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Exceptional Values in Holomorphic Families of Entire Functions

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In 1926, Julia [5] studied singularities of implicit functions defined by equations $f(z,w)=0$, where f is an entire function of two variables such that $f(\cdot,w)\not\equiv 0$ for every $w\in\mathbf{C}$. Among other things, he investigated the exceptional set P consisting of those w for which such an equation has no solutions z. In other words, P is the complement of the projection of the analytic set $\{(z,w): f(z,w)=0\}$ onto the second coordinate. Julia proved that P is closed and cannot contain a continuum unless it coincides with the w-plane. Lelong [6] and Tsuji [12; 13, Thm. VIII.37] independently improved this result by showing that the logarithmic capacity of P is zero if $P\neq\mathbf{C}$. In the opposite direction, Julia [5] proved that every discrete set $P\subset\mathbf{C}$ can occur as the exceptional set. He writes: "Resterait à voir si cet ensemble, sans être continu, peut avoir la puissance du continu." (It remains to be seen whether this set, without being a continuum, can have the power of a continuum.)

According to Alan Sokal (private communication), the same question arises in holomorphic dynamics when one tries to extend to holomorphic families of transcendental entire functions a result of Lyubich [8, Prop. 3.5] on holomorphic families of rational functions.

In this paper we show that, in general, the result of Lelong and Tsuji is best possible: every closed set of zero capacity can occur as an exceptional set (Theorem 1). Then we study a related problem of dependence of Picard exceptional values of the function $z\mapsto f(z,w)$ on the parameter w (Theorem 2).

It is known that the exceptional set P is discrete in the important case where $z\mapsto f(z,w)$ are functions of finite order. This was discovered by Lelong in [6]; the result was later generalized to the case of a multidimensional parameter w in [7, Thm. 3.44].

We also mention that the set P must be analytic in certain holomorphic families of entire functions with finitely many singular values, a situation that was considered in [1; 2]. These families may consist of functions of infinite order.

We begin with a simple proof of a version of Lelong's theorem on functions of finite order.

Proposition 1. *Let D be a complex manifold and $f:\mathbf{C}\times D\to\mathbf{C}$ an analytic function such that the entire functions $z\mapsto f(z,w)$ are not identically equal to zero and are of finite order for all $w\in D$. Then the set*

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$$P = \{w \in D : f(z,w) \neq 0 \ \forall z \in \mathbf{C}\} \tag{1}$$

is analytic.

COROLLARY. *Let D be a region in $\mathbf{C}$ and $f : \mathbf{C} \times D \to \mathbf{C}$ an analytic function such that entire functions $z \mapsto f(z,w)$ are of finite order for all $w \in D$. Then either the set P as in* (1) *is discrete or $D \backslash P$ is discrete.*

Indeed, the set $A = \{w \in D : f(\cdot, w) \equiv 0\}$ is discrete (unless $f = 0$, in which case there is nothing to prove). Hence there exists a function g holomorphic in D whose zero set is A and such that f/g satisfies all the conditions of Proposition 1. $\square$

REMARKS. 1. In general, if D is of dimension > 1 and the set A is not empty, then one can prove only that $A \cup P$ is contained in a proper analytic subset of D (unless $A \cup P = D$) [7, Thm. 3.44].

2. If the order of $f(\cdot, w)$ is finite for all $w \in D$, then this order is bounded on compact subsets of D [7, Thm. 1.41].

Proof of Proposition 1. We assume without loss of generality that

$$f(0,w) \neq 0 \quad \text{for} \ \ w \in D \tag{2}$$

(shift the origin in $\mathbf{C}_z$ and shrink D, if necessary) and that the order of the function $f(\cdot, w)$ does not exceed λ for all $w \in D$ (see Remark 2).

Let p be an integer, $p > \lambda$. Then, for each w, f has the Weierstrass representation

$$f(z,w) = e^{c_0 + \cdots + c_p z^p} \prod_{a : f(a,w)=0} \left(1 - \frac{z}{a}\right) e^{z/a + \cdots + z^p/pa^p},$$

where a are the zeros of $f(\cdot, w)$ repeated according to their multiplicities and where c_j and a depend on w. Taking the logarithmic derivative, differentiating it p times, and substituting $z = 0$, for each $w \in D$ we obtain

$$F_p(w) = \frac{d^p}{dz^p}\left(\frac{df}{f\,dz}\right)\bigg|_{z=0} = p! \sum_{a : f(a,w)=0} a^{-p-1}.$$

The series on the right-hand side is absolutely convergent owing to our choice of p. The functions F_p are holomorphic in D in view of (2). Clearly, $w \in P$ implies $F_p(w) = 0$ for all $p > \lambda$. In the opposite direction, that $F_p(w) = 0$ for all $p > \lambda$ means all but finitely many derivatives with respect to z at $z = 0$ of the function $df/f\,dz$ meromorphic in $\mathbf{C}$ are equal to zero, so this meromorphic function is a polynomial and thus $f(z,w) = \exp(c_0 + \cdots + c_p z^p)$; that is, $w \in P$. Therefore, P is the set of common zeros of F_p for $p > \lambda$. $\square$

The following result is due to Lelong and Tsuji. However, they state it only for the case $\dim D = 1$ whereas we need a multidimensional version for the proof of Theorem 2 to follow.

PROPOSITION 2. *Let D be a connected complex manifold and $f : \mathbf{C} \times D \to \mathbf{C}$ an analytic function such that the entire functions $z \mapsto f(z, w)$ are not identically equal to zero. Then the set*

$$P = \{w \in D : f(z, w) \neq 0 \ \forall z \in \mathbf{C}\}$$

either is closed and pluripolar or coincides with D.

We recall that a set X is called *pluripolar* if there is a neighborhood U of X and a plurisubharmonic function u in U such that $X \subset \{z : u(z) = -\infty\}$.

Proof of Proposition 2. Suppose that $P \neq D$. It is enough to show that every point $w_0 \in D$ has a neighborhood U such that $P \cap U$ is pluripolar. By shifting the origin in $\mathbf{C}_z$ and shrinking U, we may assume that $f(0, w) \neq 0$ for all $w \in U$. Let $r(w)$ be the smallest of the moduli of zeros of the entire function $z \mapsto f(z, w)$. If this function has no zeros, we set $r(w) = +\infty$. We shall prove that $\log r$ is a continuous plurisuperharmonic function.

First we prove the continuity of $r : U \to (0, +\infty]$. Indeed, suppose that $r(w_0) < \infty$ and let k be the number of zeros of $f(\cdot, w_0)$ on $|z| = r(w_0)$, counting multiplicity. Let $\varepsilon > 0$ be so small that the number of zeros of $f(\cdot, w_0)$ in $|z| \leq r(w_0) + \varepsilon$ equals k. Then

$$\int_{|z|=r(w_0)\pm\varepsilon} \frac{df}{f\,dz}\,dz = \begin{cases} 2\pi ik, \\ 0. \end{cases}$$

Because the integrals depend on w_0 continuously, we conclude that r is continuous at w_0. Consideration of the case $r(w_0) = +\infty$ is similar.

Now we verify that the restriction of $\log r$ to any complex line is superharmonic. Let $\zeta \to w(\zeta)$ be the equation of such a line, where $w(0) = w_0$. Let z_0 be a zero of $f(\cdot, w_0)$ of the smallest modulus. We verify the inequality for the averages of $\log r$ over the circles $|\zeta| = \delta$ where δ is small enough. According to the Weierstrass preparation theorem, the set $Q = \{(z, \zeta) : f(z, w(\zeta)) = 0\}$ is given in a neighborhood of $(z_0, 0)$ by an equation of the form

$$(z - z_0)^p + b_{p-1}(\zeta)(z - z_0)^{p-1} + \cdots + b_0(\zeta) = 0,$$

where the b_j are analytic functions in a neighborhood of 0, $b_j(0) = 0$. We rewrite this as

$$z^p + c_{p-1}(\zeta)z^{p-1} + \cdots + c_0(\zeta) = 0, \tag{3}$$

where the c_j are analytic functions and

$$c_0(0) = (-z_0)^p. \tag{4}$$

Let V be a punctured disc around 0; we choose its radius so small that $c_0(\zeta) \neq 0$ in V. For ζ in V let $z_i(\zeta)$, $i = 1, \ldots, p$, be the branches of the multivalued function $z(\zeta)$ defined by equation (3). Then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log r(w(\delta e^{i\theta}))\,d\theta &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\min_i \log|z_i(\delta e^{i\theta})|\,d\theta\\ &\le \frac{1}{2\pi p}\int_{-\pi}^{\pi}\log\prod_i|z_i(\delta e^{i\theta})|\,d\theta\\ &= \frac{1}{2\pi p}\int_{-\pi}^{\pi}\log|c_0(\delta e^{i\theta})|\,d\theta = \log|z_0| = \log r(w_0),\end{aligned}$$

where we have used (4) and the harmonicity of $\log|c_0|$ in $V\cup\{0\}$. This completes the proof of plurisuperharmonicity. Since $P=\{w : \log r(w)=+\infty\}$, we conclude that P is pluripolar. □

Our first theorem answers the question of Julia; it shows that the restriction of finiteness of order cannot be removed in Proposition 1 and that Proposition 2 is best possible—at least when $\dim D=1$.

THEOREM 1. *Let $D\subset\mathbf{C}$ be the unit disc, and let P be an arbitrary compact subset of D of zero capacity. Then there exists a holomorphic function $f:\mathbf{C}\times D\to\mathbf{C}$ such that, for every $w\in P$, the equation $f(z,w)=0$ has no solutions while for each $w\in D\backslash P$ it has infinitely many solutions.*

It is not clear whether a similar result holds with multidimensional parameter space D and arbitrary closed pluripolar set $P\subset D$.

Proof of Theorem 1. Let $\phi\colon D\to D\backslash P$ be a universal covering, and let S be the set of singular points of ϕ on the unit circle. Then S is a closed set of zero Lebesgue measure.

(We recall a simple proof of this fact. As a bounded analytic function, ϕ has radial limits almost everywhere. It is easy to see that a point where the radial limit has absolute value 1 is not a singular point of ϕ. Hence the radial limits exist and belong to P almost everywhere on S. Let u be the "Evans potential" of P: a harmonic function in $D\backslash P$ that is continuous in $\bar D$ and such that $u(\zeta)=0$ for $\zeta\in\partial D$ and $u(\zeta)=-\infty$ for $\zeta\in P$. Such a function exists for every compact set P of zero capacity. Now $v=u\circ\phi$ is a negative harmonic function in the unit disc and whose radial limits on S are equal to $-\infty$; thus $|S|=0$ by the classical uniqueness theorem.)

According to a theorem of Fatou (see e.g. [3, Chap. VI]), for every closed set S of zero Lebesgue measure on ∂D there exists a holomorphic function g in D that is continuous in $\bar D$ and such that

$$\{\zeta\in\bar D : g(\zeta)=0\}=S.$$

In particular, g has no zeros in D.

Now we define the set $Q\subset\mathbf{C}\times D$ as follows:

$$Q=\{(1/g(\zeta),\phi(\zeta)) : \zeta\in D\}.$$

It is evident that the projection of Q on the second coordinate equals $D\backslash P$. It remains to prove that the set Q is analytic. For this, it is enough to establish that the map

$$\Phi\colon D \to \mathbf{C} \times D, \qquad \Phi(\zeta) = (1/g(\zeta), \phi(\zeta)),$$

is proper. Let $K \subset \mathbf{C} \times D$ be a compact subset. Then the closure of $\Phi^{-1}(K)$ in $\bar{D}$ is disjoint from S because g is continuous and $1/g(\zeta) \to \infty$ as $\zeta \to S$. On the other hand, for every point $\zeta \in \partial D \backslash S$, the limit

$$\lim_{\zeta \to \zeta_0} \phi(\zeta)$$

exists and has absolute value 1. Hence $\Phi^{-1}(K)$ is compact in D.

Now the existence of the required function f follows from the solvability of the second cousin problem (see [4, Sec. 5.6]). □

Notice that the map Φ constructed in the proof is an immersion.

We recall that a point $a \in \mathbf{C}$ is called an *exceptional value* of an entire function f if the equation $f(z) = a$ has no solutions. Picard's little theorem states that a nonconstant entire function can have at most one exceptional value.

Let f be an entire function of z depending on the parameter w holomorphically, as in Propositions 1 and 2, and *assume in the rest of the paper that, for all $w \in D$, $f(\cdot, w) \neq$ const*. Let $n(w) \in \{0, 1\}$ be the number of exceptional values of $f(\cdot, w)$.

QUESTION 1. What can be said about $n(w)$ as a function of w?

EXAMPLE 1. $f(z, w) = e^z + wz$. We have $n(0) = 1$ and $n(w) = 0$ for $w \neq 0$.

EXAMPLE 2. $f(z, w) = (e^{wz} - 1)/w$ for $w \neq 0$, and $f(z, 0) = z$. We have $n(0) = 0$, while $n(w) = 1$ for $w \neq 0$. The exceptional value $a(w) = -1/w$ tends to infinity as $w \to 0$.

Thus n is neither upper nor lower semicontinuous.

EXAMPLE 3. Let

$$f(z, w) = \int_{-\infty}^{z} (\zeta + w) e^{-\zeta^2/2}\, d\zeta,$$

where the contour of integration consists of the negative ray, passed left to right, followed by a curve from 0 to z. We have $f(z, 0) = -e^{-z^2/2}$, which has exceptional value 0, so $n(0) = 1$. It is easy to see that there are no exceptional values for $w \neq 0$, so $n(w) = 0$ for $w \neq 0$. Thus $n(w)$ is the same as in Example 1, but this time we have the additional feature that the set of singular values of $f(\cdot, w)$ is finite for all $w \in \mathbf{C}$: there is one critical value, $f(-w, w)$, and two asymptotic values, 0 and $\sqrt{2\pi}\, w$.

QUESTION 2. Suppose that $n(w) \equiv 1$, and let $a(w)$ be the exceptional value of $f(\cdot, w)$. What can be said about $a(w)$ as a function of w?

For functions of finite order, Question 2 was addressed by Nishino, who proved the following in [9].

> *Let f be an entire function of two variables such that $z \mapsto f(z, w)$ is a nonconstant function of finite order for all w. If $n(w) = 1$ for all w*

in some set having a finite accumulation point, then there exists a meromorphic function $\tilde{a}(w)$ *such that* (i) $a(w) = \tilde{a}(w)$ *when* $a(w) \neq \infty$ *and* (ii) $f(\cdot, w)$ *is a polynomial when* $\tilde{a}(w) = \infty$.

EXAMPLE 4 (Nishino). Let $f(z,w) = (e^{we^z} - 1)/w$ for $w \neq 0$ and let $f(z,0) = e^z$. Then f is an entire function of two variables and $n(w) \equiv 1$. However, $a(w) = -1/w$ for $w \neq 0$ and also $a(0) = 0$, so a is a discontinuous function of w.

Nishino also proved that, for an arbitrary entire function f of two variables with $n(w) = 1$ in some region D, the set of discontinuity of the function $a(w)$ is closed and nowhere dense in D.

Our Theorem 2 gives a complete answer to Question 2. We first prove the following semicontinuity property of the set of exceptional values; it holds for all *meromorphic functions* that depend holomorphically on a parameter. We always assume that $z \mapsto f(z,w)$ is nonconstant for all w. Let

$$A(w) = \{a \in \bar{\mathbf{C}} : f(z,w) \neq a \ \forall z \in \mathbf{C}\}.$$

PROPOSITION 3. *For every* $w_0 \in D$ *and every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $|w - w_0| < \delta$ *implies* $A(w)$ *is contained in the* ε*-neighborhood of* $A(w_0)$ *with respect to the spherical metric.*

Proof. Let U be the open ε-neighborhood of $A(w_0)$. Then $K = \bar{\mathbf{C}} \backslash U$ is compact, so there exists an $r > 0$ such that the image of the disc $|z| < r$ under $f(\cdot, w_0)$ contains K. Then Hurwitz's theorem shows that, for every w close enough to w_0, the image of the disc $|z| < r$ under $f(\cdot, w)$ will also contain K. □

COROLLARY 1. *The set of meromorphic functions having no exceptional values on the Riemann sphere is open in the topology of uniform convergence on compact subsets of* $\mathbf{C}$ *with respect to the spherical metric in the image.*

The set of entire functions whose only exceptional value is ∞ is not open, as Example 2 shows.

COROLLARY 2. *Suppose that* $f(\cdot, w)$ *is entire and has an exceptional value* $a(w) \in \mathbf{C}$ *for all* w *on some subset* $E \subset D$. *If* $a(w)$ *is bounded on* E *then its restriction on* E *is continuous.*

Example 4 shows that a can be discontinuous.

THEOREM 2. *Let* $f : \mathbf{C} \times D \to \mathbf{C}$ *be a holomorphic function, where* D *is a region in* $\mathbf{C}$ *and* $z \mapsto f(z,w)$ *is not constant for all* $w \in D$. *Assume that, for some function* $a : D \to \mathbf{C}$, *we have* $f(z,w) \neq a(w)$ *for all* $z \in \mathbf{C}$.

Then there exists a discrete set $E \subset D$ *such that* a *is holomorphic in* $D \backslash E$ *and such that* $a(w) \to \infty$ *as* $w \to w_0$ *for every* $w_0 \in E$.

As a result, the singularities of a can only be of the type described in Example 4.

The main ingredient in the proof of Theorem 2 is the following result of Shcherbina [11].

THEOREM A. *Let h be a continuous function in a region $G \in \mathbf{C}^n$. If the graph of h is a pluripolar subset of $\mathbf{C}^{n+1}$, then h is analytic.*

Proof of Theorem 2. Consider the analytic set

$$Q = \{(z, w, a) \in \mathbf{C}_z \times D \times \mathbf{C}_a : f(z, w) - a = 0\}.$$

Let R be the complement of the projection of Q onto $D \times \mathbf{C}_a$. Then it follows from the assumptions of Theorem 2 and Picard's theorem that R is the graph of the function $w \mapsto a(w)$. Hence R is a nonempty proper subset of $D \times \mathbf{C}_a$. Proposition 2 implies that R is closed in $D \times \mathbf{C}_a$ and pluripolar.

It follows from Proposition 3 that $w \mapsto |a(w)|$ is lower semicontinuous, so the sets

$$E_n = \{w \in D : |a(w)| \leq n\}, \quad \text{where } n = 1, 2, 3, \ldots$$

are closed. We have $E_1 \subset E_2 \subset \cdots$, and the assumptions of Theorem 2 imply that $D = \bigcup E_j$. Let E_j^0 be the interiors of E_j; then $E_1^0 \subset E_2^0 \subset \cdots$, and the set $G = \bigcup E_j^0$ is open.

We claim that G is dense in D. Indeed, otherwise there would exist a disc $U \subset D$ disjoint from G. But then the closed sets $E_j \backslash E_j^0$ with empty interiors would cover D, which is impossible by the Baire category theorem. This proves the claim.

Since a is locally bounded on G, Corollary 2 implies that a is continuous in G; hence, by Shcherbina's theorem, a is analytic in G. It follows from the definition of G that a does not have an analytic continuation from any component of G to any boundary point of G. Our goal is to prove that $D \backslash G$ consists of isolated points.

If G has an isolated boundary point w_0, then

$$\lim_{w \to w_0} a(w) = \infty. \tag{5}$$

Indeed, by Proposition 3, the limit set of $a(w)$ as $w \to w_0$ ($w \in G$) consists of at most two points, $a(w_0)$ and ∞. On the other hand, this limit set is connected and so the limit exists. If the limit is finite, then it is equal to $a(w_0)$ and the removable singularity theorem gives an analytic continuation of a to $G \cup w_0$, contradicting our previous statement that there is no such continuation. Therefore, the limit is infinite and (5) holds.

We add to G all its isolated boundary points, thus obtaining a new open set G' containing G. Our function a has a *meromorphic* continuation to G' that we call $\tilde{a}$. This meromorphic continuation coincides with a in G but does not coincide at the added points $G' \backslash G$.

We claim that G' has no isolated boundary points. Indeed, suppose that w_0 is an isolated boundary point of G'. By the same argument as before, the limit (5) exists and is infinite. Then $\tilde{a}$ can be extended to w_0 such that the extended function has a pole at w_0, but then w_0 would be an isolated boundary point of G (poles cannot accumulate to a pole) and so $w_0 \in G'$ by definition of G'. This contradiction proves the claim.

Let F' be the complement of G' with respect to D. Then F' is a closed and nowhere dense subset of D. Furthermore, F' has no isolated points because such points would be isolated boundary points of G'. Hence F' is perfect or empty. Our goal is to prove that F' is empty.

Assume, to the contrary, that F' is perfect. Then the closed sets E_n would cover the locally compact space F' and so, by the Baire category theorem, one of these E_n contains a relatively open part of F'. This means that there exists a positive integer n and an open disc $U \subset D$ intersecting F' and such that

$$|a(w)| \leq n \quad \text{for } w \in U \cap F'. \tag{6}$$

By Corollary 2, this implies that the restriction of a on $U \cap F'$ is continuous.

We shall prove that $\tilde{a}$ has a continuous extension from G' to $U \cap F'$ and that this extension agrees with the restriction of a on $U \cap F'$. Let W be a point of $U \cap F'$, and let (w_k) be a sequence in G' tending to W. Choosing a subsequence, we may assume that there exists a limit

$$\lim_{k\to\infty} \tilde{a}(w_k), \tag{7}$$

finite or infinite. By a small perturbation of the sequence that does not change the limit of $\tilde{a}(w_k)$, we may assume the w_k are not poles of $\tilde{a}$ and so $a(w_k) = \tilde{a}(w_k)$. By Proposition 3, the limit (7) can only be $a(W)$ or ∞.

In order to prove continuity, we must exclude the latter case. So suppose that

$$\lim_{k\to\infty} \tilde{a}(w_k) = \infty. \tag{8}$$

Let C_k be the component of the set

$$G' \cap \{w \in U : |w - W| < 2|w_k - W|\}$$

that contains w_k, and let

$$m_k = \inf\{|\tilde{a}(w)| : w \in C_k\}. \tag{9}$$

We claim that $m_k \to \infty$. Indeed, suppose this is not so; then, by choosing a subsequence, we may assume that $m_k \leq m$ for some $m > n$. Then there exists a curve in C_k connecting w_k to some point $w_k' \in C_k$ such that $|\tilde{a}(w_k')| \leq m + 1$. Because $|a|$ is continuous in C_k, (8) implies that this curve contains a point y_k such that $|\tilde{a}(y_k)| = m + 1$. By selecting another subsequence we obtain

$$\lim_{k\to\infty} \tilde{a}(y_k) = y, \quad \text{where } |y| = m + 1 > n.$$

Since $|a(W)| \leq n$, we obtain a contradiction with Proposition 3, proving our claim that $m_k \to \infty$ in (9).

We can thus assume that

$$m_k \geq n + 1 \ \text{ for all } k. \tag{10}$$

Let us show that this leads to a contradiction. Fix k, and consider the limit set of $\tilde{a}(w)$ as $w \to \partial C_k \cap U$ from $C_k \cap U$. In view of (6), (10), and Proposition 3, this limit set consists of a single point—namely, ∞. To see that this is impossible, we use the following lemma.

LEMMA. *Let V and C be two intersecting regions in $\mathbf{C}$, and let g be a meromorphic function in C such that*

$$\lim_{w \to W,\, w \in C} g(w) = \infty \quad \textit{for all } W \in V \cap \partial C.$$

Then $V \cap \partial C$ consists of isolated points in V, and g has a meromorphic extension from C to $V \cup C$.

Proof. By shrinking V, we may assume that $|g(w)| \geq 1$ for $w \in C \cap V$. Then $h = 1/g$ has a continuous extension from C to $C \cup V$ if we set $h(w) = 0$ for $w \in V \setminus C$. The extended function is holomorphic on the set

$$\{w \in C \cap V : h(w) \neq 0\}$$

and so, by Rado's theorem [10, Thm. 3.6.5], h is analytic in $V \cup C$; hence $1/g$ gives the required meromorphic extension of g. □

Resuming the proof of Theorem 2, we now apply this lemma with

$$C = C_k, \qquad V = \{w \in U : |w - W| < 2|w_k - W|\},$$

and $g = \tilde{a}$ while taking into account that $\partial C_k \cap V \subset F'$ and F' has no isolated points. We thus arrive at a contradiction, which completes the proof that $\tilde{a}$ has a continuous extension to $F' \cap U$ that agrees with a on $F' \cap U$.

By Theorem A, a is analytic on $F' \cap U$, which contradicts the fact (stated in the beginning of the proof) that a has no analytic continuation from G. This contradiction shows that $F' = \emptyset$, proving the theorem. □

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References

[1] A. E. Eremenko and M. Yu. Lyubich, *Structural stability in some families of entire functions,* Funct. Anal. Appl. 19 (1985), 323–324.

[2] ———, *Dynamical properties of some classes of entire functions,* Ann. Inst. Fourier (Grenoble) 42 (1992), 989–1020.

[3] K. Hoffman, *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, NJ, 1962.

[4] L. Hörmander, *An introduction to complex analysis in several variables,* Van Nostrand, Princeton, NJ, 1966.

[5] G. Julia, *Sur le domaine d'existence d'une fonction implicite définie par une relation entière $G(x, y) = 0$,* Bull. Soc. Math. France 54 (1926), 26–37; C. R. Acad. Sci. Paris Sér. I Math. 182 (1926), 556.

[6] P. Lelong, *Sur les valeurs lacunaires d'une relation a deux variables,* Bull. Sci. Math. 66 (1942), 103–108, 112–125.

[7] P. Lelong and L. Gruman, *Entire functions of several complex variables,* Springer-Verlag, Berlin, 1986.

[8] M. Yu. Lyubich, *Investigation of the stability of the dynamics of rational functions,* Teor. Funktsiĭ Funktsional. Anal. i Prilozhen 42 (1984), 72–91 (in Russian); translated in Selecta Math. (N.S.) 9 (1990), 69–90.

[9] T. Nishino, *Sur les valeurs exceptionnelles au sens de Picard d'une fonction entière de deux variables,* J. Math. Kyoto Univ. 2/3 (1963), 365–372.

[10] T. Ransford, *Potential theory in the complex plane,* Cambridge Univ. Press, Cambridge, 1995.

[11] N. V. Shcherbina, *Pluripolar graphs are holomorphic,* Acta Math. 194 (2005), 203–216.

[12] M. Tsuji, *On the domain of existence of an implicit function defined by an integral relation* $G(x, y) = 0$, Proc. Imp. Acad. Tokyo 19 (1943), 235–240.

[13] ———, *Potential theory in modern function theory,* Maruzen, Tokyo, 1959.

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