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Degree Estimates for Polynomials Constant on a Hyperplane

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I. Introduction

We are interested in the complexity of real-valued polynomials that are defined on real Euclidean space $\mathbf{R}^n$ and are constant on a hyperplane. This issue arises as a simplified version of a difficult question in CR geometry, which we discuss shortly and also in Section VI. We intend to fully address the CR issues in a subsequent paper.

Let H denote the hyperplane in $\mathbf{R}^n$ defined by $\{x : s(x) = \sum_{j=1}^n x_j = 1\}$. We write $\mathbf{R}[x] = \mathbf{R}[x_1, \dots, x_n]$ for the ring of real-valued polynomials in n real variables. Suppose $p \in \mathbf{R}[x]$ and that p is constant on H. How complicated can p be? Two possible measurements of the complexity of a polynomial are its degree d and the number N of its distinct monomials. We always have the standard estimate

$$N \le \binom{n+d}{n}, \tag{1}$$

which estimates d from below. Even when p is constant on H, no upper estimate for d in terms of N is possible without additional assumptions. For example, for $d \ge 2$ consider

$$p(x) = x_1^{d-1} s(x) - x_1^{d-1} + 1. \tag{2}$$

It is evident that $p = 1$ on H, that p has $n+2$ distinct monomials, and that its degree d can be arbitrarily large. On the other hand, such degree estimates become possible when we assume that $n \ge 2$ and that the coefficients of p are nonnegative. We prove such results in this paper.

Before describing our results we briefly discuss the motivation behind them; see Section VI for additional information. In a future paper we will say more about this connection with CR geometry. Let $f : \mathbf{C}^n \to \mathbf{C}^N$ be a rational mapping such that f maps the unit ball in its domain properly to the unit ball in its target. It follows that f maps the unit sphere in $\mathbf{C}^n$ to the unit sphere in $\mathbf{C}^N$. For $n \ge 2$, the work of Forstneric [F1] implies that the degree of f is bounded in terms of n and N. The bound in [F1] is not sharp, and finding a sharp bound seems to be difficult. Meylan [M] has improved the bound when $n = 2$.

The problem simplifies somewhat by assuming that f is a monomial mapping—that is, a polynomial mapping for which (after a coordinate change if necessary)

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each component is a monomial. The condition $\|f(z)\|^2 = 1$ on $\|z\|^2 = 1$ then depends upon only the real variables $|z_1|^2, \dots, |z_n|^2$, and all coefficients involved appear as $|c|^2$ for complex numbers c. The relationship between the degree of f and the domain and target dimensions then becomes the combinatorial issue described in Problem 1.

We need to consider various subsets of $\mathbf{R}[x_1, \dots, x_n]$. Let $\mathcal{J}(n)$ denote the subset of polynomials p in $\mathbf{R}[x_1, \dots, x_n]$ for which $p(x) = 1$ on the hyperplane H. The set $\mathcal{J}(n)$ is closed under multiplication, convex combinations, and the operation X described in Section II. Let $\mathcal{P}(n)$ denote those polynomials in $\mathbf{R}[x_1, \dots, x_n]$ whose coefficients are *nonnegative*. The set $\mathcal{P}(n)$ is closed under addition and multiplication. Let $\mathcal{P}(n,d)$ denote the subset of $\mathcal{P}(n)$ whose elements are of degree d. The crucial sets for us are $\mathcal{H}(n)$ and $\mathcal{H}(n,d)$:

$$\mathcal{H}(n) = \mathcal{J}(n) \cap \mathcal{P}(n),$$
$$\mathcal{H}(n,d) = \mathcal{J}(n) \cap \mathcal{P}(n,d).$$

Thus the elements of $\mathcal{H}(n,d)$ are polynomials of degree d in n real variables, with nonnegative coefficients, and whose values are 1 on the set $\sum x_j = 1$. For $p \in \mathbf{R}[x]$, we write $N(p)$ for the number of distinct monomials occurring in p. Our goal is to prove sharp estimates relating the degree of p to $N(p)$ when $p \in \mathcal{H}(n)$.

PROBLEM 1. Assume $n \geq 2$. For $p \in \mathcal{H}(n)$, find a sharp upper bound for $d(p)$ in terms of $N(p)$ and n.

There is no such upper bound when $n = 1$, as we note in Section II. When $n = 2$, the sharp upper bound is given by $d(p) \leq 2N(p) - 3$, a result from [DKR] also discussed in Section II. For $n \geq 3$, the first author has conjectured the bound

$$d(p) \leq \frac{N(p) - 1}{n - 1}. \tag{3}$$

Example 4 provides polynomials of each degree where equality holds in (3).

In Proposition 4 we pull back to the two-dimensional case via a Veronese mapping to obtain a general but crude bound. For $n \geq 2$ and $p \in \mathcal{H}(n,d)$ we obtain

$$d(p) \leq \frac{2N(p) - 3}{n - 1}. \tag{4}$$

This result is not sharp unless $n = 2$. In Section IV we improve (4) by pulling back via the optimal mappings in two dimensions. In Theorem 1 we obtain

$$d(p) \leq \frac{2n(2N(p) - 3)}{3n^2 - 3n - 2} \leq \frac{4}{3}\,\frac{2N(p) - 3}{2n - 3}. \tag{5}$$

In Theorem 2 we prove our main result: for n sufficiently large compared with d, the estimate (3) holds, and we can find all polynomials for which equality holds in (3). We remark and later demonstrate that, when $n = 3$ (for example), there are additional polynomials for which equality holds. Hence it is reasonable to think of Theorem 2 as a stabilization result: certain complicated issues arise in low dimensions but become irrelevant as the dimension n rises. Corollary 2 also lends support to the conjecture. When $n \geq 3$ we show that the conjecture holds for degree up to 4, and we also show that the conjecture holds when $N < 4n - 3$.

This paper may be briefly summarized as follows. In Theorem 1 we prove a general bound that is not sharp unless $n = 2$. Lemmas 4 and 5 show how to sharpen that bound in specific situations. In Corollary 2 we prove a sharp bound for all n when either $d \leq 4$ or $N < 4n - 3$. In Theorem 2 we establish the sharp bound when n is sufficiently large given d.

We close this introduction with one additional comment. When $p \in \mathcal{J}(n)$, the function

$$Q(p) = \frac{p - 1}{s - 1} \tag{6}$$

is a polynomial; its coefficients need not be nonnegative even if $p \in \mathcal{H}(n)$. The polynomial Q plays a crucial role in the proof in two dimensions and thus plays an implicit role here. Perhaps some of our results can be better understood in terms of $Q(p)$.

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II. The Situations in One and Two Dimensions

The situation in one dimension is not interesting, so we dispense with it now and assume thereafter that $n \geq 2$. When $n = 1$, observe that $p \in \mathcal{H}(1)$ when p has nonnegative coefficients and $p(1) = 1$. The particular polynomial $p(x_1) = x_1^d$ lies in $\mathcal{H}(1, d)$, and $N(p) = 1$. Furthermore, for any fixed value of N, we can find a polynomial p of arbitrarily large degree with $N(p) = N$. Hence no upper bound for $d(p)$ is possible.

When $n = 2$, a sharp result is known [DKR].

Theorem 0. *Let p be a polynomial in two real variables (x, y) such that*

(1) *$p(x, y) = 1$ when $x + y = 1$, and*

(2) *each coefficient of p is nonnegative.*

Let N be the number of distinct monomials in p, and let d be the degree of p. Then $d \leq 2N - 3$. Furthermore, for each $N \geq 2$ there exists a polynomial p_d satisfying (1) and (2) and whose degree is $2N - 3$.

The estimate $d \leq 2N - 3$ can of course be rewritten $N \geq \frac{d+3}{2}$. The proof of Theorem 0 shows that a slightly stronger conclusion holds. If p satisfies (1) and (2) then p must have at least $\frac{d-1}{2}$ mixed terms (those containing both x and y) and at least two pure terms.

There is an interesting family of polynomials providing the sharp bound in Theorem 0. The polynomials in this family have integer coefficients, are group-invariant, and exhibit many interesting combinatorial and number-theoretic properties. We mention for example that $p_d(x,y) \cong x^d + y^d$ if and only if d is prime. See [D1; D2; D3; D5; DKR] for this fact and much additional information. Here is an explicit formula for these polynomials for d odd:

$$p_d(x,y) = y^d + \left(\frac{x+\sqrt{x^2+4y}}{2}\right)^d + \left(\frac{x-\sqrt{x^2+4y}}{2}\right)^d. \tag{7}$$

We also provide a recurrence formula relating these polynomials as the degree varies. Put $g_0(x,y) = x$ and $g_1(x,y) = x^3 + 3xy$. Define g_{k+2} and then p_{2k+1} by

$$\begin{aligned} g_{k+2}(x,y) &= (x^2+2y)g_{k+1}(x,y) - y^2 g_k(x,y), \\ p_{2k+1}(x,y) &= g_k(x,y) + y^{2k+1}. \end{aligned} \tag{8}$$

The equations in (8) determine the polynomials in (7). For odd d, the polynomial defined by (7) has precisely $\frac{d+3}{2}$ terms and so the bound in Theorem 0 is sharp. We can obtain a second sharp example by interchanging the roles of x and y. Other examples exhibiting the sharp bound exist for some but not all N. See Example 3 where $N = 5$.

Each p_{2r+1} is group-invariant; we have $p_{2r+1}(\omega x, \omega^2 y) = p_{2r+1}(x,y)$ whenever ω is a $(2r+1)$th root of unity. There are analogous group-invariant polynomials of even degree, but these have a single negative coefficient and will not be discussed in this paper.

The proof of the inequality $d \le 2N - 3$ in Theorem 0 is quite complicated. It relies on an analysis of certain directed graphs arising from the Newton diagram of the polynomial $Q(p)$ and their interaction with Proposition 1.

We close this section by indicating how one can use Theorem 0 to study the higher-dimensional case. Let $\phi\colon \mathbf{R}^2 \to \mathbf{R}^n$ be a polynomial mapping, and suppose that ϕ maps the line defined by $u + v = 1$ to the hyperplane H. If $p \in \mathcal{J}(n)$, then the composite map $\phi^*(p)$ is in $\mathcal{J}(2)$. To see this, observe that for $u + v = 1$ we have

$$\phi^*(p)(u,v) = p(\phi(u,v)) = 1 \tag{9}$$

because $p = 1$ on H.

We will apply this idea of pulling back to two dimensions for various functions ϕ. We give some examples. Assume $n \ge 3$. Set $x_i = u$ and $x_j = v$ for some i, j with $i \ne j$. Otherwise, set $x_k = 0$. Another possibility is to set k of the variables equal to u/k, set l of the other variables equal to v/l, and set the remaining variables equal to zero; in these cases, ϕ is linear. In the proof of Proposition 4 we let ϕ be a Veronese mapping, where ϕ is homogeneous of degree > 1. One can also gain information by pulling back via more complicated mappings (see Sections IV and V for details).

III. General Information

We begin with several formal algebraic observations. Suppose that $p \in \mathcal{J}$ and that u is an arbitrary polynomial. We define a polynomial $X_u(p)$ by

$$X_u(p) = p - u + su. \tag{10}$$

When $p \in \mathcal{J}$ we can always write $p = (1 - Q) + sQ$ with Q as in (6), so $p = X_Q(1)$. In general we will drop the dependence on u from the notation and write $X(p)$ for $X_u(p)$. The following simple but crucial result suggests decomposing elements in $\mathcal{H}$ using the operation in (10).

LEMMA 1. *Suppose that $p \in \mathcal{J}$ and that u is an arbitrary polynomial. Define $X(p)$ by* (10). *Then $X(p) \in \mathcal{J}$. Suppose $p \in \mathcal{H}$ and also that both u and $p - u$ are in $\mathcal{P}$. Then $X(p) \in \mathcal{H}$.*

Proof. It is immediate from (10) that $X(p) = p - u + u = p$ on the set $s = 1$; hence $X(p) \in \mathcal{J}$. Suppose that both u and $p - u$ are in $\mathcal{P}$ and that $s \in \mathcal{P}$. Since $\mathcal{P}$ is closed under addition and multiplication, it follows that $X(p) \in \mathcal{P}$. Since we have also shown that $X(p) \in \mathcal{J}$, it follows that $X(p) \in \mathcal{H}$. $\square$

Our concern with nonnegative coefficients leads us to make the following definition.

DEFINITION 1. Suppose that $p, g \in \mathcal{P}(n)$. We say that $g \subset p$ if $p - g \in \mathcal{P}(n)$. In other words, $g \subset p$ holds if and only if both g and $p - g$ have nonnegative coefficients. We call g a *subpolynomial* of p.

When u is a subpolynomial of p, Lemma 1 tells us that the operation X maps $\mathcal{H}$ to itself (though of course it need not preserve degree). The operation defined by replacing p with $X(p)$ is a simple special case of a tensor product operation defined in [D1].

DEFINITION 2. An element p of $\mathcal{H}(n, d)$ is called a *generalized Whitney mapping* if there exist elements $g_0, \ldots, g_d$ of $\mathcal{H}(n)$ such that:

(1) $g_0 = 1$ and $g_d = p$;
(2) for each j, the degree of g_j is j;
(3) for each $j > 0$, we have $g_j = X(g_{j-1})$.

We say that $g_0, \ldots, g_d$ defines a *Whitney chain* from 1 to p.

At each step along the way of a Whitney chain, we replace g_j with $g_j - u + su$, where u has degree j and hence g_k has degree k for all k.

EXAMPLE 1. The polynomial $x + xy + xy^2 + y^3$ is a generalized Whitney mapping with $d = 3$. We have

$$\begin{aligned} g_0 = 1 \mapsto g_1 = x + y \mapsto x + y(x + y) = g_2 = x + xy + y^2 \\ \mapsto x + xy + y^2(x + y) = g_3 = p = x + xy + xy^2 + y^3. \end{aligned} \tag{11}$$

We can rewrite (11) using the operation X:

$$x + xy + xy^2 + y^3 = X(x + xy + y^2) = X(X(x + y)) = X(X(X(1))).$$

LEMMA 2. *Suppose that $p \in \mathcal{H}(n, d)$ is a generalized Whitney mapping. Then $N(p) \geq d(n - 1) + 1$.*

Proof. We induct on d. When $d = 0$ we have $p = 1$ and the conclusion holds. Suppose that we know the result in degree $d - 1$. Then $p = X(g) = g - u + su$, where g is of degree $d-1$. By the induction hypothesis, $N(g) \geq (d-1)(n-1)+1$. Suppose first that u consists of a single monomial m. Then m is eliminated in passing from g to $g - u$, but m gets replaced with the n new monomials $x_1 m, \dots, x_n m$. Therefore,

$$N(X(g)) \geq N(g) + n - 1 \geq (d-1)(n-1) + 1 + n - 1 = d(n-1) + 1. \tag{12}$$

If u consists of several monomials then, because the coefficients are nonnegative, (12) remains true. $\square$

We make a few simple remarks. First, the operation in (10) can be generalized by replacing s with any element of $\mathcal{J}$. Next, we will show that not all elements of $\mathcal{H}(n)$ are generalized Whitney maps. On the other hand, if we allow negative coefficients along the way then all such maps can be built up in this fashion. We provide a simple example.

EXAMPLE 2. Consider $p(x, y) = x^3 + 3xy + y^3$. Then $p \in \mathcal{H}(2, 3)$. We can write $p = X_3(X_2(X_1(1)))$ as follows:

$$\begin{aligned} 1 \mapsto s \mapsto s^2 &= 3xy + s^2 - 3xy \\ &\mapsto 3xy + (s^2 - 3xy)s = 3xy + x^3 + y^3 = p(x, y). \end{aligned}$$

In the notation of (10), we have $g = s^2$ and $u = s^2 - 3xy$. In using $s^2 - 3xy$ we have introduced a negative coefficient that was eliminated by the final multiplication by s. One can easily show that we cannot construct p by iterating this process while keeping all coefficients nonnegative. As stated previously, if we allow negative coefficients then all elements of $\mathcal{H}(n)$ are obtained via iterations analogous to those in Example 2. We now prove this assertion.

Proposition 1 describes all elements of $\mathcal{H}(n)$ via *undoing* the operation in (10). Proposition 2 uses only the operation (10) but requires negative coefficients at intermediate steps. In proving these results it is convenient to expand polynomials in terms of their homogeneous parts. When p is of degree d we write

$$p = \sum_{j=0}^{d} p_j, \tag{13}$$

where each p_j is homogeneous of degree j and we allow the possibility that $p_j = 0$.

PROPOSITION 1. *Suppose $p \in \mathcal{H}(n, d)$. Then there is an integer k such that*

$$s^d = X^k(p) = \sum_{j=0}^{d} p_j s^{d-j}. \tag{14}$$

Proof. Write $p = \sum p_j$ as in (13). Suppose first that p is not already homogeneous. It is evident for each j that $p_j \subset p$. Let ν be the smallest index for which $p_\nu \neq 0$. Then p_ν is a subpolynomial of p and we may consider $X(p)$ defined as in (10) by

$$X(p) = (p - p_\nu) + sp_\nu.$$

Then $X(p)$ also lies in $\mathcal{H}(n, d)$, and $X(p)$ vanishes to higher order than p does. We iterate Lemma 1 in this way until we obtain the polynomial

$$h = \sum_{j=\nu}^{d} s^{d-j} p_j, \tag{15}$$

which lies in $\mathcal{H}(n, d)$. Now h is homogeneous of degree d. The only homogeneous polynomial of degree d that is identically equal to unity on the hyperplane $\{x : s(x) = 1\}$ is s^d. Therefore, (14) holds. □

Formula (14) holds even when $p \in \mathcal{J}$, and we obtain the following version where negative coefficients are allowed.

PROPOSITION 2. *Suppose $p \in \mathcal{J}(n, d)$. Then there is a finite list of maps $X_1, \ldots, X_t$ from $\mathcal{J}$ to itself of the form*

$$X_j(v) = (v - r) + sr \tag{16}$$

such that

$$p = X_t \circ X_{t-1} \circ \cdots \circ X_1(1). \tag{17}$$

Proof. We induct on the degree. When the degree is zero, the only example is $p = 1$. Suppose that the result holds for all elements of $\mathcal{J}(n, k)$ for $k \le d - 1$. Let $p \in \mathcal{J}(n, d)$. We expand p into its homogeneous parts as before and then use (14) to rewrite the highest-order part p_d. We obtain for a homogeneous polynomial r of degree $d - 1$ that

$$\begin{aligned} p = \sum_{j=0}^{d-1} p_j + p_d &= \sum_{j=0}^{d-1} p_j + s^d - \sum_{j=0}^{d-1} p_j s^{d-j} \\ &= \sum_{j=0}^{d-1} p_j + s\left(s^{d-1} - \sum_{j=0}^{d-1} p_j s^{d-j-1}\right) \\ &= \sum_{j=0}^{d-1} p_j + sr = (p - p_d) + sr. \end{aligned} \tag{18}$$

Note that $p - p_d + r \in \mathcal{J}(n, d - 1)$ and hence, by the induction hypothesis, can be factored as in (17). Since

$$p = (p - p_d) + sr = X(p - p_d + r), \tag{19}$$

the induction step is complete. □

We repeat one subtle point regarding Proposition 2. Given $p \in \mathcal{H}(n, d)$, it follows from (19) that there exists an r of degree $d - 1$ such that $p = u + sr$. In general, neither r nor u must have nonnegative coefficients. The next mapping provides an example where negative coefficients arise and where the sharp bound from Theorem 0 arises without group invariance.

EXAMPLE 3. Put $p(x,y) = x^7 + y^7 + \frac{7}{2}x^5y + \frac{7}{2}xy^5 + \frac{7}{2}xy$. Then $p \in \mathcal{H}(2,7)$. Following the proof of Proposition 2, we obtain

$$\begin{aligned} p(x,y) &= p_2(x,y) + p_6(x,y) + p_7(x,y) \\ &= p_2(x,y) + p_6(x,y) + (x+y)^7 \\ &\quad - (x+y)^5 p_2(x,y) - (x+y)p_6(x,y) \end{aligned}$$

and hence

$$p = p_2 + p_6 + s(s^6 - p_2 s^4 - p_6) = p - p_7 + sr, \tag{20}$$

where $r = s^6 - p_2 s^4 - p_6$. Expanding r yields

$$r(x,y) = x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6, \tag{21}$$

which has negative coefficients. Furthermore, $(p - p_7) + r$ has a negative coefficient.

The operation X replaces u with $u - r + sr$. When we want to remind the reader that we want both r and $u - r$ to have nonnegative coefficients, we write W instead of X. To repeat, one cannot realize all elements of $\mathcal{H}(n)$ by successive application of W. Let $\mathcal{W}$ denote the subset of $\mathcal{H}$ that can be obtained by repeated application of the operation W beginning with the constant function 1. We give one more simple example. Let $n = 3$ with variables (x, y, z). Applying W always to the "last" monomial, we obtain

$$\begin{aligned} W^3(1) &= W^2(x+y+z) = W(x+y+xz+yz+z^2) \\ &= x + y + xy + xz + xz^2 + yz^2 + z^3. \end{aligned} \tag{22}$$

We next give, without proof, another example of an element of $\mathcal{H}(n)$ that is not in $\mathcal{W}$:

$$x^3 + 3xy + 3xz + y^3 + 3y^2z + 3yz^2 + z^3. \tag{23}$$

The polynomial defined by (23) occurs also in the discussion after Proposition 4. It plays an important role because it satisfies the sharp estimate from Problem 1, yet it is not in $\mathcal{W}$. In some sense it can exist because the dimension 3 is too small for stabilization to have taken place. Observe that both (22) and (23) are of degree 3, and each has seven monomials.

It is easy to see that polynomials formed by the process in (22) have $N = d(n-1) + 1$ terms. The first author has conjectured, for $n \geq 3$, that the inequality

$$N \geq d(n-1) + 1 \tag{24}$$

always holds. Theorem 2 yields this inequality for all n that are large enough relative to d. Given d, for such sufficiently large n we prove a stronger result by identifying all polynomials for which equality holds in (24); these are precisely the generalized Whitney polynomials. The stronger assertion fails in dimension 3, but we believe that (24) still holds.

We next observe that there are always at least n terms of degree d.

LEMMA 3. *Suppose $f \in \mathbf{R}[x_1, \dots, x_n]$ and f is not identically 0. Then the polynomial sf has at least n monomials.*

Proof. We claim first it suffices to assume that f is homogeneous. Assuming that the homogeneous case is known, we then write $f = f' + f_d$, where f_d consists of the highest-degree terms. Then $sf = sf' + sf_d$, where sf_d has at least n terms. All the terms in sf' are of lower degree and hence cannot cannot cancel the terms in sf_d. Thus the claim holds.

To prove the homogeneous case we proceed by induction on n. When $n = 1$ the result is trivial. Suppose $n \geq 2$ and suppose the result is known in $n-1$ variables. Given a homogeneous f in n variables, we write

$$f(x) = x_n^d f\left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}, 1\right) = x_n^d f(y_1, \dots, y_{n-1}, 1). \tag{25}$$

It follows that

$$s(x)f(x) = (y_1 + \cdots + y_{n-1} + 1)x_n^{d+1} f(y_1, \dots, y_{n-1}, 1). \tag{26}$$

The number of terms in sf is the same as the number of terms in the right-hand side (RHS) of (26) after dividing by x_n^{d+1}. Hence the number of terms in sf is the number of terms in

$$(y_1 + \cdots + y_{n-1}) f(y_1, \dots, y_{n-1}, 1) + f(y_1, \dots, y_{n-1}, 1). \tag{27}$$

The first expression in (27) has at least $n-1$ terms (by the induction hypothesis), and the second expression has at least one additional term. □

COROLLARY 1. *If $d > 0$ and $p \in \mathcal{J}(n)$ has degree d, then p has at least n terms of degree d.*

Proof. From (19) it follows that $p = p' + p_d = p' + sr_{d-1}$. By Lemma 3, sr_{d-1} has at least n terms of degree d. □

We will close this section by proving Proposition 4. First we introduce a Veronese mapping $\phi_{n-1} \colon \mathbf{R}^2 \to \mathbf{R}^n$ defined by

$$\phi_{n-1}(u, v) = \left(u^{n-1}, \dots, \binom{n-1}{j} u^j v^{n-1-j}, \dots, v^{n-1}\right). \tag{28}$$

The binomial theorem shows that the sum of the components of ϕ_{n-1} is $(u+v)^{n-1}$. Therefore, ϕ_{n-1} maps the line given by $u + v = 1$ to the hyperplane H.

Let $p \colon \mathbf{R}^n \to \mathbf{R}$ be a function. The pullback $\phi_{n-1}^*(p)$ is the composite function defined on $\mathbf{R}^2$ by $(u, v) \to p(\phi_{n-1}(u, v))$. We easily obtain the following simple facts.

PROPOSITION 3. *If $p \in \mathcal{H}(n, d)$, then $\phi_{n-1}^*(p) \in \mathcal{H}(2, d(n-1))$. Furthermore, $N(\phi_{n-1}^*(p)) \leq N(p)$.*

Proof. That $\phi_{n-1}^*(p)$ has degree $(n-1)d$ follows since ϕ_{n-1} is homogeneous and since the positivity of all coefficients prevents cancellation. By the comment after (28), we have

$$\phi_{n-1}^*(s)(u,v) = s(\phi_{n-1}(u,v)) = (u+v)^{n-1}$$

and thus ϕ_{n-1} maps the line given by $u+v=1$ to the hyperplane H. Since $p=1$ on H, we see that $\phi_{n-1}^*(p)=1$ on $u+v=1$. Since all the coefficients are nonnegative, it follows that $\phi_{n-1}^*(p) \in \mathcal{H}(2, d(n-1))$. Finally, we cannot increase the number of terms by a monomial substitution and so $N(\phi_{n-1}^*(p)) \le N(p)$. □

The proof of Proposition 3 uses the nonnegativity of the coefficients. For example, the pullback of the polynomial $x_2^2 - 4x_1x_3$ to $(u^2, 2uv, v^2)$ vanishes. Without assuming nonnegativity of the coefficients we cannot therefore conclude that the degree of $\phi_{n-1}^*(p)$ is $(n-1)d$. The same example shows that pulling back via ϕ_{n-1} can decrease the number of terms.

PROPOSITION 4. *Suppose $p \in \mathcal{H}(n,d)$. Then*

$$d(p) \le \frac{2N(p)-3}{n-1}. \tag{29}$$

Proof. By Proposition 3 and Theorem 0, we obtain the chain of inequalities

$$d(p) = \frac{d(\phi_{n-1}^*(p))}{n-1} \le \frac{2N(\phi_{n-1}^*(p))-3}{n-1} \le \frac{2N(p)-3}{n-1},$$

which gives the desired conclusion. □

The inequality in Proposition 4 is not sharp unless $n=2$. When $n \ge 3$ the bound (5) obtained in Theorem 1 is smaller than the RHS of (29). For a given polynomial we can sometimes obtain a better bound by pulling back via a mapping other than the Veronese. We illustrate with a simple example. Define the mapping $p \in \mathcal{H}(3,7)$ by

$$p(x,y,z) = x^3 + 3x(y+z) + (y+z)^3.$$

Then $d(p)=3$ and $N(p)=7$. Pulling back via the Veronese mapping ϕ given by $\phi(u,v) = (u^2, 2uv, v^2)$ gives an element of $\mathcal{H}(2,6)$ with seven terms. The inequality

$$d(\phi^*(p)) = 6 \le 11 = 2N(\phi^*(p)) - 3$$

is not sharp. Pulling back via the mapping given by $\psi(u,v) = (u^3, \sqrt{3}uv, v^3)$ yields an element of $\mathcal{H}(2,9)$ with six terms; we therefore obtain the sharp result

$$d(\psi^*(p)) = 9 = 2N(\psi^*(p)) - 3.$$

This discussion motivates the technique used to prove Theorem 1.

IV. Optimal Polynomials

We call an element p of $\mathcal{H}(n,d)$ *optimal* if, for every $f \in \mathcal{H}(n,d)$, we have $N(f) \ge N(p)$. By Theorem 0 we know that, for d odd, $p \in \mathcal{H}(2,d)$ is optimal if and only if $d = 2N(p)-3$. The polynomials in (7) are optimal. We hope to prove when $n \geq 3$ that $p \in \mathcal{H}(n)$ is optimal if $N(p) = (n-1)d(p) + 1$. We can easily exhibit polynomials in $\mathcal{H}(n, d)$ for $n \geq 3$ satisfying this equality.

EXAMPLE 4. Let $s'(x) = \sum_{j=1}^{n-1} x_j$. We define g_d by

$$g_d(x) = x_n^d + s'(x) \sum_{k=0}^{d-1} x_n^k. \tag{30}$$

It is evident from (30) and the finite geometric series that $g_d \in \mathcal{W}$ and $N(g_d) = (n-1)d + 1$.

REMARK. For a given n and d there are only finitely many optimal examples, but typically there is more than one. When $n = 2$, for example, the first author has shown the following: There are infinitely many d for which there exist optimal examples other than those given in (7) and those obtained by interchanging the roles of x and y. We omit the proof here. Example 3 gives such an optimal polynomial of degree 7.

As mentioned before, it is possible to improve Proposition 4 by pulling back to the optimal examples in two dimensions. We illustrate by establishing the next two lemmas.

LEMMA 4. *Suppose $n \geq 2$ and $p \in \mathcal{H}(n, d)$. If p contains a monomial in one or two variables of degree d, then*

$$d \leq \frac{2N - 3}{2n - 3}. \tag{31}$$

Proof. After renumbering we may assume that p contains either x_1^d or $x_1^a x_2^b$, where $a + b = d$. Set $D = 2n - 3$. We pull back using the optimal map ϕ induced by p_D as defined in (7). Order the variables such that $x_1 = u^D$ and $x_2 = v^D$; in either case, we are guaranteed a term in $\phi^*(p)$ of degree Dd. Following reasoning similar to the proof of Proposition 4 we obtain

$$d(p) = \frac{d(\phi^*(p))}{D} \leq \frac{2N(\phi^*(p)) - 3}{D} \leq \frac{2N(p) - 3}{D} = \frac{2N - 3}{2n - 3}, \tag{32}$$

which gives (31). $\square$

By assuming that the highest-degree part of p contains monomials involving few of the variables, we can generalize the preceding proof. We give two of several possible versions.

LEMMA 5. *Suppose $n \geq 2$ and $p \in \mathcal{H}(n, d)$. If p contains the monomial $m = x_1^{a_1} \cdots x_k^{a_k}$ of degree d, where $k \geq 2$, then the following inequalities hold:*

$$d(p) \leq \frac{2N - 2k + 1}{2n - 2k + 1}; \tag{33}$$

$$d(p) \leq \frac{2N - 3 + \sum_{j=3}^{k} (j-2) a_j}{2n - 3}. \tag{34}$$

Proof. First we prove (33). We set $x_j = \frac{\lambda}{k-1}$ for $2 \leq j \leq k$. In doing so we replace $k-1$ terms with one term, thus killing $k-2$ terms. We also decrease the number of variables by $k-2$. We now pull back as in the proof of Lemma 4 (or use Lemma 4 directly) to see that

$$d(p) \leq \frac{2(N-(k-2))-3}{2(n-(k-2))-3} = \frac{2N-2k+1}{2n-2k+1}.$$

Thus we have proved (33).

The proof of (34) also involves pulling back to the optimal polynomials in two dimensions. We first set $D = 2n-3$ and then consider the mapping ϕ induced by p_D as defined in (7), where the coordinates are ordered such that

$$(x_1, x_2, x_3, x_4, \dots) = (u^D, v^D, c_1 u^{D-2} v, c_2 u^{D-4} v^2, \dots) = \phi(u,v).$$

Pulling back the monomial m then guarantees a term of degree

$$a_1 D + a_2 D + a_3(D-1) + \cdots + a_k(D-k+2) = D\sum_{j=1}^{k} a_j - \sum_{j=3}^{k}(j-2)a_j$$

in $\phi^*(p)$. Since the sum of the a_j is d, we obtain

$$dD - \sum_{j=3}^{k}(j-2)a_j \leq d(\phi^*(p)) \leq 2N(\phi^*(p)) - 3 \leq 2N(p) - 3 \tag{35}$$

and hence

$$d(p) = d \leq \frac{2N(p) - 3 + \sum_{j=3}^{k}(j-2)a_j}{D} = \frac{2N(p) - 3 + \sum_{j=3}^{k}(j-2)a_j}{2n-3}. \tag{36}$$

This proves (34) and hence the lemma. □

The proof of (34) when $k = 2$ is essentially the same as the proof of Lemma 4. The proof of (34) gives the strongest result by taking D as large as possible; $D = 2n-3$ is the largest number for which ϕ takes values in n-space, as required if the proof is to make sense. Thus the choice of D itself relies on Theorem 0.

Let us write $E = \sum_{j=3}^{k}(j-2)a_j$. Our next result provides a general bound for $d(p)$ in terms of $N(p)$ in all cases. We do so by estimating the *excess* E in terms of d and n. From Theorem 1 we obtain the weaker asymptotic bound

$$d(p) \leq \frac{4}{3}\frac{2N(p)-3}{2n-3}$$

as $n \to \infty$. Our main result, Theorem 2, provides the sharp asymptotic result $d \leq \frac{N-1}{n-1}$ when n is large relative to d. On the other hand, Theorem 1 holds for all n; its proof is much simpler, but the result is sharp in two dimensions only.

THEOREM 1. *Suppose* $p \in \mathcal{H}(n,d)$. *Then*

$$d(p) \leq \frac{2n(2N(p)-3)}{3n^2-3n-2} \leq \frac{4}{3}\frac{2N(p)-3}{2n-3}. \tag{37}$$

Proof. We begin with the estimate

$$d(p) \leq \frac{2N(p) - 3 + \sum_{j=3}^{k}(j-2)a_j}{2n-3} \tag{38}$$

from Lemma 5. For notational ease we rewrite (36) as

$$d(p) \leq F + \frac{E}{D}, \tag{39}$$

where $F = \frac{2N-3}{2n-3}$. We may assume $k \geq 2$ and that $a_1 \geq a_2 \geq \cdots \geq a_k$. Then

$$\frac{E}{D} = \frac{\sum_{j=3}^{k}(j-2)a_j}{D} \leq \frac{d}{Dk}\sum_{j=3}^{k}(j-2) = \frac{d}{Dk}\binom{k-1}{2}. \tag{40}$$

Since $k \leq n$, we obtain from (40) the upper estimate

$$\frac{E}{D} \leq \frac{d}{nD}\binom{n-1}{2} = c(n)d, \tag{41}$$

where the expression $c(n)$ is defined by

$$c(n) = \frac{\binom{n-1}{2}}{n(2n-3)}. \tag{42}$$

One easily shows that $c(n) < 1$. Therefore, (39) yields

$$d(p) \leq F + \frac{E}{D} \leq F + c(n)d(p)$$

and hence

$$d(p) \leq \frac{1}{1-c(n)}F = \frac{2N-3}{2n-3}\frac{1}{1-c(n)} = \frac{2n(2N(p)-3)}{3n^2-3n-2}. \tag{43}$$

We have bounded d in terms of N and n. It is elementary to verify for $n \geq 2$ that

$$\frac{2n}{3n^2-3n-2} \leq \frac{4}{3(2n-3)};$$

as a result, the inequality on the far RHS of (37) holds. □

We pause to give the following explicit optimal example:

$$p(x,y,z) = x + y + z^2 + xz + y^2z + yz^2 + xyz(x+y+z). \tag{44}$$

The polynomial in (44) is of degree 4, but each term of degree 4 involves all three of the variables and thus Lemma 4 is not useful. Note that $N(p) = 9$. By Proposition 5, nine is the smallest possible number of terms for an element in $\mathcal{H}(3,4)$.

Before turning to Proposition 5, which verifies conjecture (3) of Problem 1 for degree up to 4, we briefly discuss one-parameter families of mappings. The following proposition will be proved and developed in [L]. A one-parameter family of polynomials is defined by

$$p_\lambda(x) = \sum c_\alpha(\lambda)x^\alpha, \tag{45}$$

where each map $\lambda \to c_\alpha(\lambda)$ is a continuous function of a real parameter λ. One simple example of a one-parameter family is given by the convex combination $f_\lambda = \lambda p + (1-\lambda)q$ of elements p and q of $\mathcal{H}(n,d)$. We observed earlier that $f_\lambda \in \mathcal{H}(n,d)$ as well.

PROPOSITION L. *Let p_t denote a one-parameter family of elements of $\mathcal{H}(n,d)$. Suppose that $N(p_t)$ is constant for t in an open interval. Then p_t is optimal for no t.*

We next include some information that supports the conjectured sharp bound. The proofs of the four statements in the following result become increasingly elaborate as the codimension increases. Hence we provide detailed proofs of statements (0), (1), and (2) but only an outline of the proof of (3). The proofs of (0) and (1) are easy; the proofs of (2) and (3) first use combinatorial reasoning to make Lemma 4 applicable and then use additional combinatorial reasoning to improve the bound from Lemma 4 in these special cases. The bounds in this result are interesting in the context of CR mappings between spheres.

PROPOSITION 5. *Suppose $p \in \mathcal{H}(n,d)$ for $n \geq 3$. Then the following statements hold.*

(0) *If $N(p) < n$, then $d = 0$.*
(1) *If $N(p) < 2n-1$, then $d \leq 1$.*
(2) *If $N(p) < 3n-2$, then $d \leq 2$.*
(3) *If $N(p) < 4n-3$, then $d \leq 3$.*

Proof. The contrapositive of (0) is easy. If $d \geq 1$ then, by Corollary 1, there must be at least n distinct monomials of degree d.

We call terms of the form x_i^k *pure* terms, and we call monomials depending on at least 2 variables *mixed* terms. Pulling back to the one-dimensional case in n ways (by setting $n-1$ of the variables equal to zero), we note that there must be at least n distinct pure terms. If $d = 1$ then all the terms are pure terms and $p = s$. We may therefore assume that $d \geq 2$ while proving the rest of the statements.

The proof of (1) proceeds as follows. If no pure term is of degree at least 2 then, as before, $p = s$. We may thus assume that the monomial x_1^a occurs for some $a \geq 2$. By setting all variables except x_1 and x_j equal to 0, we see that a mixed monomial $x_1^k x_j^l$ must occur for $2 \leq j \leq n$. Hence we have at least $n-1$ mixed terms. Counting also the n pure terms shows that $N(p) \geq (n-1)+n$, and this yields (1).

If $d = 2$ then (2) holds; we therefore assume $d \geq 3$ when proving (2). We must show that $N \geq 3n-2$. There are two cases as follows. First, if x_1^a is the only pure term of degree > 1 then p must be equal to $x_1 r(x) + s - x_1$ for some $r(x) \in \mathcal{H}(n,d)$. The polynomial r has $n-1$ fewer terms than p does, and it must have degree ≥ 2. Applying (1) shows that $N(r) \geq 2n-1$ and hence $N(p) \geq (2n-1)+(n-1) = 3n-2$. Thus (2) holds in this case.

The second case is when at least two pure terms of degree ≥ 2 occur. Hence we assume that x_2^b occurs as well with $b \geq 2$. We then have at least $2(n-2)+1$

mixed terms and n pure terms for a total of $3n-3$. We want $N \geq 3n-2$, so assume (by way of contradiction) that there are no other terms. For $d \geq 3$ the only element of $\mathcal{H}(2,d)$ that has at most three distinct monomials is $u^3+3uv+v^3$. Hence all pure terms must be of degree 3, and we obtain

$$p(x) = \sum_{j=1}^{n} x_j^3 + 3\sum_{i \neq j} x_j x_i. \tag{46}$$

We claim that the polynomial in (46) is not in $\mathcal{H}(n,3)$ unless $n=2$. To verify the claim we note that $p\left(\frac{1}{n},\ldots,\frac{1}{n}\right) > 1$ when $n \geq 3$. Thus (2) holds in this case and hence it holds in general.

To prove (3) we assume that $N \leq 4n-4$. If Lemma 4 does not apply, then there is no term of degree d involving at most two of the variables. Hence we must have at least n terms of top degree, n additional pure terms, and (as before) at least $2n-3$ additional mixed terms involving two variables. The total is $4n-3$ and thus $N \geq 4n-3$. We may therefore assume Lemma 4 applies; in particular, $d \leq 4$.

We proceed by contradiction. Assume $d=4$. We consider the cases $N \leq 4n-5$ and $N = 4n-4$ separately. If $N \leq 4n-5$, we obtain a contradiction as follows. By Lemma 4,

$$d(2n-3)+3 \leq 2N.$$

Including the information on N and d yields

$$4(2n-3)+3 \leq d(2n-3)+3 \leq 2N \leq 2(4n-5),$$

from which we obtain the contradiction $-9 \leq -10$. Thus, for $N \leq 4n-5$ we have $d \leq 3$.

The remaining case is when $N(p) = 4n-4$ and $d=4$. There are two subcases. First suppose that $n \geq 4$. As argued previously, we can assume that there exist pure monomials in x_1 and x_2 of degree >1. Setting in turn $x_1 = 0$ and $x_2 = 0$ yields polynomials in $n-1$ variables with at least n fewer terms. Hence these polynomials must have degree ≤ 3. The top-degree terms must be divisible by x_1x_2 and so $p_4 = s(x)x_1x_2q(x)$, where q is homogeneous of degree 1. We can easily check that q must have all positive coefficients, and we can undo an operation X to reduce to a previous case.

The other subcase is when $n=3$, $N(p) = 4n-4 = 8$, and $d=4$. We claim that no polynomial in $\mathcal{H}(3,4)$ has exactly eight distinct monomials. There are only finitely many possibilities that need to be checked, and we outline how to do this by hand.

If all terms of degree 4 depend on three variables, then we undo and reduce to a previous case to obtain a contradiction. After renaming variables, we consider the polynomials $p(x_1,x_2,0)$, $p(x_1,0,x_2)$, and $p(0,x_2,x_3)$. A counting argument shows that the first two of these must have exactly four terms and be of degree 4, whereas the third must have three terms and be of degree ≤ 3. By a study of the two-dimensional case we see that x_1^4 must appear. One can then check by hand that the only possible configuration of degree-4 terms is $x_1^3(x_1+x_2+x_3)$, and reducing to a previous case then produces a contradiction. $\square$

The following corollary supports the conjectured sharp bound for degree ≤ 4. We believe that these bounds are sharp for all degrees when $n \ge 3$. In the next section we establish this result when n is large enough compared with d.

COROLLARY 2. *Suppose $n \ge 3$ and $p \in \mathcal{H}(n,d)$. If $d \le 4$ or $N(p) < 4n-3$, then the following two estimates hold*:

$$N(p) \ge d(n-1)+1;$$
$$d \le \frac{N(p)-1}{n-1}. \tag{47}$$

V. Whitney Mappings and the Proof of Theorem 2

In this section we give conditions under which a polynomial $p \in \mathcal{H}(n,d)$ lies in $\mathcal{W}$. By Lemma 2, if $p \in \mathcal{W} \cap \mathcal{H}(n,d)$ then the desired bound $N(p) \ge d(n-1)+1$ holds.

The following theorem is the main result of this paper. It solves Problem 1 when the domain dimension is large enough.

THEOREM 2. *Fix d and assume that $n \ge 2d^2+2d$. If $p \in \mathcal{H}(n,d)$ then $N(p) \ge (n-1)d+1$. Furthermore, if equality holds then $p \in \mathcal{W}$.*

Before proving Theorem 2 we give a simple condition guaranteeing that $p \in \mathcal{W}$. Let $x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ and define $s'(x') := \sum_{j=1}^{n-1} x_j$. We will say that p is *affine in x_n* if we can write $p(x', x_n) = a(x') + x_n b(x')$ for some polynomials a and b.

LEMMA 6. *Suppose $p \in \mathcal{H}(n,d)$ and that p is affine in x_n. Then $p \in \mathcal{W}$.*

Proof. We induct on the degree d. When $d = 1$ the result is obvious. Suppose $d \ge 2$ and that the result is known for such affine polynomials of degree $d-1$. Assume $p(x', x_n) = a(x') + x_n b(x')$. By (18) we may write $p = (p - p_d) + s r_{d-1}$. Equating the highest part of these expressions for p gives

$$\begin{aligned} a_d(x') + x_n b_{d-1}(x') &= \left(\sum_{j=1}^{n-1} x_j + x_n\right) r_{d-1}(x') \\ &= s'(x') r_{d-1}(x') + x_n r_{d-1}(x'). \end{aligned} \tag{48}$$

Hence $r_{d-1} = b_{d-1}$ and $a_d = s' r_{d-1}$. Therefore,

$$p = p - p_d + s b_{d-1} = X(p - p_d + b_{d-1}). \tag{49}$$

Note that $p - p_d + b_{d-1} \in \mathcal{H}(n, d-1)$. It is also affine in x_n and hence lies in $\mathcal{W}$ by the induction hypothesis. Thus $p \in \mathcal{W}$ as well. $\square$

We now prove two simple results that we use in the proof of Theorem 2. The reader may wish to refer to Examples 1 and 4.

LEMMA 7. *Let $p \in \mathcal{H}(2,d)$ and suppose that $p(x,y) = a(x) + yb(x)$. Then $N(p) \geq d+1$. The monomial x^d must appear and $x^j y$ must appear for each j with $0 \leq j \leq d-1$. Furthermore, p has exactly $d+1$ distinct monomials if and only if*

$$p(x,y) = x^d + y(x^{d-1} + \cdots + x + 1).$$

Proof. By Lemma 6 we know that $p \in \mathcal{W}$, and the statement follows by induction on d. □

We define the distance between two monomials $m_1 = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $m_2 = x_1^{\beta_1} \cdots x_n^{\beta_n}$ by

$$\delta(m_1, m_2) := \sum_j |\alpha_j - \beta_j|.$$

For monomials of the same degree, $\delta(m_1, m_2)$ must be even.

LEMMA 8. *Let $p \in \mathcal{H}(3,d)$ and suppose that $p(x_1,x_2,x_3) = a(x_1,x_2) + x_3 b(x_1,x_2)$. If two monomials $m_1(x_1,x_2)$ and $m_2(x_1,x_2)$ of degree d occur in $p(x)$ with $\delta(m_1,m_2) \geq 4$, then p has at least $d+1$ distinct monomials that depend on x_3.*

Proof. It follows from Lemma 6 that $p \in \mathcal{W}$ and from Lemma 7 that p must have at least one monomial of every degree that depends on x_3. Since $\delta(m_1,m_2) \geq 4$, there must be at least two monomials of maximal degree that depend on x_3, which gives at least $d+1$ monomials. □

For the rest of this section we assume that $n \geq 2d^2 + 2d$; in particular, $n \geq 3$. Let $p \in \mathcal{H}(n,d)$ and $N = N(p)$. We assume both that $N \leq d(n-1)+1$ and that p is optimal. We will show that p must be a generalized Whitney mapping and thereby prove Theorem 2.

Let m_1 and m_2 be distinct monomials that occur in p. The main idea of the proof is to show that $\delta(m_1,m_2)$ must be equal to 2. Let k be the number of distinct variables that occur in either m_1 or m_2. Then $2 \leq k \leq 2d$. After renaming the variables if necessary, we may assume that m_1 and m_2 are independent of x_j for $j \geq k+1$.

We define new polynomials in $\mathcal{H}(2,d)$ and $\mathcal{H}(3,d)$:

$$\begin{aligned} P_j(\xi, x_j) &:= p\Big(\underbrace{\frac{\xi}{k}, \ldots, \frac{\xi}{k}}_{k \text{ times}}, 0, \ldots, 0, x_j, 0, \ldots\Big); \\ P_{ij}(\xi, x_i, x_j) &:= p\Big(\underbrace{\frac{\xi}{k}, \ldots, \frac{\xi}{k}}_{k \text{ times}}, 0, \ldots, 0, x_i, 0, \ldots, 0, x_j, 0, \ldots\Big). \end{aligned} \tag{50}$$

CLAIM. *The polynomial P_j is affine in x_j for each $j \in \{k+1, \ldots, n\}$.*

Proof. Seeking a contradiction, we assume that $k+1 \leq l \leq n$, that P_j is not affine for $k+1 \leq j \leq l$, and that P_j is affine for $l+1 \leq j \leq n$.

If P_j is affine in x_j then by Lemma 6 we have

$$P_j(\xi, x_j) = c_1\xi^d + c_2\xi^{d-1}x_j + \cdots + c_d\xi x_j + c_{d+1}x_j + q(\xi),$$

where q is a possibly zero polynomial in ξ of degree $d-1$ or less. If P_j is not affine in x_j then by Theorem 0 there must be at least $\lceil \frac{d-3}{2} \rceil$ terms.

We will proceed to find a lower estimate for the number of monomials of p, taking care not to count the same monomial twice. We first count the monomial m. For each P_j with $k+1 \leq j \leq l$ we have at least $\lceil \frac{d+3}{2} \rceil - 1$ extra monomials, and for each P_j with $1 < j$ we have at least d extra monomials.

For P_{ij} with $k+1 \leq i < j \leq l$, we know that there must be at least one monomial that depends on x_i as well as x_j (keep ξ constant to see this); hence we have at least $(l-k)(l-k-1)/2$ more monomials that we have not already counted. For the same reason we can count one extra monomial depending on both x_i and x_j for each possible choice $k+1 \leq i \leq l < j \leq n$, and this yields $(l-k)(n-l)$ more monomials.

When we add the number of all these monomials, the result is

$$N \geq 1 + (l-k)\left(\left\lceil \frac{d+3}{2} \right\rceil - 1 + \frac{l-k-1}{2} + (n-l)\right) + (n-l)d. \tag{51}$$

By assumption, $l \geq k+1$. If

$$(l-k)\left(\left\lceil \frac{d+3}{2} \right\rceil - 1 + \frac{l-k-1}{2} + (n-l)\right) > (l-1)d, \tag{52}$$

then p cannot be optimal; this happens when

$$(l-k)(d-l-k+2n) - 2(l-1)d > 0. \tag{53}$$

Once we fix k, d, n, the expression in (53) is concave down in l and therefore must achieve a minimum if $l = k+1$ or $l = n$. We know that $2 \leq k \leq 2d$ and so obtain two bounds for n:

$$\begin{aligned} n &> \frac{4d^2 + 3d + 1}{2}; \\ n &> 5d. \end{aligned} \tag{54}$$

Our assumption that $n \geq 2d^2 + 2d$ implies both bounds (noting that $d \geq 2$). We have thus proved the Claim. □

Proof of Theorem 2. Now suppose by way of contradiction that $\delta(m_1, m_2)$ is at least 4. Write $m_1 = \prod_{i=1}^k x_i^{r_i}$ and $m_2 = \prod_{i=1}^k x_i^{s_i}$. Renaming the variables again if necessary, we may assume there exists an integer t such that $r_i \geq s_i$ for $i = 1, \dots, t$ and $r_i \leq s_i$ for $i = t+1, \dots, k$. It follows from the Claim that, for $j = k+1, \dots, n$, the polynomial P_j (as defined in (50)) must be affine in x_j.

Let

$$P(y, z, x_{k+1}, \dots, x_n) := p\Big(\underbrace{\frac{y}{t}, \dots, \frac{y}{t}}_{t \text{ times}}, \underbrace{\frac{z}{k-t}, \dots, \frac{z}{k-t}}_{k-t \text{ times}}, x_{k+1}, \dots, x_n\Big).$$

It follows that P has two terms of highest degree $y^{r_1}z^{r_2}$ and $y^{s_1}z^{s_2}$ with $r_1 > s_1 + 1$ and $r_2 < s_2 - 1$. Hence, for every $j \in \{k+1, \dots, n\}$, the polynomial $P(y, z, 0, \dots, 0, x_j, 0, \dots, 0)$ is a polynomial in three variables that satisfies the conditions of Lemma 8 and so has at least $d+1$ terms that depend on x_j. As a result, P (and thus also p) has at least $(d+1)(n-2d) = dn + n - 2d^2 - 2d$ distinct monomials. We assumed that $n \geq 2d^2 + 2d$ and so the polynomial cannot be optimal, which contradicts our assumption. Therefore, $\delta(m_1, m_2) = 2$.

By Corollary 1 there exist at least n terms of highest degree. It follows that the terms of highest degree must equal $cs \cdot m$ for some constant c and some monomial m of degree $d-1$. (Recall that s denotes the sum of the variables.) Thus we can undo the operation X to obtain a new polynomial of degree $d-1$ with exactly $n-1$ terms fewer than p. The reason is that p is optimal; undoing the operation X must create a new term of degree $d-1$ (otherwise, multiplying that term by s would yield a polynomial with fewer terms than p). This new polynomial of degree $d-1$ must again be optimal, because if there existed a polynomial of degree $d-1$ with fewer terms then we could apply operation X to it and again invalidate the optimality of p.

An inductive argument with respect to the degree shows that p must be obtained by starting with s and repeatedly multiplying one of the highest-degree terms with s—in other words, $p \in \mathcal{W}$. This completes the proof of Theorem 2. □

VI. CR Mappings between Spheres

The results of this paper are closely related to a basic question in CR geometry. Let f be a rational mapping from complex Euclidean space $\mathbf{C}^n$ to $\mathbf{C}^N$, and suppose that f maps the unit sphere S^{2n-1} in its domain to the unit sphere S^{2N-1}. Can we give any estimate for the degree of f in terms of n and N? The degree of a rational map $f = p/q$ is defined to be the maximum of the degrees of p and q when f is reduced to lowest terms. It is easy to show in this context (see [D2]) that the degree of f equals the degree of p.

Many of the results described in this section do not begin by assuming that f is rational. Instead, they assume that f is a proper mapping between balls and then make some regularity assumptions at the boundary in the case of positive codimension. By the work of Forstneric [F1; F2], a proper mapping between balls—with domain dimension at least 2 and with sufficient differentiability at the boundary—must be a rational mapping. We therefore assume rationality in this section.

We return to the basic question of estimating the degree. As in this paper, if $n = 1$ then no estimate is possible. Assume next that $n \geq 2$. As in Proposition 5, when $N < n$ we can conclude by elementary considerations that f must be a constant. For $N = n \geq 2$, Pincuk [P] proved that f must be either a constant or a linear fractional transformation and hence must be of degree ≤ 1. Faran [Fa2] showed that we can draw the same conclusion when $n \leq N \leq 2n-2$. When $n = 2$ and $N = 2n - 1 = 3$, Faran [Fa1] showed that, up to composition with automorphisms of the ball on both sides, the map must be a monomial mapping of degree ≤ 3. Hence the rational mapping is of degree ≤ 3 in this case. In particular, Faran discovered the mapping $(u^3, \sqrt{3}uv, v^3)$, which is of maximum degree from the two-ball to the three-ball and is also group-invariant. In [D2; D3; D5] the first author studied the group-invariance aspects of CR mappings, discovered the maps (7), and observed many connections to other branches of mathematics.

Huang and Ji [H; HJ] have investigated aspects of the basic question. For example, they have established that, when $3 \leq n \leq N = 2n - 1$, the degree of a rational mapping between spheres is at most 2; they have also discovered various conditions somewhat analogous to those established here for guaranteeing partial linearity. One striking aspect of their work is that it does not assume rationality and the regularity assumptions are minimal. All these papers address low codimension. Meylan's result [M] gives the bound $d \leq \frac{N(N-1)}{2}$ in any codimension for which the domain dimension n is assumed to be 2. The paper [HJX] includes the following result: Let f be a rational proper mapping between balls of degree 2; if f has *geometric degree* 1, then f is a generalized Whitney map.

The expository paper [D4] includes the relationship of this complexity issue to a complex variables analogue of Hilbert's 17th Problem and includes the following result. Given a rational mapping $p/q \colon \mathbf{C}^n \to \mathbf{C}^N$ that maps the closed unit ball into the open unit ball, we can find an integer K and another rational mapping $g/q \colon \mathbf{C}^n \to \mathbf{C}^K$ (with the same denominator) such that the mapping $(p/q, g/q)$ maps S^{2n-1} to $S^{2(N+K)-1}$. We must be able to choose K large enough, and even for quadratic mappings and $n = 2$ we must choose K to be arbitrarily large. Thus, by placing no restriction on the target dimension, we can create arbitrarily complicated rational mappings between spheres. In future work we will show how the bounds in this paper, which arise by considering monomial rather than rational maps, can to some extent be extended to the rational case.

The first author has conjectured that the degree of a rational mapping sending S^{2n-1} to S^{2N-1} is at most $\frac{N-1}{n-1}$ when $n \geq 3$ and at most $2N - 3$ when $n = 2$. The results in this paper show how to obtain sharp results in the special but nontrivial case where the map is a monomial.

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