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Approximation of Plurisubharmonic Functions on Bounded Domains in $\mathbf{C}^n$

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1. Introduction

Let Ω be a domain in $\mathbf{C}^n$. An upper semicontinuous function $u\colon \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if the restriction of u to each complex line is subharmonic (we allow the function identically $-\infty$ to be plurisubharmonic). We write $\mathcal{PSH}(\Omega)$ for the set of plurisubharmonic functions on Ω, $\mathcal{PSH}^-(\Omega)$ for the set of plurisubharmonic functions bounded from above on Ω, and $\mathcal{PSH}(\bar\Omega)$ for the set of plurisubharmonic functions on neighborhoods of $\bar\Omega$. If u is a function bounded from above on Ω, then by u^* we mean the upper regularization of u; that is, if $z \in \bar\Omega$ then

$$u^*(z) = \limsup_{\xi \to z} u(\xi).$$

For a given point $z \in \bar\Omega$, we define the following class of Jensen measures:

$$J_z^1(\bar\Omega) = \left\{\mu \in \mathcal{B}(\bar\Omega) : u^*(z) \le \int_{\bar\Omega} u^*\, d\mu \ \forall u \in \mathcal{PSH}^-(\Omega)\right\},$$

where $\mathcal{B}(\bar\Omega)$ is the set of positive regular Borel measures with mass 1 on $\bar\Omega$. We can define J_z^2 and J_z^3 analogously when $\mathcal{PSH}^-(\Omega)$ is replaced by $\mathcal{PSH}^c(\Omega)$ (the set of plurisubharmonic functions on Ω, continuous on $\bar\Omega$) and $\mathcal{PSH}^c(\bar\Omega)$ (the set of continuous functions on $\bar\Omega$ that are uniform limits of continuous functions in $\mathcal{PSH}(\bar\Omega)$), respectively. For simplicity of notation, we will write J_z^i instead of $J_z^i(\bar\Omega)$ if there is no risk of confusion. It is obvious that $\delta_z \in J_z^1 \subset J_z^2 \subset J_z^3$, where δ_z is the Dirac measure at z. With a little more effort, one can prove that each J_z^i is a closed convex subset of $\mathcal{B}(\bar\Omega)$. We say that Ω is *J-regular* if $J_z^1 = J_z^3$ for all $z \in \Omega$. The classes J_z^1, J_z^2, J_z^3 are introduced and studied extensively in [CCeW; DW; P; S2; W1; W2] and elsewhere. The main reason for introducing them is a duality theorem of Edwards that allows us to express upper envelopes of plurisubharmonic functions as lower envelopes of integrals with respect to Jensen measures. Since the traditional method of constructing plurisubharmonic functions has been to take envelopes over classes of plurisubharmonic functions, Edwards's duality theorem provides alternative ways of investigating these constructions. As an illustration of this idea, we prove in [DW] that, for every bounded domain Ω in $\mathbf{C}^n$: (i) if $J_z^1 = J_z^2$ for all $z \in \Omega$ then every $u \in \mathcal{PSH}^-(\Omega)$ is the pointwise limit

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of a sequence $\{u_j\}_{j\geq 1}$ with $u_j \in \mathcal{PSH}^c(\Omega)$; and (ii) $\limsup_{j\to\infty} u_j \leq u^*$ on $\partial\Omega$. There are some drawbacks to this result. First, the regularity of u_j is not known beyond continuity (on $\bar{\Omega}$); second, we do not know whether the approximating sequence can be chosen to be decreasing. Notice that, if Ω is B-regular, then (by [W2, Thm. 4.1]) u^* can be approximated on $\bar{\Omega}$ from above by a decreasing sequence $\{u_j\}$ in $\mathcal{PSH}^c(\Omega)$. In this case, it is also an open problem whether $\{u_j\}$ can be chosen such that the Monge–Ampère mass of u_j on Ω is finite—in other words, such that $\int_\Omega (dd^c u_j)^n < \infty$ for all j. Recall that it is always possible to do so if we do not require $u_j \downarrow u^*$ on $\partial\Omega$ (see [Ce, Thm. 2.1]).

The main result of this paper is Theorem 3.2, where we show that if Ω is J-regular then every $u \in \mathcal{PSH}^-(\Omega)$ is the pointwise limit of a sequence u_j, where the u_j are smooth plurisubharmonic functions on neighborhoods of $\bar{\Omega}$. Moreover, the sequence is eventually decreasing on each compact subset of Ω. One novelty in this result is that, if u is locally bounded, then the approximating sequence u_j can be chosen to have the additional property that $(dd^c u_j)^n$ converges to $(dd^c u)^n$ in the sense of currents. The other main result of the paper is Theorem 4.4, which gives a geometric sufficient condition for the J-regularity of Ω. Roughly speaking, if we can cover the "bad" part of $\partial\Omega$ by biholomorphic images of Ω then Ω is indeed J-regular. This result is similar to Theorem 3.5 in [DW]. Nevertheless, in the proof we rely mainly on Edwards's duality theorem instead of on the gluing technique of plurisubharmonic functions introduced by Cegrell in Theorem 2.1 of [Ce], as we did in the proof of Theorem 3.5 in [DW]. We conclude the paper by giving a list of questions connected to our work.

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2. Background

Unless otherwise stated, by Ω we always mean a bounded domain in $\mathbf{C}^n$. If φ is an arbitrary function on $\bar{\Omega}$ then we define three kinds of envelopes for φ. More precisely, if $z \in \bar{\Omega}$ then we let

$$S_1\varphi = \sup\{u : u \in \mathcal{PSH}(\Omega) \cap \mathrm{USC}(\bar{\Omega}),\ u \leq \varphi \text{ on } \bar{\Omega}\}, \tag{1}$$

$$S_2\varphi = \sup\{u : u \in \mathcal{PSH}^c(\Omega),\ u \leq \varphi \text{ on } \bar{\Omega}\}, \tag{2}$$

$$S_3\varphi = \sup\{u : u \in \mathcal{PSH}^c(\bar{\Omega}),\ u \leq \varphi \text{ on } \bar{\Omega}\}, \tag{3}$$

where $\mathrm{USC}(\bar{\Omega})$ denotes the set of upper semicontinuous functions on $\bar{\Omega}$. The following result, essentially due to Wikström (see [W2, Cor. 2.2]), is a straightforward consequence of a general duality theorem of Edwards [E]. This reveals the connections between the aforementioned envelopes and the Jensen measures introduced at the beginning of the paper.

PROPOSITION 2.1. *Let* $\varphi\colon \bar\Omega \to [-\infty,\infty]$ *be a lower semicontinuous function. Then*

$$S_1\varphi(z) = \inf\left\{\int_{\bar\Omega}\varphi\,d\mu : \mu\in J_z^1\right\} \quad \forall z\in\Omega,$$

$$S_2\varphi(z) = \inf\left\{\int_{\bar\Omega}\varphi\,d\mu : \mu\in J_z^2\right\} \quad \forall z\in\bar\Omega,$$

$$S_3\varphi(z) = \inf\left\{\int_{\bar\Omega}\varphi\,d\mu : \mu\in J_z^3\right\} \quad \forall z\in\bar\Omega.$$

It is pointed out in [DW] that there is a minor gap in the statement of Corollary 2.2 in [W2], since the set $\{u^* : u\in\mathcal{PSH}^-(\Omega)\}$ considered there is *not* a convex cone. (In general, $(u+v)^*$ is strictly smaller than u^*+v^* on $\partial\Omega$.) So the first identity claimed in that corollary holds only for interior points, and this result is obtained by looking at the set $\mathcal{PSH}(\Omega)\cap\mathrm{USC}(\bar\Omega)$, which is truly a cone. We will also need to recall some elements of pluripotential theory. A set $P\in\mathbf{C}^n$ is called *pluripolar* if, for every $z\in P$, there exist a neighborhood U of z and a $u\in\mathcal{PSH}(U)$ such that $u\not\equiv-\infty$ on every connected component of U and $u\equiv-\infty$ on $U\cap P$. A set $P\subset\Omega$ is called *complete pluripolar* in Ω if there exists a $u\in\mathcal{PSH}(\Omega)$, $u\not\equiv-\infty$, such that $P=\{z\in\Omega : u(z)=-\infty\}$. A deep theorem of Josefson (see [K, Thm. 4.7.4]) shows that u can be chosen to be plurisubharmonic on $\mathbf{C}^n$. In particular, u can be made to be negative on any fixed *bounded* neighborhood of P. One of basic tools in pluripotential theory is the complex Monge–Ampère operator $(dd^c)^n$, an analogue of the classical Laplacian. According to the fundamental work of Bedford and Taylor in [BTa] (see also [K, Chap. 3]), the operator $(dd^c)^n$ is well-defined (in the sense of currents) over the class $L^\infty_{\mathrm{loc}}(\Omega)\cap\mathcal{PSH}(\Omega)$ of locally bounded plurisubharmonic functions. More precisely, if $u\in L^\infty_{\mathrm{loc}}(\Omega)\cap\mathcal{PSH}(\Omega)$ then $(dd^cu)^n$ is a positive Borel measure. Furthermore, $(dd^c)^n$ is continuous under monotone sequences in this class (see [Ce] for a more recent account of these matters). Recall also that, following Sibony [S2], a bounded domain Ω is called *B-regular* if every real-valued continuous function on $\partial\Omega$ can be extended continuously to a plurisubharmonic function on Ω. It is well known that every smoothly bounded strictly pseudoconvex domain is B-regular. There is a characterization of B-regularity in terms of Jensen measures: Ω is B-regular if and only if $J_z^2=\{\delta_z\}$ for every $z\in\partial\Omega$ (see [S2, Thm. 2.1; W2, Cor. 3.8]). We now collect some useful facts about the envelopes introduced in (1)–(3).

LEMMA 2.2. (a) *If* $\varphi\in\mathcal{C}(\bar\Omega)$ *then* $S_1\varphi\in\mathcal{PSH}^-(\Omega)$.

(b) *Let* $z_0\in\Omega$ *and* $i,j\in\{1,2,3\}$; *then* $J^i_{z_0}=J^j_{z_0}$ *if and only if* $S_i\varphi(z_0)=S_j\varphi(z_0)$ *for all* $\varphi\in\mathcal{C}(\bar\Omega)$.

Proof. (a) Since $S_1\varphi\le\varphi$ on $\bar\Omega$ and $\varphi\in\mathcal{C}(\bar\Omega)$, it follows that $(S_1\varphi)^*\le\varphi$ on $\bar\Omega$. Hence $(S_1\varphi)^*\in\mathcal{PSH}^-(\Omega)$ and so $S_1\varphi\equiv(S_1\varphi)^*\in\mathcal{PSH}^-(\Omega)$.

(b) If $J^i_{z_0} = J^j_{z_0}$ then, using Proposition 2.1, we deduce that $S_i\varphi(z_0) = S_j\varphi(z_0)$ for all $\varphi \in \mathcal{C}(\bar\Omega)$. Conversely, if there is some $\mu \in J^i_{z_0}\setminus J^j_{z_0}$ then, by applying the Hahn–Banach separation theorem for the closed convex set $J^j_{z_0}$ of $\mathcal{B}(\bar\Omega)$, we find a $\varphi \in \mathcal{C}(\bar\Omega)$ such that

$$\int_{\bar\Omega} \varphi\, d\mu < 0 < \int_{\bar\Omega} \varphi\, d\nu \quad \forall \nu \in J^j_{z_0}.$$

By Proposition 2.1 we have $S_i(z_0) < S_j(z_0)$—a contradiction. □

The following simple fact is a slight variation of [S2, Lemma 1.5].

GLUING LEMMA. *Let Ω' be a subdomain of Ω, and let $u_1 \in \mathcal{PSH}^c(\Omega')$ and $u_2 \in \mathcal{PSH}^c(\bar\Omega)$. Assume that $u_1 \le u_2$ on $\bar\Omega \cap \partial\Omega'$. Set*

$$u_3 = \begin{cases} \max\{u_1, u_2\} & \text{on } \Omega', \\ u_2 & \text{on } \bar\Omega\setminus\Omega'. \end{cases}$$

Then $u_3 \in \mathcal{PSH}^c(\bar\Omega)$.

The following lemma gives a sufficient condition for $J^3_z = \{\delta_z\}$ to hold at $z \in \partial\Omega$.

LEMMA 2.3. *Let $z_0 \in \partial\Omega$. Assume there exist a neighborhood U of z_0 and a negative plurisubharmonic function u on $U \cap \Omega$ such that* (i) $u^* < 0$ *on* $(U \cap \partial\Omega)\setminus\{z_0\}$, (ii) $\lim_{z\to z_0} u(z) = 0$, *and* (iii) $U \cap \partial\Omega$ *is of class* $\mathcal{C}^1$. *Then* $J^3_{z_0} = \{\delta_{z_0}\}$.

Proof. The main idea stems from [S2, Thm. 2.1]. By the smoothness of $\partial\Omega$ near z_0, we can find open balls B_1 and B_2 centered at z_0 and $\varepsilon_0 > 0$ (respectively) such that:

(a) $B_2 \subset\subset B_1 \subset\subset U$ and $\overline{B_2 \cap \Omega} \subset\subset (B_1 \cap \Omega) + \varepsilon\mathbf{n}$ for all $\varepsilon \in (0, \varepsilon_0)$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ at z_0; and

(b) $u^* < 0$ on $\Omega \cap \partial B_2$.

Set $u_\varepsilon(z) = u(z - \varepsilon\mathbf{n})$ and $\Omega' = B_2 \cap \Omega$. Then from (a) we infer that u_ε is plurisubharmonic on a neighborhood of $\overline{\Omega'}$. Let $\mu \in J^3_{z_0}(\overline{\Omega'})$ and fix $\varepsilon \in (0, \varepsilon_0)$. Then, for $\delta > 0$ small enough, the convolution $u_\varepsilon * \rho_\delta$ defines a smooth and plurisubharmonic function on a neighborhood of $\overline{\Omega'}$. Recall that ρ_δ denotes a standard regularizing kernel with support in $B(0, \delta)$, the open ball centered at 0 with radius δ, and that

$$(u_\varepsilon * \rho_\delta)(z) := \int u_\varepsilon(z - w)\rho_\delta(w)\, d\lambda(w),$$

where $d\lambda$ is the Lebesgue measure in $\mathbf{C}^n$. Then

$$(u_\varepsilon * \rho_\delta)(z_0) \le \int_{\overline{\Omega'}} (u_\varepsilon * \rho_\delta)\, d\mu.$$

Let $\delta \to 0$ and $\varepsilon \to 0$; then applying Fatou's lemma yields

$$0 = \lim_{\varepsilon\to 0} u_\varepsilon(z_0) \le \int_{\overline{\Omega'}} u^*\, d\mu.$$

Combining this with (b), we deduce that $\mu = \delta_{z_0}$. Now the lemma follows from [S2, Prop. 1.4]. □

We also need the following simple fact, whose proof is left to the reader.

LEMMA 2.4. *Let $\{\varphi_j\}_{j\ge 1}$ be a sequence of continuous functions on $\bar{\Omega}$ that increases to a lower semicontinuous function φ on $\bar{\Omega}$. Then, for every sequence $\{a_j\}_{j\ge 1} \to a \in \bar{\Omega}$, $a_j \in \bar{\Omega}$, we have*

$$\varphi(a) \le \liminf_{j\to\infty} \varphi_j(a_j).$$

The last preparatory fact is the classical Choquet topological lemma (see [K, Lemma 2.3.4]).

CHOQUET'S LEMMA. *Let $\{u_\alpha\}_{\alpha\in A}$ be a family of functions on $\bar{\Omega}$ that are locally bounded from above. Then there exists a countable subfamily $\{\alpha_j\} \subset A$ such that*

$$(\sup\{u_\alpha : \alpha \in A\})^* = (\sup\{u_{\alpha_j} : j \ge 1\})^*.$$

Moreover, if u_α is lower continuous for every $\alpha \in A$, then we can choose $\{\alpha_j\}$ such that

$$\sup\{u_\alpha : \alpha \in A\} = \sup\{u_{\alpha_j} : j \ge 1\}.$$

3. Equality of Jensen Measures

We start with a simple fact.

PROPOSITION 3.1. *The following assertions are equivalent:*

(i) *$J_z^2 = J_z^3$ for all $z \in \bar{\Omega}$;*

(ii) *every $u \in \mathcal{PSH}^c(\Omega)$ can be approximated uniformly on $\bar{\Omega}$ by elements in $\mathcal{PSH}^c(\bar{\Omega})$.*

Proof. (ii) $\Rightarrow$ (i) follows easily from the definitions of J_z^2 and J_z^3. For the reverse implication, let $u \in \mathcal{PSH}^c(\Omega)$; then clearly we have $S_2 u \equiv u$ on $\bar{\Omega}$. On the other hand, Proposition 2.1 gives $S_2 u \equiv S_3 u$ on $\bar{\Omega}$ and so $S_3 u \equiv u$ on $\bar{\Omega}$. Applying the Choquet topological lemma yields a sequence $u_j \in \mathcal{PSH}^c(\bar{\Omega})$ that is increasing to $S_3 u$ on $\bar{\Omega}$. By Dini's lemma, this convergence is uniform on $\bar{\Omega}$. $\square$

REMARKS. 1. If Ω has $\mathcal{C}^1$ boundary then (ii) and hence (i) always hold by [FWi, Thm. 1].

2. Consider the hyperconvex domain $\Omega := \Delta \backslash I$, where Δ is the unit disk in $\mathbf{C}$ and I is the segment $[0, 1/2] \subset \Delta$. Let u be the harmonic function on Ω, continuous on $\bar{\Omega}$, such that $u \equiv 0$ on $\partial\Delta$ and $u \equiv 1$ on I. Using the maximum principle, we can check that u cannot be approximated uniformly on $\bar{\Omega}$ by elements in $\mathcal{PSH}^c(\bar{\Omega})$. According to Proposition 3.1(ii), Ω is not J-regular.

Here is the main result of this section.

THEOREM 3.2. *Assume that $J_z^1 = J_z^3$ for all $z \in \Omega \backslash P$, where P is a subset of Lebesgue measure 0 in Ω. Then there exists a pluripolar subset P' of Ω such that the following statements hold.*

(i) $J_z^1 = J_z^3$ *for all* $z \in \Omega \backslash P'$.

(ii) *For every* $u \in \mathcal{PSH}^-(\Omega)$, *there exists a sequence* $u_j \subset \mathcal{PSH}(\bar{\Omega}) \cap \mathcal{C}^\infty(\bar{\Omega})$ *that satisfies the following conditions.*

(a) $u_j \to u$ *pointwise on* $\Omega \backslash P'$, *with* $\limsup_{j\to\infty} u_j \le u^*$ *on* $\bar{\Omega}$.

(b) *There exists an increasing sequence of compact sets* K_j *of* $\Omega \backslash P'$ *such that* $\bigcup K_j = \Omega \backslash P'$ *and* $u_j \ge u_{j+1}$ *on* K_j. *In particular, for every* $z_0 \in \Omega \backslash P'$, *the sequence* $\{u_j(z_0)\}_{j\ge 1}$ *is decreasing from a sufficiently large index* j.

(c) *If* u *is locally bounded on* Ω, *then the sequence* u_j *can be chosen to have the additional property that* $(dd^c u_j)^n \to (dd^c u)^n$ *in the sense of currents.*

Proof. (i) We first claim that there exists a pluripolar $P' \subset \Omega$ such that, for every $\varphi \in \mathcal{C}(\bar{\Omega})$, we have $S_1\varphi = S_3\varphi$ on $\Omega \backslash P'$. For this, we choose a countable dense subset $\{\varphi_j\}_{j\ge 1}$ of $\mathcal{C}(\bar{\Omega})$. By Lemma 2.2(a), $S_1\varphi_j \in \mathcal{PSH}^-(\Omega)$. Applying Proposition 2.1 now yields $S_1\varphi_j = S_3\varphi_j$ on $\Omega \backslash P$. It follows that

$$S_1\varphi_j = S_3\varphi_j = (S_3\varphi_j)^* \text{ a.e. on } \Omega.$$

Therefore, $S_1\varphi_j \equiv (S_3\varphi_j)^*$ on Ω, since these functions are plurisubharmonic on Ω. According to [BTa, Thm. 7.1], there exist pluripolar subsets P_j of Ω such that $S_3\varphi_j = (S_3\varphi_j)^*$ on $\Omega \backslash P_j$. Set $P' = \bigcup P_j$; then P' is pluripolar and $S_1\varphi_j \equiv S_3\varphi_j$ on $\Omega \backslash P'$ for all $j \ge 1$. Now let φ be an arbitrary function in $\mathcal{C}(\bar{\Omega})$; then we can find a sequence $\{\varphi_{k_j}\}_{j\ge 1}$ that converges uniformly to φ on $\bar{\Omega}$. From (1) and (3) we deduce that $S_1\varphi_{k_j}$ and $S_3\varphi_{k_j}$ converge uniformly to $S_1\varphi$ and $S_3\varphi$ (respectively) on $\bar{\Omega}$. Putting all this together, we obtain $S_1\varphi \equiv S_3\varphi$ on $\Omega \backslash P'$ and so the claim follows. Now we apply Lemma 2.2(b) to get $J_z^1 = J_z^3$ for all $z \in \Omega \backslash P'$.

(ii) Consider $u \in \mathcal{PSH}^-(\Omega)$. Since u is bounded from above, there exists a sequence $\{\varphi_j\}_{j\ge 1} \subset \mathcal{C}(\bar{\Omega})$ that decreases to u^* on $\bar{\Omega}$. Then (i) and Proposition 2.1 yield $S_1\varphi_j \equiv S_3\varphi_j$ on $\Omega \backslash P'$. Notice that the sequence $S_1\varphi_j$ is also decreasing and that

$$u \le S_1\varphi_j \le \varphi_j \quad \forall j.$$

This implies that $S_1\varphi_j$ decreases to u on Ω. Let $\{\Omega_j\}_{j\ge 1}$ be a sequence of subdomains in Ω such that $\bigcup \Omega_j = \Omega$ with $\Omega_j \subset\subset \Omega_{j+1}$. Choose $h \in \mathcal{PSH}(\Omega)$ such that $h \not\equiv -\infty$ and $h \equiv -\infty$ on P'. Fix $j \ge 1$. Then, by the Choquet topological lemma, we can find a sequence $\{v_{k,j}\}_{k\ge 1} \subset \mathcal{PSH}^c(\bar{\Omega})$ that increases to $S_3\varphi_j$ on $\bar{\Omega}$. Since $S_3\varphi_j$ is continuous on the compact $K_j := \overline{\Omega_j} \backslash V_j$, where $V_j = \{z : h(z) < -j\}$, by Dini's lemma it follows that the sequence $v_{k,j}$ converges uniformly to $S_3\varphi_j = S_1\varphi_j$ on K_j. Thus we can find k_j so large that

$$S_3\varphi_j - \frac{1}{2^{j+2}} \le v_{k_j,j} \le S_3\varphi_j \text{ on } K_j. \tag{4}$$

Convolving $v_{k_j,j}$ with a suitable regularizing kernel ρ_{δ_j}, we obtain

$$v_j \in \mathcal{PSH}(\bar{\Omega}) \cap \mathcal{C}^\infty(\bar{\Omega}) \quad \text{such that } 0 \le v_j - v_{k_j,j} \le \frac{1}{2^{j+2}} \text{ on } \bar{\Omega}.$$

Set $u_j = v_j + 1/2^j$ and $P' = h^{-1}(-\infty)$. Then $u_j \in \mathcal{PSH}(\bar{\Omega}) \cap \mathcal{C}^\infty(\bar{\Omega})$ and $u_j \ge u_{j+1}$ on K_j. It follows that $(\limsup_{j\to\infty} u_j)^* = u$ on Ω and so $\limsup_{j\to\infty} u_j \le u$ on Ω. Notice also that, for $z \in \partial\Omega$,

$$\limsup_{j\to\infty} u_j(z) = \limsup_{j\to\infty} v_{k_j,j}(z) \le \limsup_{j\to\infty} S_3\varphi_j(z) \le u^*(z).$$

Thus we have constructed a sequence u_j satisfying (a) and (b). For (c), let $u \in \mathcal{PSH}^-(\Omega)$ be locally bounded. Since the conclusion is trivial for $n = 1$, we may assume that $n \ge 2$. Retaining the notation used until (4), we claim that there exists a sequence $\{u_j\}_{j\ge1} \subset \mathcal{PSH}(\bar{\Omega}) \cap \mathcal{C}^\infty(\bar{\Omega})$ with the following properties:

$$u_j \ge u_{j+1} \ \text{ on } K_j; \tag{5}$$

$$\left| \int_{\Omega_{j-1}} (u_j - S_1\varphi_j)(dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cu_j)^m \wedge \omega \right| < \frac{3}{j} \tag{6}$$

for all $1 \le m \le n-1$ and $j \ge 2$, where ω is the Kähler form $dd^c|z|^2$.

To see this, first observe that, since $S_1\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$,

$$0 \le A_j = \int_{\bar{\Omega}_j} (dd^cS_1\varphi_j)^{n-1} \wedge \omega < \infty.$$

Let $\{\lambda_j\}_{j\ge1}$ be a sequence satisfying

$$\lambda_1 = 1, \qquad 0 < \lambda_{j+1} < \min\left(\lambda_j, \frac{1}{2^{j+2}}, \frac{1}{jA_j + 1}\right).$$

Define a sequence $\{a_j\}_{j\ge1}$ as $a_1 = 0$ and $a_j = (\lambda_j - \lambda_{j+1})/2$ for all $j \ge 2$. Fix $j \ge 2$. Then, by Dini's lemma, there exist k_j so large that

$$S_3\varphi_j - a_j \le v_{k,j} \le S_3\varphi_j \ \text{ on } K_j$$

for all $k \ge k_j$.

Let θ be a nonnegative smooth $(1,1)$-form with compact support in Ω such that $\theta = \omega$ on a neighborhood of $\overline{\Omega_j}$. Then, for all $1 \le m \le n-1$,

$$\int_{\overline{\Omega_j}} (S_1\varphi_j - v_{k,j})(dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cv_{k,j})^m \wedge \omega$$
$$\le \int_{\Omega} (S_1\varphi_j - v_{k,j})(dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cv_{k,j})^m \wedge \theta.$$

Observe that $\{v_{k,j}\}_{k\ge1}$ increases pointwise to $S_1\varphi_j$ except on a pluripolar set. The monotone convergence theorem of Bedford and Taylor implies that the sequences of currents

$$S_1\varphi_j(dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cv_{k,j})^m,\ v_{k,j}(dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cv_{k,j})^m$$

converge to $S_1\varphi_j(dd^cS_1\varphi_j)^{n-1}$ for $1 \le m \le n-1$ and that $(dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cv_{k,j})^m \wedge \omega$ converges to $(dd^cS_1\varphi_j)^{n-1} \wedge \omega$. It follows that

$$\lim_{k\to\infty} \int_{\Omega} (S_1\varphi_j - v_{k,j})(dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cv_{k,j})^m \wedge \theta = 0$$

and

$$\limsup_{k\to\infty} \int_{\overline{\Omega_j}} (dd^cS_1\varphi_j)^{n-m-1} \wedge (dd^cv_{k,j})^m \wedge \omega \le \int_{\overline{\Omega_j}} (dd^cS_1\varphi_j)^{n-1} \wedge \omega.$$

Hence we can find $k_{j'} \ge k_j$ such that

$$\int_{\overline{\Omega_j}}(S_1\varphi_j - v_{k_{j',j}})(dd^cS_1\varphi_j)^{n-m-1}\wedge(dd^cv_{k_{j',j}})^m\wedge\omega<\frac{1}{j}$$

for all $1\le m\le n-1$ and such that

$$\int_{\overline{\Omega_j}}(dd^cS_1\varphi_j)^{n-m-1}\wedge(dd^cv_{k,j})^m\wedge\omega\le A_j+\frac{1}{2j}.$$

Convolving $v_{k_{j',j}}$ with a suitable regularizing kernel and using again the monotone convergence theorem of Bedford and Taylor, we find $\tilde v_j\in\mathcal{PSH}(\bar\Omega)\cap\mathcal{C}^\infty(\bar\Omega)$ such that $0\le\tilde v_j-v_{k_{j',j}}<a_{j-1}$ on $\bar\Omega$ and such that:

$$\left|\int_{\Omega_{j-1}}(S_1\varphi_j-\tilde v_j)(dd^cS_1\varphi_j)^{n-m-1}\wedge(dd^c\tilde v_j)^m\wedge\omega\right|<\frac{2}{j};\tag{7}$$

$$\int_{\Omega_{j-1}}(dd^cS_1\varphi_j)^{n-m-1}\wedge(dd^c\tilde v_j)^m\wedge\omega<A_j+\frac{1}{j}.\tag{8}$$

Set $u_1\equiv\sup\varphi_1$ and $u_j=\tilde v_j+\lambda_j$ for $j\ge 2$. Now (6) follows from (7) and (8). For (5) we notice that, by the choices of a_j, b_j, λ_j, on K_j we have

$$u_j=\tilde v_j+\lambda_j\ge v_{k_{j',j}}+\lambda_j\ge S_3\varphi_j-a_j+\lambda_j\ge S_3\varphi_{j+1}+\lambda_{j+1}+a_j\ge u_{j+1}.$$

The claim is proved. Observe also that $u_j\to u$ pointwise on $\Omega\backslash P'$ and that $\limsup u_j\le u^*$ on $\bar\Omega$.

It remains to show that $(dd^cu_j)^n\to(dd^cu)^n$ in the sense of currents. Toward this end, let χ be a smooth $\mathcal{C}^\infty$-function with compact support in Ω. Choose $C>0$ and $j_0\ge 1$ so large that $dd^c(C|z|^2+\chi(z))\ge 0$ on Ω and $\operatorname{supp}\chi\subset\Omega_{j_0}$. Integrating by parts, for $j\ge j_0+1$ we obtain

$$\begin{aligned}\int\chi(dd^cu_j)^n&=\int u_j(dd^cu_j)^{n-1}\wedge dd^c\chi\\&=\int S_1\varphi_j(dd^cu_j)^{n-1}\wedge dd^c\chi+\int(u_j-S_1\varphi_j)(dd^cu_j)^{n-1}\wedge dd^c\chi.\end{aligned}$$

Given (6), we see that the last term in the second equality is smaller in absolute value than $3C/j$. We also have

$$\begin{aligned}\int S_1\varphi_j(dd^cu_j)^{n-1}\wedge dd^c\chi&=\int u_j(dd^cS_1\varphi_j)\wedge(dd^cu_j)^{n-2}\wedge dd^c\chi\\&=\int S_1\varphi_j(dd^cS_1\varphi_j)\wedge(dd^cu_j)^{n-2}\wedge dd^c\chi\\&\quad+\int(u_j-S_1\varphi_j)(dd^cu_j)^{n-1}\wedge(dd^cS_1\varphi_j)\wedge dd^c\chi.\end{aligned}$$

We again infer by (6) that the last term in the second equality is smaller in absolute value than $3C/j$. Continuing in this manner, we obtain

$$\left|\int\chi(dd^cu_j)^n-\int S_1\varphi_j(dd^cS_1\varphi_j)^{n-1}\wedge dd^c\chi\right|<\frac{3nC}{j}.$$

Since $S_1\varphi_j \downarrow u$ on Ω, it follows that $S_1\varphi_j(dd^cS_1\varphi_j)^{n-1}$ converges in the sense of currents to $u(dd^cu)^{n-1}$. Putting all this together, we obtain

$$\lim_{j\to\infty}\int \chi(dd^cu_j)^n = \int \chi(dd^cu)^n.$$

This proves the theorem. □

REMARKS. 1. If the set of points z where u is not locally bounded at z is contained in a pseudoconvex domain Ω' such that $\Omega' \subset\subset \Omega$, then—by following the lines of the proof of Theorem 3.2 and using convergence theorems for the Monge–Ampère operator from [De] and [S1]—we can see that part (ii)(c) of Theorem 3.2 is still valid. More generally, if u belongs to some subclass of $\mathcal{PSH}^-(\Omega)$ where the Monge–Ampère operator can be reasonably defined, then the conclusion of Theorem 3.2(ii)(c) still holds. See [Ce] for a recent account of this matter.

2. We do not know whether it is possible to conclude that $P = \emptyset$ when P is contained in some pluripolar subset of Ω.

COROLLARY 3.3. *The following assertions are equivalent.*

(i) Ω *is J-regular.*

(ii) *For every* $u \in \mathcal{PSH}^-(\Omega)$, *there exists a sequence* $\{u_j\}_{j\ge1}$ *in* $\mathcal{PSH}^c(\bar\Omega)$, *uniformly bounded from above, such that* $u_j \to u$ *pointwise on* Ω *and*

$$\limsup_{j\to\infty} u_j(z) \le u^* \ \text{ on } \partial\Omega.$$

(iii) *For every* $u \in \mathcal{PSH}^-(\Omega)$ *that is bounded from below, there exists a sequence* $\{u_j\}_{j\ge1}$ *in* $\mathcal{PSH}^c(\bar\Omega)$, *uniformly bounded from above, such that* $u_j \to u$ *pointwise on* Ω *and*

$$\limsup_{j\to\infty} u_j(z) \le u^* \ \text{ on } \partial\Omega.$$

Proof. (i) ⇒ (ii) follows from the proof given in part (ii) of Theorem 3.2. (ii) ⇒ (iii) is trivial. It remains to show (iii) ⇒ (i). For this, fix $z_0 \in \Omega$, $\mu \in J^3_{z_0}$, and $u \in \mathcal{PSH}^-(\Omega)$. We must show that

$$u(z_0) \le \int_{\bar\Omega} u^*\,d\mu. \tag{9}$$

For each $j \ge 1$, set $u_j = \max(u, -j)$. Fix $j \ge 1$. Let $\{u_{j,k}\}_{k\ge1}$ be a sequence in $\mathcal{PSH}^c(\bar\Omega)$, uniformly bounded from above, such that

$$\lim_{k\to\infty} u_{j,k}(z) = u_j(z), \quad z \in \Omega,$$

$$\limsup_{k\to\infty} u_{j,k}(z) \le u_j^*(z), \quad z \in \partial\Omega.$$

Then

$$u_{j,k}(z_0) \le \int_{\bar\Omega} u_{j,k}\,d\mu \quad \forall k \ge 1.$$

Letting k tend to ∞ and applying Fatou's lemma yields

$$u_j(z_0) \le \int_{\bar{\Omega}} u_j^* \, d\mu.$$

Notice that $\limsup_{j\to\infty} u_j^* \le u^*$ on $\partial\Omega$. Letting j tend to ∞ and again using Fatou's lemma, we obtain (9), which finishes the proof. □

REMARKS. 1. Using an argument given in the proof of Theorem 4 in [FWi], it is possible to prove (iii) $\Rightarrow$ (ii) directly.

2. We will present an example of Fornaess and Wiegerinck on a smoothly bounded domain Ω that is *not* J-regular. The example is contained in [FWi, p. 260]; for the reader's convenience we offer some details here. Let

$$\Omega = \{(z, w) \in \mathbf{C}^2 : |w - e^{i\varphi(|z|)}| < r(|z|)\},$$

where r and φ are real-valued $\mathcal{C}^\infty$-functions on $\mathbf{R}$. Fornaess and Wiegerinck show that, if r and φ are well chosen, then we can find a bounded continuous plurisubharmonic function f on Ω and a compact subset K of Ω such that f cannot be approximated uniformly on K by plurisubharmonic functions in neighborhoods of $\bar{\Omega}$. By Theorem 3.2(ii), Ω is not J-regular. It is an open and interesting problem to see if such an example can be found in the class of smoothly bounded *pseudoconvex* domains.

As we observed in the remark following Proposition 3.1, Ω may not be J-regular even if Ω is a regular domain in $\mathbf{C}$. In view of this, the following result is somewhat surprising.

PROPOSITION 3.4. *Let Ω be a bounded domain in $\mathbf{C}$. Assume that, for every irregular point z_0 in $\partial\Omega$, there exists a neighborhood U such that $U \cap \partial\Omega$ is polar. Then $J_z^1 = J_z^2$ for all $z \in \Omega$.*

Proof. We split the proof into two steps.

Step 1. We show that there is a subharmonic function u on $\mathbf{C}$ such that $J_z^1 = J_z^2$ for all $z \in \Omega$ when $u(z) \ne -\infty$. Let Q be the set of irregular points of $\partial\Omega$. According to Kellog's theorem (see [R, Thm. 4.2.5]), Q is polar. Fix $z_0 \in (\partial\Omega)\setminus Q$. Let $f \in \mathcal{C}(\partial\Omega)$ be such that $f(z_0) = 0$ and $f < 0$ elsewhere. Denote by u the (unique) solution of the generalized Dirichlet problem with boundary data f. Clearly, u is negative and harmonic on Ω. Since every irregular point on $\partial\Omega$ admits a neighborhood U such that $U \cap \partial\Omega$ is polar, we see that u extends continuously to every point of $\partial\Omega$ and is strictly negative everywhere except at z_0. It follows that $J_z^2 = \{\delta_z\}$ for all $z \in Q$. Choose a subharmonic function u in $\mathbf{C}$ such that $u \equiv -\infty$ on Q and $u \not\equiv -\infty$. By [DW, Thm. 3.5] (see also the remark following Theorem 4.4), we have $J_z^1 = J_z^2$ for all $z \in \Omega$ with $u(z) \ne -\infty$.

Step 2. Since the set $\{z : u(z) = -\infty\}$ has zero length, we can find a sequence of domains Ω_j such that $\Omega_j \subset\subset \Omega_{j+1}$, $\bigcup \Omega_j = \Omega$, and $u(z) \ne -\infty$ for $z \in \bigcup \partial\Omega_j$. Fix $\varphi \in \mathcal{C}(\bar{\Omega})$. We claim that $S_1\varphi \equiv S_2\varphi$ on Ω. Applying Proposition 2.1, we obtain $S_1\varphi \equiv S_2\varphi$ on $\bigcup \partial\Omega_j'$. Fix $j \ge 1$ and $\varepsilon > 0$; then the Choquet topological lemma yields a sequence $v_m \uparrow S_2\varphi$ on $\bar{\Omega}$. Now using Lemma 2.4, for $\delta > 0$ small enough and m sufficiently large we have

$$(S_1\varphi + u) * \rho_\delta \le v_m + \varepsilon \ \text{ on } \partial\Omega_j'.$$

Set

$$v_{m,\delta} = \begin{cases} \max((S_1\varphi + u) * \rho_\delta, v_m + \varepsilon) & \text{on } \Omega_j', \\ v_m + \varepsilon & \text{on } \bar\Omega \backslash \Omega_j'. \end{cases}$$

By the gluing lemma, $v_{m,\delta}$ is subharmonic on Ω and continuous on $\bar\Omega$. Moreover, for δ small enough we also have $v_{m,\delta} \le \varphi + \varepsilon$ on $\bar\Omega$. It follows that $v_{m,\delta} \le S_2\varphi + \varepsilon$. Letting m and δ go to ∞ and 0, respectively, we get $S_1\varphi \equiv S_2\varphi$ on Ω_j' and so the claim follows. By Lemma 2.2(b) we conclude that $J_z^1 = J_z^2$ for all $z \in \Omega$.

4. Examples of *J*-Regular Domains

It seems hard to find a good geometric characterization of domains on which we have equality between Jensen measures. The proof of Theorem 4.10 in [W2] implies that, if Ω is *strongly star shaped* in the sense that $t\Omega \subset\subset \Omega$ for all $t \in (0,1)$, then $J_z^1 = J_z^2$ for all $z \in \Omega$; the same proof shows that $J_z^1 = J_z^3$ for all $z \in \Omega$. Notice that there is a flaw in the formulation of Theorem 4.10 in [W2], since Ω is assumed to be merely *star shaped* there. Observe also that a strongly star-shaped domain may be very far from being pseudoconvex. Our Theorem 4.4 is a generalization of this fact. Before formulating it, we recall some terminology from [DW].

Definition 4.1. Let E be a pluripolar subset of $\bar\Omega$. The (relative) *pluripolar hull* of E, $\mathrm{pph}(E)$, is defined as

$$\mathrm{pph}(E) = \{z \in \Omega,\ u \in \mathcal{PSH}(\bar\Omega),\ u|_E \equiv -\infty \Rightarrow u(z) = -\infty\},$$

where $\mathcal{PSH}(\bar\Omega)$ denotes the set of plurisubharmonic functions on neighborhoods of $\bar\Omega$.

Definition 4.2. By an *isotopy family of biholomorphic mappings* defined on Ω, we mean a continuous map $\Phi\colon [0,1] \times \bar\Omega \to \mathbf{C}^n$ with the following properties.

(a) $\Phi_t := \Phi(t,\cdot)$ maps Ω biholomorphically onto its image; moreover, Φ_t is a homeomorphism between $\bar\Omega$ and $\overline{\Phi_t(\Omega)}$.
(b) $\Phi_t^{-1}(z)$ is real analytic in t on a neighborhood of 0 for all $z \in \Omega$.
(c) Φ_t^{-1} converges uniformly to $\Phi_0^{-1} = \mathrm{Id}$ on $\bar\Omega$ as $t \to 0$.

Definition 4.3. Let Φ_t be an isotopy family of biholomorphic mappings on Ω. Then the *boundary cluster set* of Φ_t is defined as the set of limit points of sequences of elements in $\bar\Omega \cap \Phi_t(\partial\Omega)$ as $t \to 0$.

Theorem 4.4. *Let Φ_t be an isotopy family of biholomorphic maps on Ω, and let X be the boundary cluster set of Φ_t. Assume there exists a pluripolar subset P of X such that $J_z^3 = \{\delta_z\}$ for all $z \in X \backslash P$. Then the following statements hold.*

(i) $J_z^1 = J_z^3$ *for all* $z \in \Omega \backslash \mathrm{pph}(P)$; *in particular,* Ω *is J-regular if* $\mathrm{pph}(P) = \emptyset$.
(ii) *If* $\mathrm{pph}(P)$ *is of* F_σ *and* G_δ *type and if* Y *is a compact subset of* $\partial\Omega$ *satisfying* $J_z^3 = \{\delta_z\}$ *for all* $z \in Y$, *then for every* $u \in \mathcal{PSH}^-(\Omega)$ *there exists*

a sequence $\{u_j\}_{j\ge 1} \subset \mathcal{PSH}(\bar\Omega)\cap\mathcal{C}^\infty(\bar\Omega)$ *such that* $u_j \to u^*$ *pointwise on* $(\Omega\cup Y)\backslash \mathrm{pph}(P)$.

(iii) *If* $\mathrm{pph}(P)=\emptyset$ *and* Q *is a* G_δ *subset of* $\partial\Omega$ *satisfying* $J_z^3=\{\delta_z\}$ *for all* $z\in(\partial\Omega)\backslash Q$, *then for every* $u\in\mathcal{PSH}^-(\Omega)$ *there exists a sequence* $\{u_j\}_{j\ge1}\subset\mathcal{PSH}(\bar\Omega)\cap\mathcal{C}^\infty(\bar\Omega)$ *such that* $u_j\to u^*$ *pointwise on* $\bar\Omega\backslash Q$. *In addition, if* u^* *is continuous on* $\bar\Omega\backslash Q$ *then the convergence is uniform on compact sets of* $\bar\Omega\backslash Q$.

Proof. (i) Fix $z_0\in\Omega\backslash\mathrm{pph}(P)$. We will prove that $S_1\varphi(z_0)=S_3\varphi(z_0)$ for all $\varphi\in\mathcal{C}(\bar\Omega)$. Given $\varphi\in\mathcal{C}(\bar\Omega)$, by Proposition 2.1 we have

$$S_1\varphi\equiv S_3\varphi\equiv\varphi \text{ on } X\backslash P. \tag{10}$$

By the Choquet topological lemma, there exists a sequence $\{v_j\}_{j\ge1}\subset\mathcal{PSH}^c(\bar\Omega)$ that increases to $S_3\varphi$ on $\bar\Omega$. Since $z_0\notin\mathrm{pph}(P)$, we can find $v\in\mathcal{PSH}(\bar\Omega)$ such that $v<0$, $v(z_0)\ne-\infty$, and $v\equiv-\infty$ on P. Since $\varphi\in\mathcal{C}(\bar\Omega)$, by Lemma 2.2(a) we have $S_1\varphi\in\mathcal{PSH}^-(\Omega)$. For $t\in[0,1]$ and $\varepsilon>0$ we set

$$u_{t,\varepsilon}=(S_1\varphi)\circ\Phi_t^{-1}+\varepsilon v.$$

Now we will use a reasoning similar in spirit to the proof of Proposition 3.4. It is clear that $u_{t,\varepsilon}\in\mathcal{PSH}^-(\Phi_t(\Omega))$ for all t sufficiently close to 0. Let Ω_k be a sequence of subdomains in Ω satisfying $\Omega_k\subset\subset\Omega_{k+1}$ and $\bigcup\Omega_k=\Omega$. Fix $\varepsilon>0$. We claim that there exist $t_0\in(0,1)$, $k_0\ge1$, and $m\ge1$ such that for all $t\in(0,t_0)$, $k\ge k_0$, there are

$$0<\delta_k<a_k:=\min_{0\le t\le t_0}\mathrm{dist}(\Phi_t(\partial\Omega),\Phi_t(\partial\Omega_k))$$

satisfying

$$u_{t,\varepsilon}*\rho_{\delta_k}\le v_m+\varepsilon \text{ on } \bar\Omega\cap\Phi_t(\partial\Omega_k). \tag{11}$$

Assume otherwise; then we obtain sequences $t_j\downarrow0$, $k_j\uparrow\infty$, $m_j\uparrow\infty$, $\{\xi_j\}\subset\bar\Omega\cap\Phi_{t_j}(\partial\Omega_{k_j})$, and $0<\delta_{k_j}<a_{k_j}$ such that

$$(u_{t_j,\varepsilon}*\rho_{\delta_{k_j}})(\xi_j)>v_{m_j}(\xi_j)+\varepsilon \quad \forall j.$$

Since Φ is uniformly continuous on $[0,1]\times\bar\Omega$, we may assume (after switching to a subsequence) that $\xi_j\to\xi^*\in X$. Since $v\equiv-\infty$ on P, it follows that $\xi^*\in X\backslash P$. Using Lemma 2.4 and (10), we see that the lim inf of the right-hand side of the preceding inequality is larger than $S_3\varphi(\xi^*)+\varepsilon=S_1\varphi(\xi^*)+\varepsilon$ when $j\to\infty$. Since $v<0$ on $\bar\Omega$ and since Φ_t^{-1} converges uniformly to the identity map on $\bar\Omega$, the definition of the convolution operator enables us to check that the lim sup of the left-hand side is smaller than $S_1\varphi(\xi^*)$. Thus we have a contradiction and the claim follows. By increasing k_0 and shrinking t_0, we may also assume that $z_0\in\Phi_t(\Omega_k)$ for all $k\ge k_0$ and $0\le t<t_0$.

By the gluing lemma and (11), for $k\ge k_0$ and $t\in(0,t_0)$ the function

$$\tilde v_{t,k}=\begin{cases}\max(v_m+\varepsilon,u_{t,\varepsilon}*\rho_{\delta_k}) & \text{on } \Omega\cap\Phi_t(\Omega_k),\\ v_m+\varepsilon & \text{on } \bar\Omega\backslash\Phi_t(\Omega_k)\end{cases}$$

belongs to $\mathcal{PSH}^c(\bar{\Omega})$. Since $v < 0$, $S_1\varphi \le \varphi$ on $\bar{\Omega}$, and $\varphi \in \mathcal{C}(\bar{\Omega})$, we can choose $k \ge k_0$ large enough and $t \in (0, t_0)$ so small that

$$u_{t,\varepsilon} * \rho_{\delta_k} \le \varphi + \varepsilon \quad \text{on } \Phi_t(\overline{\Omega_k}).$$

Fix such k, t. Notice also that $v_m + \varepsilon \le \varphi + \varepsilon$ on $\bar{\Omega}$. Therefore, $\tilde{v}_{t,k} \le S_3\varphi + \varepsilon$. In particular,

$$u_{t,\varepsilon}(z_0) \le (u_{t,\varepsilon} * \rho_{\delta_k})(z_0) \le \tilde{v}_{t,k}(z_0) \le S_3\varphi(z_0) + \varepsilon.$$

Taking the lim sup of the left-hand side as $t \to 0$ and observing that the curve $\Phi_t^{-1}(z_0)$, being real analytic near 0, is not plurithin at $t = 0$, we infer that

$$S_1\varphi(z_0) + \varepsilon v(z_0) \le S_3\varphi(z_0) + \varepsilon.$$

If we let $\varepsilon \to 0$ then, since $v(z_0) \ne -\infty$, we obtain $S_1\varphi(z_0) \le S_3\varphi(z_0)$. Thus $S_1\varphi(z_0) = S_3\varphi(z_0)$ for all $\varphi \in \mathcal{C}(\bar{\Omega})$. Now by Lemma 2.2(a) we conclude that $J^1_{z_0} = J^3_{z_0}$.

(ii) First we show there exists an $h \in \mathcal{PSH}^-(\Omega)$ such that

$$\text{pph}(P) = \{z \in \Omega : h(z) = -\infty\}.$$

To see this, we will use an argument similar to the one given in the proof of Theorem 3.2 in [HDL]. More precisely: since pph(P) is of F_σ and G_δ type, we can choose increasing sequences of compact sets $P_j \subset \text{pph}(P)$ and $Q_j \subset \Omega\backslash\text{pph}(P)$ such that

$$\bigcup P_j = \text{pph}(P), \qquad \bigcup Q_j = \Omega\backslash\text{pph}(P).$$

Fix $j \ge 1$ and a point $a \in Q_j$; then we can find $u_a \in \mathcal{PSH}(\bar{\Omega})$ such that

$$u_a < 0, \quad u_a(a) > -\infty, \quad u_a|_{P_j} \equiv -\infty.$$

After regularizing u_a and then composing with an increasing convex function, we obtain a continuous plurisubharmonic function u'_a on a neighborhood of $\bar{\Omega}$ and on a neighborhood U_a of a such that

$$u'_a < 0, \quad u'_a|_{U_a} > -1, \quad u'_a|_{P_j} < -2^j.$$

Using the compactness of Q_j yields a continuous plurisubharmonic function v_j on a neighborhood of $\bar{\Omega}$ such that

$$v_j < 0, \quad v_j|_{P_j} < -2^j, \quad v_j|_{Q_j} > -1.$$

Set

$$h(z) = \sum_{j \ge 1} 2^{-j} v_j(z) \quad \forall z \in \Omega.$$

It is easy to check that h is the desired function, and the claim follows.

Next, we pick a sequence Ω_j of subdomains in Ω such that $\Omega_j \subset\subset \Omega_{j+1}$ and $\bigcup \Omega_j = \Omega$. Let $K_j = (Y \cup \overline{\Omega_j})\backslash V_j$, where $V_j = \{z \in \Omega : h(z) < -j\}$. Now we can proceed as in the proof of Theorem 3.2(ii). The details are omitted.

(iii) Let $\bar{\Omega}\backslash Q = \bigcup_{j\ge1} K_j$, where K_j is compact. Since pph$(P) = \emptyset$, we have that Ω is J-regular and so $J^1_z = J^3_z$ for all $z \in K_j$. We again follow the lines of the proof of Theorem 3.2(ii) to reach the desired conclusion. □

REMARKS. 1. The same proof as in (i) shows that a similar conclusion holds if we consider J_z^2 instead of J_z^3. This is precisely Theorem 3.5 in [DW].

2. Let Ω be the "worm" domain constructed by Diederich and Fornaess in [DiF]; it is a smoothly bounded pseudoconvex domain in $\mathbf{C}^2$. Moreover, $\partial\Omega$ is strictly pseudoconvex except on the disk $P = \{(z,0) : 1 \leq |z| \leq r\}$. It follows that $J_z^3 = \{\delta_z\}$ for all $z \in (\partial\Omega)\backslash P$. Since the complex line $w = 0$ does not intersect Ω, we have $\mathrm{pph}(P) = \emptyset$. Consider the family $\Phi_t \equiv \mathrm{Id}$ for all $t \in [0,1]$; by Theorem 4.4(i), Ω is J-regular. Moreover, Theorem 4.4(iii) implies that, for every $u \in \mathcal{PSH}^-(\Omega)$ such that u^* is continuous on $\bar{\Omega}\backslash P$, there exists a sequence $\{u_j\}_{j\geq 1} \subset \mathcal{PSH}(\bar{\Omega}) \cap \mathcal{C}^\infty(\bar{\Omega})$ that converges uniformly to u^* on compact sets of $\bar{\Omega}\backslash P$. On the other hand, there exists a holomorphic function f on Ω that is $\mathcal{C}^\infty$-smooth on $\bar{\Omega}$ which cannot be approximated uniformly on $\bar{\Omega}$ by holomorphic functions on neighborhoods of $\bar{\Omega}$. (For more details see [DiF, p. 280].) I am grateful to Professor François Bertheloot for directing my attention to this worm domain. □

We conclude this section with another consequence of Theorem 4.4.

COROLLARY 4.5. *Let Ω_1 be a bounded B-regular domain with $\mathcal{C}^1$ boundary in $\mathbf{C}^n$, and let K be a compact subset of Ω_1 such that $tK \subset \mathrm{int}(K)$ for all $t \in [0,1)$. Let $\Omega_2 = \Omega_1\backslash K$. Then, for every $u \in \mathcal{PSH}^-(\Omega_2)$, there exists a sequence $\{u_j\}_{j\geq 1}$ of $\mathcal{C}^\infty$-smooth plurisubharmonic functions on neighborhoods of $\bar{\Omega}_2$ such that $u_j^* \to u$ pointwise on $\Omega_2 \cup \partial\Omega_1$.*

This result should be compared to [FS, Thm. 3.1], where a smoothing theorem is obtained when Ω_1 is a ball.

Proof of Corollary 4.5. Let $\Phi_t(z) = (1+t)z$. Clearly, Φ_t is an isotopy family of biholomorphic maps on Ω_2. Observe that the condition on K implies that the boundary cluster set X is contained in $\partial\Omega_1$. By Lemma 2.3 we have $J_z^3 = \{\delta_z\}$ for all $z \in X$. Hence, Theorem 4.4(ii) applies. □

5. Open Problems

We mention some questions connected to our work.

1. Is it possible to make the sequence u_j in Theorem 3.2(ii) decreasing to u on Ω?
2. Is there any smoothly bounded pseudoconvex domain that is not J-regular?
3. Is the product of two J-regular domains again J-regular?
4. Is J-regularity an invariant property under proper holomorphic mappings? We remark that J-regularity is invariant under any biholomorphism that extends beyond the boundaries of the domains.
5. Is there any description of J_z^1 in terms of measures that are pushforwards of the Lebesgue measure on the circle under closed analytic disks? In this connection, see [DW, Prop. 5.6] for a partial result; for J_z^2 and J_z^3, such descriptions can be found in [P].
6. Is Ω J-regular if the set $\{z \in \Omega : J_z^1 = J_z^3\}$ is pluripolar?

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