ. We have a decomposition

$$\Lambda^k H^1(\Sigma_g \times S^1) \cong \Lambda^k H^1(\Sigma_g) \oplus (\sigma \wedge \Lambda^{k-1} H^1(\Sigma_g)),$$

where σ is Poincaré dual to $[\Sigma_g \times pt]$. With respect to this decomposition, the boundary in $C^\infty_*(\Sigma_g \times S^1)$ is trivial on the first factor and is given by

$$\partial(\sigma \wedge \alpha \otimes U^n) = \omega \lrcorner \alpha \otimes U^{n-1} \in \Lambda^{k-2} H^1(\Sigma_g) \otimes U^{n-1}$$

on the second factor for $\alpha \in \Lambda^k H^1(\Sigma_g)$. Let $E_k(g)$ denote the abelian group determined by the long exact sequence

$$\cdots \Lambda^{k+1} H^1(\Sigma_g) \xrightarrow{\omega \lrcorner \cdot} \Lambda^{k-1} H^1(\Sigma_g) \to E_k(g) \to \Lambda^k H^1(\Sigma_g) \xrightarrow{\omega \lrcorner \cdot} \Lambda^{k-2} H^1(\Sigma_g) \cdots .$$

That is, with appropriate grading conventions $E_*(g)$ is the homology of the mapping cone of $\omega \lrcorner \cdot$ acting on $\Lambda^* H^1(\Sigma_g)$, thought of as a complex with trivial differential. (Because the latter is free abelian, $E_*(g)$ is uniquely determined by the sequence given.) It follows from the previous discussion that

$$HC^\infty_*(\Sigma_g \times S^1) \cong E_*(g) \otimes \mathbb{Z}[U, U^{-1}]. \tag{4}$$

To give an explicit expression for $E_k(g)$, observe that there is a natural duality isomorphism $\star\colon \Lambda^k H^1(\Sigma_g) \to \Lambda^{2g-k} H_1(\Sigma_g)$ induced by interior product with the orientation form $(1/g!)\omega^g \in \Lambda^{2g} H_1(\Sigma_g)$. Identifying $H_1(\Sigma_g)$ with $H^1(\Sigma_g)$ via Poincaré duality, it is an easy exercise to see that $\star(\omega \lrcorner \alpha) = \omega \wedge \star\alpha$ for any $\alpha \in \Lambda^* H^1(\Sigma_g)$. Therefore, if we define $E^k(g)$ by the sequence

$$\cdots \Lambda^{k-2} H_1(\Sigma_g) \xrightarrow{\omega \wedge \cdot} \Lambda^k H_1(\Sigma_g) \to E^k(g) \to \Lambda^{k-1} H_1(\Sigma_g) \xrightarrow{\omega \wedge \cdot} \Lambda^{k+1} H_1(\Sigma_g) \cdots,$$

then $E^k(g) \cong E_{2g+1-k}(g)$. Now, $\Lambda^k H_1(\Sigma_g) \cong H^k(T^{2g})$, where T^{2g} is the Jacobian torus of Σ_g. As a result, the foregoing sequence can be identified with the

Gysin sequence associated to the circle bundle $\mathcal{E}(g)$ over T^{2g} having Euler class ω, so that $E^*(g) = H^*(\mathcal{E}(g))$.

The cohomology of $\mathcal{E}(g)$ was determined (in a different guise) by Lee and Packer [4] using this Gysin sequence together with combinatorial matrix theory. Taking the Poincaré dual of their result shows:

$$E_k(g) \cong \begin{cases} \mathbb{Z}^{\binom{2g}{k}-\binom{2g}{k-2}} \oplus \bigoplus_{j=2}^{\lfloor (k+1)/2 \rfloor} \mathbb{Z}_j^{\binom{2g}{k-2j+1}-\binom{2g}{k-2j-1}} & \text{for } 0 \le k \le g, \\ \bigoplus_{j=0}^{\lfloor (2g+1-k)/2 \rfloor} \mathbb{Z}_j^{\binom{2g}{k+2j-1}-\binom{2g}{k+2j+1}} & \text{for } g+1 \le k \le 2g+1, \end{cases}$$

where $\mathbb{Z}_0 = \mathbb{Z}$ and $\mathbb{Z}_1$ is the trivial group.

From this it follows in particular that, for $k \le g$,

$$\mathrm{rk}_{\mathbb{Z}}(E_k(g)) = \mathrm{rk}_{\mathbb{Z}}(E_{2g+1-k}(g)) = \binom{2g}{k} - \binom{2g}{k-2},$$

though it is possible to obtain the latter directly as well. Given this and (4), one may derive that

$$h(\Sigma_g \times S^1) = \binom{2g+1}{g}.$$

We remark that the Floer homology $HF^\infty(\Sigma_g \times S^1, \mathfrak{s})$ for $c_1(\mathfrak{s}) = 0$ was calculated in [2] for coefficients in $\mathbb{C}$ and in $\mathbb{Z}_2$. Both these results are consistent with the hypothesis that $HF^\infty(\Sigma_g \times S^1, \mathfrak{s}) \cong HC_*^\infty(\Sigma_g \times S^1)$ with coefficients in $\mathbb{Z}$.

More generally, consider a 3-manifold obtained as the mapping torus of a diffeomorphism $f\colon \Sigma_g \to \Sigma_g$. That is, Y is constructed by gluing the boundaries of $\Sigma_g \times [0,1]$ via f. Then $H^1(Y) \cong \mathbb{Z} \oplus V$, where $\mathbb{Z}$ is generated by the Poincaré dual of $[\Sigma_g]$ and $V = \ker(1 - f^*)$, with f^* denoting the action of f on the first cohomology of Σ_g. It is not hard to see that the cup product form of Y is given in this case by

$$\mu_Y = s \wedge (\omega|_V),$$

where ω is the intersection form on Σ_g as before and s is represented by a section of the obvious fibration $Y \to S^1$. Working over the rationals for simplicity, we can write $V = W \oplus V_0$, where W is a maximal symplectic subspace of V and $\omega|_{V_0} = 0$. Clearly this induces a decomposition

$$C_*^\infty(Y) \cong C_*^\infty(\Sigma_w \times S^1) \otimes_{\mathbb{Z}[U,U^{-1}]} C_*^\infty(\#^{v_0} S^2 \times S^1)$$

of chain complexes, where $2w = \dim(W)$ and $v_0 = \dim(V_0)$. Applying both the Künneth formula as in the proof of Theorem 2 (see Section 3) and our preceding results for $\#^n S^1 \times S^2$ and $\Sigma_w \times S^1$, we infer that

$$h(Y) = 2^{v_0}\binom{2w+1}{w}.$$

One can see that this number is at least as large as the corresponding value of h for a trivially fibered 3-manifold having the same first Betti number.

3. Proofs

3.1. Proof of Theorem 1

Theorem 1 follows from a straightforward estimate of the size of the homology of (C_*^∞, ∂). To state what we need explicitly, note that for each i, $0 \le i \le b_1(Y)$, the differential restricts as a map

$$\partial\colon \Lambda^i H^1(Y) \to \Lambda^{i-3} H^1(Y).$$

The differential commutes with the action of U, so we are reduced to considering three chain complexes C_0, C_1, C_2, where

$$C_j = \bigoplus_{i \equiv j \bmod 3} \Lambda^i H^1(Y). \tag{5}$$

We wish to bound the size of the homology of the C_j from below, and we do so by observing that the total rank of the homology of a chain complex must be at least the absolute value of the Euler characteristic of the complex.

PROPOSITION 14. *Fix an integer $b \ge 1$ and let H denote a free abelian group of rank b. Define graded groups C_j $(j = 0, 1, 2)$ by equation* (5), *where the grading on the factor $\Lambda^{3k+j}H$ is given by k.*

(a) *If b is odd, then*

$$|\chi(C_0)| + |\chi(C_1)| + |\chi(C_2)| = 2 \cdot 3^{(b-1)/2}.$$

(b) *If b is even, then*

$$|\chi(C_0)| + |\chi(C_1)| + |\chi(C_2)| = 4 \cdot 3^{b/2-1}.$$

Proof of Theorem 1. The upper bound on $h(Y)$ was proved in Lemma 11. To obtain a lower bound, note that it suffices to consider two adjacent values of k, say $k = 0$ and $k = 1$. It is easy to see that

$$HC_0^\infty \oplus HC_1^\infty \cong H_*(C_0) \oplus H_*(C_1) \oplus H_*(C_2).$$

Since $\chi(H_*(C_j)) = \chi(C_j)$, the proposition, together with the obvious bound $\mathrm{rk}(H_*(C_j)) \ge |\chi(H_*(C_j))|$, gives

$$\mathrm{rk}(HC_0^\infty \oplus HC_1^\infty) \ge |\chi(C_0)| + |\chi(C_1)| + |\chi(C_2)| = 2L(b),$$

where $b = b_1(Y)$, and Theorem 1 follows. □

Proof of Proposition 14. The proof is an exercise in the binomial theorem. We begin by noting that, for $j = 0, 1, 2$,

$$\chi(C_j) = \sum_k (-1)^k \binom{b}{3k+j}.$$

To facilitate our discussion, the preceding sum will be denoted by $S(b, j)$. Now, if ξ satisfies $\xi^3 = 1$ then the binomial theorem gives

$$(1-\xi)^b = S(b,0) - S(b,1)\xi + S(b,2)\xi^2. \tag{6}$$

In particular, taking $\xi = 1$ yields

$$S(b,0) - S(b,1) + S(b,2) = 0. \tag{7}$$

Now we take $\xi = e^{2\pi i/3}$ and apply (6) to the identity

$$(1-\xi)^b = (1-\xi)(1-\xi)^{b-1}.$$

Using $1 - \xi = 3/2 - i\sqrt{3}/2$ and equating real and imaginary parts gives the relations

$$2S(b,0) + S(b,1) - S(b,2) = 3S(b-1,0) - 3S(b-1,2),$$
$$S(b,1) + S(b,2) = S(b-1,0) + 2S(b-1,1) + S(b-1,2).$$

Together with (7), this leads quickly to the recursion relations

$$S(b,0) = S(b-1,0) - S(b-1,2), \tag{8}$$
$$S(b,1) = S(b-1,0) + S(b-1,1), \tag{9}$$
$$S(b,2) = S(b-1,1) + S(b-1,2). \tag{10}$$

Thus the values of $S(b, j)$ for b even are determined by those for b odd. We focus on the latter case.

First, an easy exercise using the symmetry $\binom{b}{n} = \binom{b}{b-n}$ shows that

$$S(6n+1,2) = 0 \quad \text{and} \quad S(6n+1,0) = S(6n+1,1), \tag{11}$$
$$S(6n+3,0) = 0 \quad \text{and} \quad S(6n+3,1) = S(6n+3,2), \tag{12}$$
$$S(6n+5,1) = 0 \quad \text{and} \quad S(6n+5,0) = -S(6n+5,2). \tag{13}$$

Next, we obtain a "2-level" recursion formula by applying (6) to the identity

$$(1-\xi)^b = (1-\xi)^2(1-\xi)^{b-2},$$

using $(1-\xi)^2 = 3(1/2 - i\sqrt{3}/2)$, and equating real and imaginary parts as before. The result is

$$S(b,0) + \tfrac{1}{2}S(b,1) - \tfrac{1}{2}S(b,2) = 3\big(\tfrac{1}{2}S(b-2,0) - \tfrac{1}{2}S(b-2,1) - S(b-2,2)\big),$$
$$S(b,1) + S(b,2) = 3(S(b-2,0) + S(b-2,1)),$$

which can be rearranged to yield

$$S(b,0) + S(b,1) = 3(S(b-2,0) - S(b-2,2)),$$
$$S(b,0) - S(b,2) = -3(S(b-2,1) + S(b-2,2)),$$
$$S(b,1) + S(b,2) = 3(S(b-2,0) + S(b-2,1)).$$

Substituting these equations into each other, we obtain

$$S(b,0) + S(b,1) = -3^3(S(b-6,0) + S(b-6,1)),$$
$$S(b,0) - S(b,2) = -3^3(S(b-6,0) - S(b-6,2)),$$
$$S(b,1) + S(b,2) = -3^3(S(b-6,1) + S(b-6,2)).$$

Now suppose that $b \equiv 1 \bmod 6$. Then by (11) we have $S(b,2) = 0$, so the last of the equations just displayed shows that, in this case, $S(b,1)$ satisfies

$$S(b,1) = -3^3 S(b-6,1).$$

Since $S(1,1)$ is obviously 1, it follows that if $b \equiv 1 \bmod 6$ then $S(b,1) = \pm 3^{(b-1)/2}$, where the sign depends on the value of b modulo 12. Similar reasoning for the other odd values of b modulo 6 gives the following statements.

$$\text{If } b \equiv 1 \bmod 6: \quad S(b,2) = 0 \text{ and } S(b,0) = S(b,1) = \pm 3^{(b-1)/2},$$

$$\text{If } b \equiv 3 \bmod 6: \quad S(b,0) = 0 \text{ and } S(b,1) = S(b,2) = \pm 3^{(b-1)/2},$$

$$\text{If } b \equiv 5 \bmod 6: \quad S(b,1) = 0 \text{ and } S(b,0) = -S(b,2) = \pm 3^{(b-1)/2}.$$

In particular, since $S(b,j) = \chi(C_j)$, we have proved the proposition for the case of b odd. The even case follows from this and the recursion relations (8)–(10). For example, if $b \equiv 0 \bmod 6$ then

$$S(b,0) = 2 \cdot 3^{b/2-1} \quad \text{and} \quad S(b,1) = -S(b,2) = \pm 3^{b/2-1}.$$

This completes the proof of Proposition 14. □

3.2. Behavior under Connected Sum

Proof of Theorem 2. If $Y = Y_1 \# Y_2$ is a connected sum, then we have a decomposition $H^1(Y) = H^1(Y_1) \oplus H^1(Y_2)$ and therefore

$$\Lambda^* H^1(Y) \cong \Lambda^* H^1(Y_1) \otimes \Lambda^* H^1(Y_2).$$

Under this decomposition, the cup product form μ_Y satisfies

$$\mu_Y = \mu_{Y_1} \otimes 1 + 1 \otimes \mu_{Y_2},$$

and since contraction is a derivation, $\partial_Y = \partial_{Y_1} \otimes 1 + (-1)^p 1 \otimes \partial_{Y_2}$ on $C_p^\infty(Y_1) \otimes C_q^\infty(Y_2)$. We thus have a decomposition of chain complexes

$$C_*^\infty(Y) \cong C_*^\infty(Y_1) \otimes_{\mathbb{Z}[U,U^{-1}]} C_*^\infty(Y_2).$$

Working with coefficients in a field $\mathbb{F}$, it follows that

$$HC_0^\infty(Y_1 \# Y_2) \cong (HC_0^\infty(Y_1) \otimes_{\mathbb{F}} HC_0^\infty(Y_2)) \oplus (HC_1^\infty(Y_1) \otimes_{\mathbb{F}} HC_1^\infty(Y_2))$$

and

$$HC_1^\infty(Y_1 \# Y_2) \cong (HC_0^\infty(Y_1) \otimes_{\mathbb{F}} HC_1^\infty(Y_2)) \oplus (HC_0^\infty(Y_1) \otimes_{\mathbb{F}} HC_1^\infty(Y_2)).$$

In the notation of Section 1, this gives

$$h_{\mathrm{e}}(Y_1 \# Y_2) = h_{\mathrm{e}}(Y_1) h_{\mathrm{e}}(Y_2) + h_{\mathrm{o}}(Y_1) h_{\mathrm{o}}(Y_2),$$
$$h_{\mathrm{o}}(Y_1 \# Y_2) = h_{\mathrm{e}}(Y_1) h_{\mathrm{o}}(Y_2) + h_{\mathrm{o}}(Y_1) h_{\mathrm{e}}(Y_2).$$

Adding these equations yields

$$h_{\mathrm{e}}(Y_1 \# Y_2) + h_{\mathrm{o}}(Y_1 \# Y_2) = (h_{\mathrm{e}}(Y_1) + h_{\mathrm{o}}(Y_1))(h_{\mathrm{e}}(Y_2) + h_{\mathrm{o}}(Y_2)),$$

which—because $h = \frac{1}{2}(h_e + h_o)$—is equivalent to

$$2h(Y_1 \# Y_2) = 4h(Y_1)h(Y_2). \qquad \square$$

As a simple illustration of this result we show that, if Y is a 3-manifold with $b_1(Y) = 5$ that is not rationally irreducible, then $h(Y)$ can only be 12 or 16. Indeed, suppose Y decomposes as $Y = Y' \# Y''$ and assume first that $b_1(Y') = 2$ and $b_1(Y'') = 3$. Then $h(Y') = 2$, and $h(Y'')$ is either 3 or 4. These two cases give $h(Y) = 12$ or 16, according to Theorem 2. The other possibility is that $b_1(Y') = 1$ and $b_1(Y'') = 4$; here $h(Y') = 1$ and $h(Y'')$ is 6, 7, or 8. The case $h(Y'') = 7$ is ruled out by Lemma 12, and the other two cases again give $h(Y) = 12$ and $h(Y) = 16$.

Proof of Theorem 6. We are given 3-manifolds Y_1 and Y_2 with nonvanishing first Betti numbers $x = b_1(Y_1)$ and $y = b_1(Y_2)$ that are not both odd. First suppose that x and y are of opposite parity: say x is odd and y is even. Then

$$h(Y_1 \# Y_2) = 2h(Y_1)h(Y_2) \geq 2(3^{(x-1)/2} \cdot 2 \cdot 3^{y/2-1}) = 4 \cdot 3^{(x+y-1)/2-1} = \frac{4}{3}L(b),$$

where $b = x + y = b_1(Y_1 \# Y_2)$ is odd.

Similarly, if both $x = b_1(Y_1)$ and $y = b_1(Y_2)$ are even, we have

$$h(Y_1 \# Y_2) = 2h(Y_1)h(Y_2) \geq 8 \cdot 3^{x/2-1} \cdot 3^{y/2-1} = \frac{8}{3} \cdot 3^{(x+y)/2-1} = \frac{4}{3}L(b). \qquad \square$$

4. Relation to Floer Homology

We next outline some results of Ozsváth and Szabó concerning the structure of $HF^\infty(Y, \mathfrak{s})$ for $\mathfrak{s}$ a torsion spin^c structure and then describe the relationship to $HC^\infty_*(Y)$.

Recall that there is a version of Heegaard Floer homology with "universal" coefficients in the ring $R_Y = \mathbb{Z}[H^1(Y; \mathbb{Z})]$. We have the following general result of Ozsváth and Szabó.

THEOREM 15 [10, Thm. 10.12]. *If $(Y, \mathfrak{s})$ is a closed* spin^c *3-manifold and $c_1(\mathfrak{s})$ is torsion, then there is an isomorphism*

$$HF^\infty(Y, \mathfrak{s}; R_Y) \cong \mathbb{Z}[U, U^{-1}]$$

of R_Y-modules, where elements of $H^1(Y; \mathbb{Z})$ act as the identity on $\mathbb{Z}[U, U^{-1}]$.

There is an additional structure on Heegaard Floer homology in a spin^c structure with torsion Chern class: a grading that takes values in the rational numbers, constructed by Ozsváth and Szabó in [11]. In the case of HF^∞, Theorem 15 provides an integral grading, natural up to an integer shift, with respect to which nonvanishing homogeneous parts lie in even degrees. The relationship between these two gradings is determined by the spin^c structure $\mathfrak{s}$ via the formula for the shift in grading induced by cobordisms [11]: explicitly, write the spin^c 3-manifold $(Y, \mathfrak{s})$ as the

spin^c boundary of a 4-manifold $(Z, \mathfrak{t})$. Then the rational-valued grading on the Heegaard Floer homology of $(Y, \mathfrak{s})$ takes values in $\mathbb{Z} + r$, where $r \in \mathbb{Q}$ is given by

$$r = \frac{1}{4}(c_1^2(\mathfrak{t}) - 3\sigma(Z) - 2e(Z)) + \frac{1}{2}.$$

Here $\sigma(Z)$ is the signature of the intersection form of Z and $e(Z)$ is the Euler characteristic.

For our purposes the rational-valued grading is unimportant, so we simply impose an integer grading on HF^∞ in such a way that the universally twisted homology is supported in even degrees as before. This choice induces an integer grading on $HF^\infty(Y, \mathfrak{s}, M)$ for any coefficient module M—in particular, for $M = \mathbb{Z}$—that is well-defined up to a shift by an even integer.

The $\mathbb{Z}$-coefficient Floer homology can be calculated from the universal version by making use of a change-of-coefficients spectral sequence. For the benefit of readers not familiar with this construction, we outline it here in general before considering the case at hand.

Let (C_*, d) be a free chain complex of modules over a commutative ring R and let M be an R-module. We wish to compute the homology of the complex $C_* \otimes_R M$. Under reasonable circumstances we may assume that M has free resolution

$$\cdots \xrightarrow{\delta_2} P_2 \xrightarrow{\delta_1} P_1 \xrightarrow{\delta_0} P_0 \longrightarrow M \longrightarrow 0;$$

in other words, each P_j is a free R-module and the sequence shown is exact (a projective resolution would also suffice). We can form the double complex $E^0_{i,j} = C_i \otimes_R P_j$, with the obvious pair of differentials $d\colon E^0_{i,j} \to E^0_{i-1,j}$ (the "horizontal" differential) and $\delta\colon E^0_{i,j} \to E^0_{i,j-1}$ (the "vertical" differential) and "total complex" $(\mathrm{tot}(E)_*, D)$ where $\mathrm{tot}(E)_k = \bigoplus_{i+j=k} E^0_{i,j}$ and $D = d + (-1)^j\delta$ on $E^0_{i,j}$.

The double complex gives rise to a spectral sequence (E^n, d_n) converging to the associated graded module $\mathrm{Gr}(H_*(\mathrm{tot}(E)))$ determined by one of two natural filtrations on $\mathrm{tot}(E)$: the horizontal filtration

$$\cdots \subset F^h_0 \subset F^h_1 \subset \cdots, \quad F^h_j = \bigoplus_{j' \le j} E^0_{i,j'},$$

and the vertical filtration

$$\cdots \subset F^v_0 \subset F^v_1 \subset \cdots, \quad F^v_i = \bigoplus_{i' \le i} E^0_{i',j}.$$

In fact, each of these filtrations gives rise to its own spectral sequence, but (as we shall see) one of them is essentially trivial. Hence we will suppress the filtration from the notation for the sequence (E^n, d_n).

The spectral sequence is constructed as follows. Choose one of the filtrations F^h_j or F^v_i and take the homology of E^0 with respect to the corresponding differential. The result is called $E^1_{i,j}$, and it inherits a differential from the original complex that corresponds to the differential in the direction not used at first. For example, with the double complex $E^0_{i,j} = C_i \otimes P_j$, consider the homology with respect to the differential δ (i.e., the vertical homology). Because the C_i are free modules, the result is $E^1_{i,j} = C_i \otimes H_j(P_*)$. Of course, by construction $H_j(P_*) = 0$

except when $j = 0$, and $H_0(P_*) = M$. Thus, starting with the vertical differential gives a spectral sequence with

$$E^1_{i,j} = \begin{cases} 0 & \text{if } j > 0, \\ C_i \otimes M & \text{if } j = 0. \end{cases}$$

Taking the homology with respect to the remaining differential—the horizontal one—gives the E^2 term of this sequence, which is obviously

$$E^2_{i,j} = \begin{cases} 0 & \text{if } j > 0, \\ H_i(C_* \otimes M) & \text{if } j = 0. \end{cases}$$

The general machinery of spectral sequences would now give a differential d_2: $E^2_{i,j} \to E^2_{i-2,j+1}$ whose homology is the E^3-term, but from the structure of E^2 just shown, this differential must be trivial (and the same holds for all subsequent differentials). The spectral sequence corresponding to this filtration therefore collapses at the E^2 term, and we infer that $H_*(\mathrm{tot}(E)) = H_*(C_* \otimes M)$ is the homology we wish to calculate.

To calculate this homology, we turn to the other filtration; that is, we use the other differential first. Returning to E^0, we take the homology with respect to the horizontal differential, the differential of C_*. Again, since the P_j are free, it follows that

$$E^1_{i,j} = H_i(C_*) \otimes_R P_j.$$

When we take the next homology to get E^2—that is, the homology in the vertical direction coming from the δ_j—we can no longer commute homology and tensor product (since the $H_i(C_*)$ need not be free R-modules). By definition, this vertical homology is

$$E^2_{i,j} = \mathrm{Tor}^R_j(H_i(C_*), M).$$

We have thus recovered the following standard fact.

PROPOSITION 16. *Given a free chain complex of R-modules C_* and another module M, there exists a "universal coefficients spectral sequence" converging to the homology $H_*(C_* \otimes_R M)$ whose E^2 term is given by $E^2_{i,j} = \mathrm{Tor}^R_j(H_i(C_*), M)$ and with differential $d_2\colon E^2_{i,j} \to E^2_{i+1,j-2}$.*

Returning now to Heegaard Floer homology, Theorem 15 states that the Floer homology $HF^\infty(Y, \mathfrak{s}, R_Y)$ for $\mathfrak{s}$ a spin^c structure with $c_1(\mathfrak{s})$ torsion is equal to 0 or $\mathbb{Z}$ in alternating degrees. Hence the E^2 term of the universal coefficient spectral sequence (taking $M = \mathbb{Z}$) has zeros in each odd column, while the remaining columns are given by

$$E^2_{i,*} = \mathrm{Tor}^{R_Y}_*(\mathbb{Z}, \mathbb{Z})$$

for each even i. It is a standard fact that if G is an abelian group then

$$\mathrm{Tor}^{\mathbb{Z}[G]}_*(\mathbb{Z}, \mathbb{Z}) = H_*(G; \mathbb{Z}) = H_*(K(G,1); \mathbb{Z}).$$

In our case, $G = H^1(Y; \mathbb{Z})$ is free abelian of rank $b_1(Y)$ and $K(G,1)$ is the torus T^b of dimension $b = b_1(Y)$. Thus we have a natural identification $\mathrm{Tor}^{R_Y}_*(\mathbb{Z}, \mathbb{Z}) = H_*(T^b; \mathbb{Z}) \cong \Lambda^* H^1(Y; \mathbb{Z})$, and the E^2 term of the universal coefficients spectral sequence can be written as

$$\mathrm{Tor}_*^{R_Y}(HF_*^\infty(Y,\mathfrak{s};\,R_Y),\mathbb{Z}) \cong \Lambda^* H^1(Y;\mathbb{Z}) \otimes \mathbb{Z}[U,U^{-1}] \cong C_*^\infty(Y). \qquad (14)$$

Observe that, by Proposition 16, the d_2 differential maps one column to the right and two rows down. Since every other column in the E^2 complex vanishes, we infer that $d_2 = 0$ and so (14) is also the E^3 term of the spectral sequence. The d_3 differential restricts as a map $d_3\colon \Lambda^k H^1(Y) \otimes U^n \to \Lambda^{k-3} H^1(Y) \otimes U^{n-1}$. Ozsváth and Szabó conjecture in [8] that (E^3, d_3) is the complex $(C_*^\infty(Y), \partial)$ considered in this paper. It is also conjectured in [8] that all subsequent differentials in the spectral sequence vanish, so that the E^∞ term in the universal coefficients spectral sequence is our $HC_*^\infty(Y)$.

Although we do not address these conjectures here, we observe that—since the arguments in the proof of Theorem 1 do not use the differential on $C_*^\infty(Y)$ but only the ranks of the chain groups—the bounds obtained there apply to the rank in each degree of $HF^\infty(Y,\mathfrak{s};\mathbb{Z})$ provided that the universal coefficients sequence collapses after the E_3 stage (see also the remarks on monopole Floer homology at the end of the Introduction). Let us say that $(Y,\mathfrak{s})$ is *regular* if $c_1(\mathfrak{s})$ is a torsion class and all differentials d_r $(r \geq 4)$ in that spectral sequence vanish.

PROPOSITION 17. *If $(Y,\mathfrak{s})$ is a regular* spin^c *3-manifold then the rank of $HF_k^\infty(Y,\mathfrak{s})$ satisfies*

$$L(b) \leq \mathrm{rk}(HF_k^\infty(Y;\mathfrak{s})) \leq 2^{b-1},$$

where $b = b_1(Y)$. □

The even differentials d_{2r} in the sequence we are considering vanish for dimensional reasons. Therefore, the first differential past d_3 that may be nontrivial is $d_5\colon \Lambda^k H^1(Y) \otimes U^n \to \Lambda^{k-5} H^1(Y) \otimes U^{n-2}$. Hence our next proposition follows.

PROPOSITION 18. *If $b_1(Y) \leq 4$, then $(Y,\mathfrak{s})$ is regular for any torsion* spin^c *structure $\mathfrak{s}$.*

This proposition—together with Theorems 1 and 2—proves Corollary 4.

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