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On the Homotopy Lie Algebra of an Arrangement

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1. Definitions and Statement of Results

1.1. Holonomy and Homotopy Lie Algebras

Fix a field $\Bbbk$ of characteristic 0. Let A be a graded, graded-commutative algebra over $\Bbbk$ with graded piece A_k, $k \geq 0$. We will assume throughout that A is locally finite, connected, and generated in degree 1. In other words, $A = T(V)/I$, where V is a finite-dimensional $\Bbbk$-vector space, $T(V) = \bigoplus_{k\geq 0} V^{\otimes k}$ is the tensor algebra on V, and I is a two-sided ideal generated in degrees 2 and higher. To such an algebra A, one naturally associates two graded Lie algebras over $\Bbbk$ (see e.g. [3; 14]).

Definition 1.1. The *holonomy Lie algebra* $\mathfrak{h}_A$ is the quotient of the free Lie algebra on the dual of A_1 modulo the ideal generated by the image of the transpose of the multiplication map $\mu\colon A_1 \wedge A_1 \to A_2$; thus,

$$\mathfrak{h}_A = \mathrm{Lie}(A_1^*)/\mathrm{ideal}(\mathrm{im}(\mu^*\colon A_2^* \to A_1^* \wedge A_1^*)). \tag{1}$$

Note that $\mathfrak{h}_A$ depends only on the quadratic closure of A: if we put $\bar{A} = T(V)/(I_2)$ then $\mathfrak{h}_A = \mathfrak{h}_{\bar{A}}$.

Definition 1.2. The *homotopy Lie algebra* $\mathfrak{g}_A$ is the graded Lie algebra of primitive elements in the Yoneda algebra of A:

$$\mathfrak{g}_A = \mathrm{Prim}(\mathrm{Ext}_A(\Bbbk,\Bbbk)). \tag{2}$$

In other words, the universal enveloping algebra of the homotopy Lie algebra is the Yoneda algebra:

$$U(\mathfrak{g}_A) = \mathrm{Ext}_A(\Bbbk,\Bbbk). \tag{3}$$

The algebra $U = \mathrm{Ext}_A(\Bbbk,\Bbbk)$ is a bigraded algebra; let us write U^{pq} to denote cohomological degree p and polynomial degree q. Then $U^{pq} = 0$ unless $-q \geq p$. The subalgebra $R = \bigoplus_{p\geq 0} U^{p,-p}$ is called the *linear strand* of U. For convenience we will let $U_q^p = U^{p,-p-q}$, where the lower index q is called the *internal*

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degree. Then U is a graded R-algebra with $R = U_0$. Note that $U_+ = \bigoplus_{q>0} U_q$ is an ideal in U with $U/U_+ \cong R$.

The relationship between the holonomy and homotopy Lie algebras of A is provided by the following well-known result.

LEMMA 1.3 (Löfwall [19]). *The universal enveloping algebra of the holonomy Lie algebra, $U(\mathfrak{h}_A)$, equals the linear strand, $R = \bigoplus_{p\geq 0} U_0^p$, of the Yoneda algebra $U = U(\mathfrak{g}_A)$.*

Particularly simple is the case when A is a Koszul algebra. By definition this means that the homotopy Lie algebra $\mathfrak{g}_A$ coincides with the holonomy Lie algebra $\mathfrak{h}_A$ (i.e., $U = R$). Alternatively, A is quadratic (i.e., $A = \bar{A}$) and its quadratic dual, $A^! = T(V)/(I_2^\perp)$, coincides with the Yoneda algebra: $A^! = U$. For an expository account of Koszul algebras, see [13].

As a simple (yet basic) example, take $E = \bigwedge V$, the exterior algebra on V. Then E is Koszul and its quadratic dual is $E^! = \mathrm{Sym}(V^*)$, the symmetric algebra on the dual vector space. Moreover, $\mathfrak{g}_A = \mathfrak{h}_A$ is the abelian Lie algebra on V.

1.2. Main Result

The computation of the homotopy Lie algebra of a given algebra A is, in general, a very hard problem. Our goal here is to determine $\mathfrak{g}_A$ under certain homological hypotheses. First, we need one more definition.

Let $B = \bar{A}$ be the quadratic closure of A. View $J = \ker(B \twoheadrightarrow A)$ as a graded left module over B.

DEFINITION 1.4. The *homotopy module* of a graded algebra A is

$$M_A = \mathrm{Ext}_B(J, \Bbbk) \tag{4}$$

viewed as a bigraded left module over the ring $R = U(\mathfrak{h}_A) = \mathrm{Ext}_B(\Bbbk, \Bbbk)$ via the Yoneda product.

THEOREM 1.5. *Let A be a graded algebra over a field $\Bbbk$ with quadratic closure $B = \bar{A}$ and homotopy module $M = M_A$. Assume B is a Koszul algebra and assume there exists an integer ℓ such that $M_q = 0$ unless $\ell \leq q \leq \ell + 1$. Then, as graded Hopf algebras,*

$$U(\mathfrak{g}_A) \cong T(M_A[-2]) \mathbin{\widehat{\otimes}_{\Bbbk}} U(\mathfrak{h}_A). \tag{5}$$

Here $M[q]$ is the graded R-module with $M[q]^r = M^{q+r}$; $T(M[-2]) \mathbin{\widehat{\otimes}_{\Bbbk}} R$ is the "twisted" tensor product of algebras with underlying vector space $T(M[-2]) \otimes_{\Bbbk} R$ and multiplication $(m \otimes r) \cdot (n \otimes s) = (-1)^{|r||n|}((m \otimes n) \otimes rs + (m \otimes nr) \otimes s)$. In turn, R acts on $T(M[-2])$ by extending its action on $M[-2]$ via the Leibniz rule.

Taking the Lie algebras of primitive elements in the respective Hopf algebras, we obtain the following.

COROLLARY 1.6. *Given Theorem 1.5, the homotopy Lie algebra of A splits as a semi-direct product of the holonomy Lie algebra with the free Lie algebra on the (shifted) homotopy module*:

$$\mathfrak{g}_A \cong \mathrm{Lie}(M_A[-2]) \rtimes \mathfrak{h}_A, \tag{6}$$

where the action of $\mathfrak{h}$ *on* $\mathrm{Lie}(M)$ *is given by* $[h, m] = hm$ *for* $h \in \mathfrak{h}$ *and* $m \in M$.

As pointed out to us by S. Iyengar, Theorem 1.5 implies that, in the case we consider, the projection map $U(\mathfrak{g}_A) \to U(\mathfrak{h}_A)$ is a Golod homomorphism. Therefore, the semi-direct product structure of $\mathfrak{g}_A$ also follows from results of Avramov [1; 2].

1.3. Hyperplane Arrangements

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$ with intersection lattice $L(\mathcal{A})$ and complement $X(\mathcal{A})$. The cohomology ring $A = H^\bullet(X(\mathcal{A}), \Bbbk)$ admits a combinatorial description (in terms of $L(\mathcal{A})$) due to Orlik and Solomon:

$$A = E/I, \tag{7}$$

where E is the exterior algebra over $\Bbbk$ on generators $e_1, \dots, e_n$ in degree 1 and where I is the ideal generated by all elements of the form $\sum_{q=1}^{r} (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$ for which $\mathrm{rk}(H_{i_1} \cap \cdots \cap H_{i_r}) < r$; see [22].

The holonomy Lie algebra of the Orlik–Solomon (OS) algebra also admits an explicit presentation, this time solely in terms of $L_{\le 2}(\mathcal{A})$. Identify $\mathrm{Lie}(A_1^*)$ with the free Lie algebra over $\Bbbk$ on generators $x_H = e_H^*$, $H \in \mathcal{A}$. Then

$$\mathfrak{h}_A = \mathrm{Lie}(A_1^*) \Big/ \mathrm{ideal}\left\{ \left[x_H, \sum_{H' \in \mathcal{A} \,:\, H' \supset F} x_{H'} \right] \,\middle|\, F \in L_2(\mathcal{A}) \text{ and } F \subset H \right\}. \tag{8}$$

As shown in Section 5, the homotopy Lie algebra $\mathfrak{g}_A$ also admits a finite presentation for a certain class of hypersolvable arrangements that we shall define.

QUESTION 1.7. Do there exist arrangements for which $\mathfrak{g}_A$ is not finitely presented or for which the (bigraded) Hilbert series of $U(\mathfrak{g}_A)$ is not a rational function?

1.4. Hypersolvable Arrangements

An arrangement $\mathcal{A}$ is called *supersolvable* if its intersection lattice admits a maximal modular chain. The OS algebra of a supersolvable arrangement has a quadratic Gröbner basis and thus is a Koszul algebra (this result, implicit in Björner and Ziegler [4], was proved in Shelton and Yuzvinsky [30]).

An arrangement $\mathcal{A}$ is called *hypersolvable* if it has the same intersection lattice up to rank 2 as that of a supersolvable arrangement. This "supersolvable deformation", $\mathcal{B}$, is uniquely defined and has the property that the two complements have isomorphic fundamental groups (see Jambu and Papadima [16; 17]). Let $A = H^\bullet(X(\mathcal{A}), \Bbbk)$ and $B = H^\bullet(X(\mathcal{B}), \Bbbk)$ be the respective OS algebras. It is readily seen that $B = \bar{A}$; thus, A and B share the same holonomy Lie algebra: $\mathfrak{h} = \mathfrak{h}_A = \mathfrak{h}_B$. Furthermore, since B is Koszul it follows that $\mathfrak{g}_B = \mathfrak{h}$.

The hypotheses of Theorem 1.5 hold in two nice situations, which can be checked combinatorially; see Sections 4.2 and 4.3 for precise definitions.

THEOREM 1.8. *Let* $\mathcal{A}$ *be an arrangement, and let* A *be its Orlik–Solomon algebra. Suppose*:

(i) *$\mathcal{A}$ is hypersolvable and its singular range has length* 0 *or* 1; *or*
(ii) *$\mathcal{A}$ is obtained by fibered extensions of a generic slice of a supersolvable arrangement.*

Then $\mathfrak{g}_A \cong \mathrm{Lie}(M_A[-2]) \rtimes \mathfrak{h}_A$.

An explicit finite presentation for $\mathfrak{g}_A$ is given in Theorem 5.1 for the case when $\mathcal{A}$ is a generic slice of a supersolvable arrangement. The Eisenbud–Popescu–Yuzvinsky resolution [8] permits us to compute the Hilbert series of M_A (and hence that of $\mathfrak{g}_A$) for the case when $\mathcal{A}$ is a 2-generic slice of a Boolean arrangement.

Theorem 1.8 allows us to distinguish between hyperplane arrangements whose holonomy Lie algebras are isomorphic. In Example 6.2, we exhibit a pair of 2-generic 4-dimensional sections of the Boolean arrangement in $\mathbb{C}^7$; the two arrangements have the same fundamental group, the same Poincaré polynomial, and the same holonomy Lie algebra, yet they have different homotopy Lie algebras.

In Section 7 we provide some topological interpretations. As noted in [5; 24], the holonomy Lie algebra of a supersolvable arrangement equals (up to a rescaling factor) the topological homotopy Lie algebra of the corresponding "redundant" subspace arrangement. We extend this result, relating the homotopy Lie algebra of an arbitrary hyperplane arrangement to the topological homotopy Lie algebras of the redundant subspace arrangement. As a consequence, we find a pair of codimension-2 subspace arrangements in $\mathbb{C}^8$ whose complements are simply connected and have the same homology groups yet have distinct higher homotopy groups.

2. Some Homological Algebra

2.1. The Homotopy Module

Let A be graded, graded-commutative, connected, locally finite algebra. Assume that A is generated in degree 1 and that its quadratic closure, $B = \bar{A}$, is a Koszul algebra. Let E be the exterior algebra on $A_1 = B_1$. Let I and J be the kernels of the respective natural surjections $E \twoheadrightarrow B$ and $B \twoheadrightarrow A$, giving the exact sequences

$$0 \longrightarrow I \longrightarrow E \longrightarrow B \longrightarrow 0, \tag{9}$$

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0. \tag{10}$$

In what follows we will record some homological properties of the ring A when viewed as a B-module. Recall that, if N is a B-module, then the Yoneda product gives $\mathrm{Ext}_B(N, \Bbbk)$ the structure of a left module over the ring $R = U(\mathfrak{h}_A) = \mathrm{Ext}_B(\Bbbk, \Bbbk)$. An object of primary interest for us will be the *homotopy module* of A:

$$M = M_A = \mathrm{Ext}_B(J, \Bbbk). \tag{11}$$

This bigraded R-module will play a crucial role in the determination of the homotopy Lie algebra $\mathfrak{g}_A$.

Our grading conventions shall be as follows. Suppose V and W are $\mathbb{Z}$-graded $\Bbbk$-vector spaces. Then $f \in \mathrm{Hom}_{\Bbbk}(V, W)$ has degree r if $f\colon V^q \to W^{q+r}$ for all

q. For any $\mathbb{Z}$-graded $\Bbbk$-vector space V we shall let V^* denote the graded $\Bbbk$-dual of V. In particular, then, $(V^*)^q = \mathrm{Hom}_{\Bbbk}(V^{-q}, \Bbbk)$. If V has finite $\Bbbk$-dimension in each graded piece, then $(V^*)^* \cong V$.

We shall treat all boundary maps in chain complexes as having polynomial degree 0 and homological degree +1. Then chain complexes will be regarded as cochain complexes in negative degree. We shall indicate shifts of polynomial grading by defining $V(q)^r = V^{q+r}$ and (analogously) shifts of homological grading by $V[q]$.

Following these conventions, $M^{pq} = \mathrm{Ext}_B^p(J, \Bbbk)^q$ is nonzero only for $q \leq -p$. Then, taking $M_q^p = M^{p,-p-q}$ (the internal grading), we have $M_q^p \neq 0$ only for $q \geq 0$. The grading is such that, for each fixed q, the action of R on M satisfies $R^r \otimes M_q^p \to M_q^{r+p}$.

LEMMA 2.1. $\mathrm{Ext}_B(A, \Bbbk) \cong \Bbbk \oplus M[-1]$ *as graded R-modules.*

Proof. Consider the long exact sequence for $\mathrm{Ext}_B(-, \Bbbk)$ applied to (10):

$$\cdots \longrightarrow \mathrm{Ext}_B^{q-1}(J,k) \longrightarrow \mathrm{Ext}_B^q(A,k) \longrightarrow \mathrm{Ext}_B^q(B,k) \longrightarrow \cdots. \qquad (12)$$

Since $\mathrm{Ext}_B^0(A, \Bbbk) \cong \mathrm{Ext}_B^0(B, \Bbbk) \cong \Bbbk$ and $\mathrm{Ext}_B^q(B, \Bbbk) = 0$ for all $q > 0$, it follows that the map $\mathrm{Ext}_B(B, \Bbbk) \to \mathrm{Ext}_B(J, \Bbbk)$ is zero. Hence the long exact sequence breaks into short exact sequences that we will write, using (11), as a single short exact sequence of graded R-modules:

$$0 \longrightarrow M[-1] \longrightarrow \mathrm{Ext}_B(A, \Bbbk) \longrightarrow \Bbbk \longrightarrow 0. \qquad (13)$$

For each q, one of the two maps is zero and the other is an isomorphism, so the short exact sequence splits. □

2.2. *Injective Resolutions*

For any E-module N, let

$$N^\circ = \{a \in E : ax = 0 \text{ for all } x \in N\}, \qquad (14)$$

the annihilator of N in E. Later we shall require explicit injective resolutions.

LEMMA 2.2. *Suppose the ring $B = E/I$ is an arbitrary quotient of a finitely generated exterior algebra E. If*

$$0 \longleftarrow \Bbbk \longleftarrow B \otimes_{\Bbbk} F^0 \longleftarrow B \otimes_k F^1 \longleftarrow \cdots \qquad (15)$$

is a minimal free resolution of $\Bbbk$ over B, then

$$0 \longrightarrow \Bbbk \longrightarrow B^* \otimes_{\Bbbk} (F^0)^* \longrightarrow B^* \otimes_k (F^1)^* \longrightarrow \cdots \qquad (16)$$

is an injective resolution of $\Bbbk$ over B.

Proof. The resolution (15) is an acyclic complex of E-modules and so its vector space dual (16) is an acyclic complex as well, since each F^i has finite $\Bbbk$-dimension.

Now $B^* \cong I^\circ(n)$ as E-modules via the determinantal pairing in E. On the other hand, E is injective as a module over itself and so I° is injective as an E-module;

see [29, Prop. 2.27]. Since each F^i has finite $\Bbbk$-dimension, each $B^* \otimes_{\Bbbk} (F^i)^*$ is injective. □

LEMMA 2.3. *Let A and B be two algebras with $\bar{A} = B$ Koszul. Write $B = E/I$, $A = B/J$, $\mathfrak{h} = \mathfrak{h}_A = \mathfrak{h}_B$, and $R = U(\mathfrak{h})$. Then*:

(i) *the complex*

$$0 \longrightarrow \Bbbk \longrightarrow I^\circ(n) \otimes_{\Bbbk} R^0 \longrightarrow I^\circ(n) \otimes_{\Bbbk} R^1 \longrightarrow \cdots \tag{17}$$

is an injective resolution of $\Bbbk$ over B with boundary map as described in the proof; *and*

(ii) $\mathrm{Ext}^q_B(A, \Bbbk) \cong H^q(J^\circ(n) \otimes_{\Bbbk} R^\bullet)$ *for all $q \geq 0$.*

Proof. The Koszul complex $\mathcal{K}^* = B \otimes_{\Bbbk} R^*$ is a free B-module resolution of $\Bbbk$, so it is also an acyclic complex of E-modules with boundary map induced from

$$\partial^*\colon 1 \otimes x_i^* \mapsto e_i \otimes 1. \tag{18}$$

Then $\mathrm{Hom}_{\Bbbk}(B \otimes_{\Bbbk} R^*, \Bbbk) = B^* \otimes_{\Bbbk} R$ is an injective resolution by Lemma 2.2.

To establish (2), it suffices to note that $\mathrm{Hom}_B(A, I^\circ) \cong J^\circ$. □

3. Proof of the Main Result

Our approach to the proof of Theorem 1.5 is to construct a spectral sequence comparing the minimal resolution and the Koszul complex of A. In Proposition 3.2 we show that the spectral sequence collapses at E_2 under suitable hypotheses though not in general (Example 3.3). This collapsing is enough to prove the theorem via Proposition 3.1.

3.1. A Spectral Sequence

Using the previous notation, $A \otimes_{\Bbbk} U^* \to \Bbbk \to 0$ is a minimal free resolution of $\Bbbk$ over A. It is filtered by degree, and the linear strand is $A \otimes_{\Bbbk} R^*$. That is, there exists a short exact sequence of chain complexes

$$0 \longrightarrow A \otimes_{\Bbbk} R^* \xrightarrow{1\otimes\varepsilon^*} A \otimes_{\Bbbk} U^* \longrightarrow A \otimes_{\Bbbk} U_+^* \longrightarrow 0. \tag{19}$$

Now $B \otimes_{\Bbbk} R^*$ is a free resolution of $\Bbbk$ over B, since B is Koszul. Using Lemma 2.1, we find that the homology of the linear strand (Koszul complex) is

$$\begin{aligned} H_\bullet(A \otimes R^*) &\cong \mathrm{Tor}^B(A, \Bbbk) \\ &\cong \mathrm{Ext}_B(A, \Bbbk)^* \\ &\cong \Bbbk \oplus M[-1]^*. \end{aligned} \tag{20}$$

The long exact sequence in homology then reveals that

$$H_\bullet(A \otimes_{\Bbbk} U_+^*) \cong M[-2]^* \tag{21}$$

as A-modules. Recall that A acts trivially on M (and hence on $M[-2]^*$), so

$$\mathrm{Hom}_A(H_\bullet(A \otimes_{\Bbbk} U_+^*), \Bbbk) \cong M[-2]. \tag{22}$$

On the other hand, since our complex is a quotient of a minimal resolution,

$$H_\bullet(\mathrm{Hom}_A(A \otimes_{\Bbbk} U_+^*, \Bbbk)) \cong U_+. \tag{23}$$

Comparing (22) and (23) gives a universal coefficients spectral sequence of the form

$$E_2^{pq} = \mathrm{Ext}_A^p((M[-2]^*)^q, \Bbbk) \cong M[-2]^q \otimes_{\Bbbk} U^p \implies U_+^{p+q}. \tag{24}$$

The spectral sequence is used as follows.

PROPOSITION 3.1. *If $E_\infty = E_2$ in the spectral sequence* (24), *then*

$$0 \longrightarrow M[-2] \otimes_{\Bbbk} U \xrightarrow{\ \phi\ } U \xrightarrow{\ \varepsilon\ } R \longrightarrow 0$$

is an exact sequence of (*left*) *R-modules, and the conclusion of Theorem 1.5 holds.*

Proof. If $E_\infty = E_2$ then $M[-2] \otimes U \cong U_+$ as a (left) R-module. Now $U_+ = \ker \varepsilon$, giving the short exact sequence. Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, it follows that $R = U(\mathfrak{h})$ is a Hopf subalgebra of $U = U(\mathfrak{g})$ and so the sequence splits. The isomorphism of Theorem 1.5 can then be obtained by induction. □

3.2. Collapsing Conditions

In order to show that the higher differentials in the spectral sequence (24) vanish, we use a degree argument that begins by considering the E_0 term. Since

$$0 \longrightarrow \Bbbk \longrightarrow A^* \otimes_{\Bbbk} U^0 \longrightarrow A^* \otimes_{\Bbbk} U^1 \longrightarrow \cdots \tag{25}$$

is an injective resolution of $\Bbbk$ over A (Lemma 2.2), we consider the double complex

$$\begin{aligned} C^{pq} &= \mathrm{Hom}_A(A \otimes_{\Bbbk} (U^q)_+^*, A^* \otimes_{\Bbbk} U^p) \\ &\cong U_+^q \otimes_{\Bbbk} A^* \otimes_{\Bbbk} U^p, \end{aligned} \tag{26}$$

with induced boundary maps ∂_h and ∂_v. Then our spectral sequence (24) is obtained by filtering $C^{\bullet\bullet}$ by columns. Checking the grading, we see that

$$\partial_v \colon U_+^q \otimes_{\Bbbk} (A^*)^s \otimes_{\Bbbk} U^p \to U_+^{q+1} \otimes_{\Bbbk} (A^*)^{s+1} \otimes_{\Bbbk} U^p \tag{27}$$

and

$$\partial_h \colon U_+^q \otimes_{\Bbbk} (A^*)^s \otimes_{\Bbbk} U^p \to U_+^q \otimes_{\Bbbk} (A^*)^{s+1} \otimes_{\Bbbk} U^{p+1}. \tag{28}$$

By inspection of E_2 and ∂_v we see that necessarily $E_1 = E_2$.

We first consider the case where the ideal J has a (shifted) linear resolution.

PROPOSITION 3.2. *Suppose $\mathcal{A}$ is a hypersolvable arrangement for which $M_q^p = 0$ unless $q = \ell$ for some fixed ℓ. Then $E_2 = E_\infty$.*

Proof. In this case, $M[-2]_r^q = 0$ unless $r = \ell - 2$. Then $H^q(C^{p\bullet}, \partial_v)_r = 0$ unless $r = \ell - 2$.

First we observe that $(U_+)_t^q = 0$ unless $t \geq \ell - 2$. This follows because, by (24), U_+ is a graded subquotient of $M[-2] \otimes_{\Bbbk} U$; the support of $M[-2]$ is as just described, and $U_q^p = 0$ unless $q \geq 0$.

Regard A^* as a chain complex concentrated in homological degree 0. Then note that, by our first observation, the internal degree of a nontrivial cocycle representative in $(U_+)^q_t \otimes_{\Bbbk} (A^*)_s$ is $s + t = \ell - 2$, and it follows that $s \leq 0$. However, $(A^*)_s = 0$ unless $0 \leq s \leq \ell$, so the representative of a nonzero homogeneous E_2-cocycle in E_0 must have $s = 0$.

Now suppose that $x \in E_2^{pq}$ is such a cocycle with representative $\tilde{x}$ in C^{pq}. The foregoing implies that $\tilde{x} \in U_+^q \otimes_{\Bbbk} (A^*)_0$, so $\partial_h(\tilde{x}) = 0$ in $C^{p+1,q}$ by (28). This means that $d_2(x) = 0$, and similarly for higher differentials.

Proof of Theorem 1.5. Given Proposition 3.1, we need only show that the spectral sequence collapses when $M_q^p = 0$ unless $0 \leq \ell - q \leq 1$ for some ℓ. In this case, let $N = M_\ell$ denote the R-submodule of M of internal degree ℓ.

By the same reasoning as in the proof of Proposition 3.2, $N[-2] \otimes U \subseteq \ker d_k$ for $k \geq 2$. Now $N[-2] \cong N[-2] \otimes U^0$ is a submodule of the $p = 0$ column of E_2. Since $N[-2]$ is (trivially) not in the image of any nonzero differentials, it must be an R-submodule of U.

Let K denote the Hopf subalgebra of U generated by R and $N[-2]$. By [20, Thm. 4.4], U is a free K-algebra. It follows that $K \cong T(N[-2]) \otimes_{\Bbbk} R$. In the notation of Proposition 3.2, any nontrivial differential d_k with $k \geq 2$ would lift in E_0 to a map $U_+ \otimes (A^*)_1 \otimes U \to U_+ \otimes (A^*)_0 \otimes U$. We have shown that the targets of these maps are unchanged between E_2 and E_∞, so it follows that also the maps themselves must all be zero.

3.3. A Noncollapsing Spectral Sequence

Calculations with the Macaulay 2 package [15] show that the hypotheses of Theorem 1.5 cannot in general be relaxed: differentials in the spectral sequence (24) may not be zero.

EXAMPLE 3.3. Consider the arrangement defined by the polynomial

$$Q = xyz(x - w)(y - w)(z - w)(x - u)(y - u).$$

Let A be the Orlik–Solomon algebra and $M = M_A$ its homotopy module. It is readily seen that $M_q \neq 0$ for $q = 3, 4, 5$. An Euler characteristic calculation shows that the spectral sequence (24) must have a nonzero differential

$$d_2^{04} \colon M[-2]_6^4 \otimes_{\Bbbk} U^0 \to M[-2]_5^3 \otimes_{\Bbbk} U^2.$$

It follows that the Hopf algebra $U(\mathfrak{g}_A)$ will not have the structure we find in Theorem 1.5.

4. Hypersolvable Arrangements

In this section, we apply our main result to certain classes of hypersolvable arrangements.

4.1. Solvable Extensions

We start by reviewing in more detail the notion of a hypersolvable arrangement, introduced by Jambu and Papadima in [16]. Roughly, a hypersolvable arrangement is a linear projection of a supersolvable arrangement that preserves intersections through codimension 2.

DEFINITION 4.1 [16]. An arrangement $\mathcal{A}$ is *hypersolvable* if there exist subarrangements $\{0\} = \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_m = \mathcal{A}$ such that each inclusion $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is solvable. In turn, an inclusion of hyperplane arrangements $\mathcal{A} \subset \mathcal{B}$ is called a *solvable extension* if:

(i) there are no hyperplanes $H \in \mathcal{B} \setminus \mathcal{A}$ and $H', H'' \in \mathcal{A}$ with $H' \neq H''$ and $\operatorname{rk}(H \cap H' \cap H'') = 2$;
(ii) for any $H, H' \in \mathcal{B} \setminus \mathcal{A}$, there is exactly one $H'' \in \mathcal{A}$ with $\operatorname{rk}(H \cap H' \cap H'') = 2$, denoted by $f(H, H')$; and
(iii) for any $H, H', H'' \in \mathcal{B} \setminus \mathcal{A}$, one has $\operatorname{rk}(f(H, H') \cap f(H, H'') \cap f(H', H'')) \leq 2$.

It turns out that, if $\mathcal{A}$ is hypersolvable with a sequence of solvable extensions as defined here, then for all i the rank of $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ differ by at most 1. If the ranks are equal, the extension is said to be singular; otherwise, the extension is nonsingular (or fibered in the sense of Falk and Randell [11]).

If s denotes the number of singular extensions, then $\operatorname{rk} \mathcal{A} = m - s$. One can replace the singular extensions by nonsingular ones in order to construct a supersolvable arrangement $\mathcal{B}$ of rank m that projects onto $\mathcal{A}$, preserving the intersection lattice through rank 2. This was shown by Jambu and Papadima [17] as follows.

THEOREM 4.2. *An arrangement $\mathcal{A}$ is hypersolvable if and only if there exists a supersolvable arrangement $\mathcal{B}$ and a linear subspace W for which $\mathcal{A} = \mathcal{B} \cap W$ and $L(\mathcal{A})_{\leq 2} \cong L(\mathcal{B})_{\leq 2}$.*

Proof. The “only if” claim is Theorem 2.4 of [17]. The “if”, due to Jambu (private communication), runs as follows. Suppose $\mathcal{B}$ is supersolvable and there exists a subspace W as described in the theorem. By definition, $\mathcal{B}$ has a maximal modular chain $F_1 < F_2 < \cdots < F_m$. Putting $\mathcal{B}_i = \mathcal{B}_{X_i}$ yields a sequence of solvable extensions for $\mathcal{B}$, all of which are fibered. For $1 \leq i \leq m$, let $\mathcal{A}_i = \mathcal{B}_i \cap W$. Since collinearity relations are preserved, each $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is also a solvable extension and so $\mathcal{A}$ is hypersolvable. □

We remark that, in the above proof, $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is a singular extension if and only if $F_i \cap W = F_{i+1} \cap W$. The arrangement $\mathcal{B}$ is called the *supersolvable deformation* of $\mathcal{A}$. For example, any arrangement $\mathcal{A}$ for which no three hyperplanes intersect in codimension 3 is hypersolvable, and its supersolvable deformation is the Boolean arrangement in $\mathbb{C}^n$, where $n = |\mathcal{A}|$.

LEMMA 4.3. *Suppose $\mathcal{A}' \subset \mathcal{A}$ is a fibered extension. The projection $p\colon X(\mathcal{A}) \to X(\mathcal{A}')$ induces an inclusion $A' \hookrightarrow A$ of the respective Orlik–Solomon algebras that makes A into a free A'-module of rank $k = |\mathcal{A} \setminus \mathcal{A}'|$.*

Proof. The projection $p\colon X \to X'$ is a bundle map with fiber $\mathbb{C} \setminus \{k \text{ points}\}$. As noted by Falk and Randell [11], this bundle admits a section and thus the Serre spectral sequence collapses at the E_2 term. Hence $H^\bullet(X) \cong H^\bullet(X') \otimes H^\bullet\big(\bigvee^k S^1\big)$. The result follows. □

4.2. *Singular Range*

We now give some easy-to-check combinatorial conditions ensuring that a hypersolvable arrangement satisfies the hypotheses of Theorem 1.5. We start by attaching a pair of relevant integers to a hypersolvable arrangement.

DEFINITION 4.4. Suppose that $\mathcal{A}$ is hypersolvable with supersolvable deformation $\mathcal{B}$ and that $\mathcal{A} \neq \mathcal{B}$. Let c be the least integer for which $L(\mathcal{A})_{\leq c} \not\cong L(\mathcal{B})_{\leq c}$. Since $\mathcal{A} \neq \mathcal{B}$, there is a largest integer i for which the extension $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is singular. Let d the rank of these two arrangements. We will call the pair (c, d) the *singular range* of the arrangement $\mathcal{A}$ and call $|d - c|$ the *length* of this range.

LEMMA 4.5. *If $\mathcal{A}$ is hypersolvable with singular range (c, d), then $3 \leq c \leq d$.*

Proof. The inequality $c \geq 3$ follows from Theorem 4.2. Suppose $d < c$; then $L(\mathcal{A})_{\leq d} \cong L(\mathcal{B})_{\leq d}$. It follows that $L(\mathcal{A}_{d+1})_{\leq d} \cong L(\mathcal{B}_{d+1})_{\leq d}$, whence $\mathcal{A}_{d+1} = \mathcal{B}_{d+1}$ because the arrangements are central. However, since d is greater than or equal to the index of the last singular extension, it follows that $\mathcal{A}_i = \mathcal{B}_i$ for $d+1 \leq i \leq m$ and so $\mathcal{A} = \mathcal{B}$, a contradiction. □

Let $A = H^\bullet(X(\mathcal{A}), \Bbbk)$ and $B = H^\bullet(X(\mathcal{B}), \Bbbk)$ be the respective Orlik–Solomon algebras. Since $L(\mathcal{A})_{\leq 2} \cong L(\mathcal{B})_{\leq 2}$ and since the OS algebra of a supersolvable arrangement is quadratic, the algebra $B = E/I$ is the quadratic closure of A. Let $J = \ker(B \twoheadrightarrow A)$, and let $M = \operatorname{Ext}_B(J, \Bbbk)$ viewed as a module over $R = \operatorname{Ext}_B(\Bbbk, \Bbbk)$. Since $\mathcal{B}$ is supersolvable, the algebra B is Koszul (see [30]); thus, $R = B^!$.

LEMMA 4.6. *If $\mathcal{A}$ is a hypersolvable arrangement with singular range (c, d), then $M^p_q = 0$ unless $p \geq 0$ and $c \leq q \leq d$.*

Proof. The ideal J has a minimal (infinite) free resolution over B of the form

$$0 \longleftarrow J \longleftarrow B \otimes_\Bbbk (M^{0,-})^* \longleftarrow B \otimes_\Bbbk (M^{1,-})^* \longleftarrow \cdots. \tag{29}$$

Recall that J is generated by Orlik–Solomon relations. By Definition 4.4, the least degree of a generator of J is c; hence $M^0_c \neq 0$ and $M^0_q = 0$ for $q < c$. Thus $M^p_q = 0$ for $q < c$, establishing the first inequality.

In order to show that $M^p_q = 0$ for $q > d$, too, let i be the largest index of a singular extension $\mathcal{A}_i \subset \mathcal{A}_{i+1}$. Let $B_{i+1} = H^\bullet(X(\mathcal{B}_{i+1}), \Bbbk)$ and $A_{i+1} = H^\bullet(X(\mathcal{A}_{i+1}), \Bbbk)$, and let $B'_{i+1} = H^\bullet(X(\mathcal{B}'_{i+1}), \Bbbk)$ be the cohomology ring of the projectivization (decone) of $\mathcal{B}_{i+1}$. Recall from [22] that $X(\mathcal{B}_{i+1}) = X(\mathcal{B}'_{i+1}) \times \mathbb{C}^\times$. From the Künneth formula we obtain the following exact sequence of B'_{i+1}-modules:

$$0 \longrightarrow B'_{i+1} \longrightarrow B_{i+1} \longrightarrow B'_{i+1}(-1) \longrightarrow 0. \tag{30}$$

Let J_{i+1} denote the kernel of the canonical projection $B_{i+1} \twoheadrightarrow A_{i+1}$. If we let $J' = J_{i+1} \cap B'_{i+1}$ then $J_{i+1} = B_{i+1} \otimes_{B'_{i+1}} J'$ as a module over B_{i+1}. Since $\mathcal{A}, \mathcal{B}$ are obtained from $\mathcal{A}_{i+1}, \mathcal{B}_{i+1}$ (respectively) by a sequence of fibered extensions, we have $J = B \otimes_{B_{i+1}} J_{i+1}$.

On the other hand, B_{i+1} is a free module over B'_{i+1}, and applying Lemma 4.3 inductively shows that B is free over B_{i+1}. Hence $B'_{i+1} \to B$ is a flat change of rings, and it is enough to check that

$$\mathrm{Ext}^p_{B'_{i+1}}(J', \Bbbk)_q = 0 \tag{31}$$

if $q > d$. By Lemma 2.1, $\mathrm{Ext}^p_{B'_{i+1}}(J', \Bbbk)_q = \mathrm{Ext}^{p+1}_{B'_{i+1}}(A'_{i+1}, \Bbbk)_{q-1}$. Since B'_{i+1} is Koszul and since $(A'_{i+1})_q = 0$ for $q > d-1$, the rank of the arrangement, it follows by [18, Lemma 2.2] that the groups (31) are zero for $q > d$. □

Lemma 4.1 means in particular that the B-module $J(-c)$ has Castelnuovo–Mumford regularity no greater than the length of the singular range, $d - c$. Moreover, the lemma gives a combinatorial condition for the hypotheses of Theorem 1.5 to be satisfied.

Corollary 4.7. *If $\mathcal{A}$ is hypersolvable and its singular range has length* 0 *or* 1, *then* $\mathfrak{g}_A \cong \mathrm{Lie}(M[-2]) \rtimes \mathfrak{h}_A$.

Example 4.8 (2-generic arrangements of rank 4). Suppose $\mathcal{A}$ is a central arrangement in $\mathbb{C}^4$ with the property that no three hyperplanes contain a common plane. Such an arrangement is hypersolvable, by Theorem 4.2, with supersolvable deformation $\mathcal{B}$ a Boolean arrangement. From Definition 4.4 and Lemma 4.5 we have $3 \le c \le d \le 4$, so the singular range has length 0 or 1.

On the other hand, the arrangement from Example 3.3 is hypersolvable with singular range $(3, 5)$, and Corollary 4.7 does not apply (indeed, its conclusion fails).

4.3. Generic Slices of Supersolvable Arrangements

Lemma 4.6 provides bounds on the polynomial degrees of the homotopy module M, bounds that cannot be improved without imposing further restrictions on the arrangement. In general, it is not obvious how to characterize the support of M combinatorially; the problem seems similar to that of characterizing which arrangements have quadratic defining ideals (see [6; 10]). Toward this end, we isolate a class of hypersolvable arrangements for which the situation is more manageable.

Definition 4.9. A codimension-k linear space W is said to be *generic* with respect to an arrangement $\mathcal{B}$ if $\mathrm{rk}(X \cap W) = \mathrm{rk}\, X + k$ for all $X \in L(\mathcal{B})$ with $\mathrm{rk}\, X \le \mathrm{rk}\, \mathcal{B} - k$.

If $\mathcal{B}$ is an essential and supersolvable arrangement of rank m and if W is a proper linear space of dimension $\ell \ge 3$, then by Theorem 4.2 the arrangement $\mathcal{A} = \mathcal{B} \cap W$ is hypersolvable. We call such an arrangement a *generic* (*hypersolvable*) *slice* of rank ℓ.

Not every hypersolvable arrangement is a generic slice—see Example 4.15 from [23].

Lemma 4.10. *Let $\mathcal{B}$ be a rank-m supersolvable arrangement and $\mathcal{A}$ a rank-ℓ generic slice. Then the singular range of $\mathcal{A}$ is* (ℓ, ℓ).

Proof. The assumption of genericity means that $L(\mathcal{A})_{\leq \ell-1} \cong L(\mathcal{B})_{\leq \ell-1}$. However, $X \cap W = 0$ for all $X \in L(\mathcal{B})_\ell$ and so, since W is proper and $\mathcal{B}$ is essential, the singular range of $\mathcal{A}$ is (ℓ, d) for some d. On the other hand, $\operatorname{rk} \mathcal{A}_\ell = \operatorname{rk} \mathcal{A}_m = \ell$, so the last $m - \ell$ extensions are all singular and $d = \ell$. □

This is to say that, for generic slice arrangements, the module $J(-\ell)$ has a linear resolution. We can state this somewhat more generally as follows.

PROPOSITION 4.11. *Let $\mathcal{A}$ be a rank-ℓ hypersolvable arrangement, and suppose there exist a generic slice $\mathcal{C}$ and fibered extensions $\mathcal{C} = \mathcal{A}_{m-i} \subset \cdots \subset \mathcal{A}_{m-1} \subset \mathcal{A}_m = \mathcal{A}$ for some $i \geq 0$. Then the singular range of $\mathcal{A}$ is (ℓ, ℓ).*

Proof. We may (as in the proof of Lemma 4.6) reduce to the case where $\mathcal{A} = \mathcal{C}$, a generic slice of rank ℓ. Let $\mathcal{B}$ be the supersolvable deformation of $\mathcal{A}$. Denote by A' and B' the OS algebras of the respective decones, and let $J' = \ker(B' \twoheadrightarrow A')$.

Let $R' = (B')^!$, and let $\mathcal{K} = R' \otimes_{\Bbbk} (B')^*$ be the corresponding Koszul complex. That is, $\mathcal{K}^q = R'(-q) \otimes_{\Bbbk} (B'^q)^*$ for $q \geq 0$ with differential $\partial \colon e_i^* \otimes 1 \mapsto 1 \otimes x_i$. Since B' is a Koszul algebra, it follows that $\mathcal{K}$ is a free resolution of $\Bbbk$ over R'.

Let M' be the ℓth syzygy module in the resolution $\mathcal{K} \to \Bbbk \to 0$. That is, M' is the cokernel of $\partial_{\ell+1}$, a left R'-module, which means M' has minimal free resolution

$$0 \longrightarrow \mathcal{K}^m \xrightarrow{\partial_m} \cdots \longrightarrow \mathcal{K}^{\ell+1} \xrightarrow{\partial_{\ell+1}} \mathcal{K}^\ell \xrightarrow{\eta} M' \longrightarrow 0. \tag{32}$$

From this we see that M' is concentrated in internal degree ℓ and $\operatorname{Ext}_{R'}(M', \Bbbk) \cong J'$ as a B'-module. Since Koszul duality is an involution we have $\operatorname{Ext}_{B'}(J', \Bbbk) \cong M'$ as a left R'-module, and M' is bigraded as claimed. □

The proposition gives another criterion for the hypotheses of Theorem 1.5 to be satisfied, as follows.

COROLLARY 4.12. *If $\mathcal{A}$ is obtained by fibered extensions of a generic slice of a supersolvable arrangement, then $\mathfrak{g}_A \cong \operatorname{Lie}(M[-2]) \rtimes \mathfrak{h}_A$.*

4.4. Hilbert Series

Expressions for the Hilbert series of the graded module $M = \operatorname{Ext}_B(J, \Bbbk)$ are not known in general (cf. [28]). However, a simple formula exists for generic slices that can be extended to fibered extensions of generic slices.

Let β_i denote the ith Betti number of B', so that $h(B', t) = \sum_{i=0}^m \beta_i t^i$ is its Hilbert series. The following is well known (see [22]).

LEMMA 4.13. *There exist positive integers $1 = d_1 \leq d_2 \leq \cdots \leq d_m$ for which*

$$h(B', t) = \prod_{j=2}^m (1 + d_j t).$$

Taking the Euler characteristic of (32) we note that, for a generic slice of dimension ℓ,

$$h_R(M,t) = h_{R'}(M',t) = h(R',t)\sum_{i=0}^{m-\ell}(-1)^i\beta_{i+\ell}t^i. \tag{33}$$

More generally, a fibered extension results in the same formula.

The hypotheses of Theorem 1.5, together with (33), yield the following.

COROLLARY 4.14. *If $h(U,t,u) = \sum_{p,q}\dim_{\Bbbk} U_q^p t^p u^q$ is the bigraded Hilbert series of $U = U(\mathfrak{g}_A)$, then*

$$h(U,t,u) = h(R,t)(1-u^{-2}h_R(M,t,u))^{-1}. \tag{34}$$

In the case of a generic slice of dimension ℓ,

$$h(U,t,u) = h(R,t)\left(1 - t^2u^{-2}h(R,t)\sum_{i=0}^{m-\ell}(-1)^i\beta_{i+\ell}t^i\right)^{-1}. \tag{35}$$

5. A Presentation for the Homotopy Lie Algebra

For the hypersolvable arrangements satisfying the hypotheses of Theorem 1.8, the problem of writing an explicit presentation for the homotopy Lie algebra $\mathfrak{g}_A$ is equivalent to that of presenting the homotopy module $M_A = \mathrm{Ext}_B(J,\Bbbk)$. We carry out this computation for generic slices of supersolvable arrangements.

Let $\mathcal{A} = \{H_1,\dots,H_n\}$ be a hypersolvable arrangement with supersolvable deformation $\mathcal{B}$. As usual, let $\mathfrak{h}$ denote the holonomy Lie algebra and $R = U(\mathfrak{h})$ its enveloping algebra. Recall that $\mathfrak{h}$ has a presentation with n generators $x_1,\dots,x_n$ in degree $(1,0)$, one for each hyperplane $H_i \in \mathcal{A}$, and for each flat $F \in L_2(\mathcal{A}) = L_2(\mathcal{B})$, the relations

$$\left[x_i, \sum_{j\in F} x_j\right] = 0 \tag{36}$$

for all i for which $i \in F$ (i.e., $F \subset H_i$).

Now assume that $\mathcal{A}$ is a generic slice of a supersolvable arrangement. Then the resolution (32) gives a presentation of the (deconed) homotopy module M' as an R'-module. In order to use this presentation explicitly, we will choose the basis for B'^* given by identifying it with the flag complex of $\mathcal{B}'$ (see [6] for details).

Recall that Fl_p is a free $\Bbbk$-module on "flags" $(F_1,\dots,F_p)$, where $F_i \in L_i(\mathcal{B}')$ for $1 \le i \le p$ and where $F_i < F_{i+1}$ modulo the relations

$$\sum_{G\,:\,F_{i-1}<G<F_{i+1}} (F_1,\dots,F_{i-1},G,F_{i+1},\dots,F_p) \tag{37}$$

for each i, $1 < i < p$. Moreover, the map $f\colon \mathrm{Fl}_p \to (B'^p)^*$ given by

$$f\colon (F_1,\dots,F_p) \mapsto \left(\sum_{i\in F_1} e_i^*\right)\left(\sum_{i\in F_2-F_1} e_i^*\right)\cdots\left(\sum_{i\in F_p-F_{p-1}} e_i^*\right) \tag{38}$$

is an isomorphism (cf. [27, dual of (2.3.2)]).

Under the identification $\mathrm{Fl} \cong B'^*$, the boundary map in the Koszul complex is transformed as follows. Given a flag $\mathbf{F} = (F_1, \dots, F_p)$ and $i \in F_p$, define an element $\mathbf{F} - i \in \mathrm{Fl}^{p-1}$ by finding the integer j for which $i \in F_j - F_{j-1}$ and letting

$$\mathbf{F} - i = (-1)^{j-1} \sum (F_1, \dots, F_{j-1}, G_j, G_{j+1}, \dots, G_{p-1}), \tag{39}$$

where the sum is taken over all flags with the property that $i \notin G_{p-1}$ and $G_k < F_{k+1}$ for all k $(j \le k < p)$. Then the boundary map is given by extending

$$\partial \colon (F_1, \dots, F_p) \mapsto \sum_{i \in F_p} (\mathbf{F} - i) \otimes x_i \tag{40}$$

R-linearly.

For each element $\mathbf{F} \in \mathrm{Fl}^{\ell}$, let $y_{\mathbf{F}}$ denote the corresponding element of M'; that is, $y_{\mathbf{F}} = \eta \circ (f \otimes 1)(\mathbf{F} \otimes 1)$. In particular, we find a minimal generating set for M' by choosing a set of β_ℓ flags of length ℓ in $L(\mathcal{B})$ appropriately. In particular, one may construct a basis for Fl^{ℓ} using "nbc" sets (see e.g. [6, Lemma 3.2]).

Then the relations in M' are given by the image of $\partial_{\ell+1}$ in (32). For each flag $\mathbf{F} = (F_1, \dots, F_{\ell+1})$ we have a relation in M' of the form

$$\sum_{i \in F_{\ell+1}} y_{\mathbf{F}-i} x_i. \tag{41}$$

It follows that in $\mathfrak{g}_A$, for each flag $\mathbf{F} = (F_1, \dots, F_{\ell+1})$, we have a relation

$$\sum_{i \in F_{\ell+1}} [x_i, y_{\mathbf{F}-i}]. \tag{42}$$

Because M' is the restriction of the module M from R to R', we can use the preceding paragraphs to derive a presentation for M as well, noting that the central element $\sum_{i=1}^n x_i$ in R acts trivially. One can find a minimal set of relations just by taking the flags $\mathbf{F}$ to come from a basis of $\mathrm{Fl}^{\ell+1}$. We summarize this discussion as follows.

THEOREM 5.1. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a generic slice of a supersolvable arrangement, and let A be the Orlik–Solomon algebra of $\mathcal{A}$. Then the homotopy Lie algebra $\mathfrak{g}_A$ has presentation with generators*

- x_i *in degree* $(1, 0)$ *for each* $i \in [n]$ *and*
- $y_{\mathbf{F}}$ *in degree* $(2, \ell - 2)$ *for each* $\mathbf{F} \in \mathrm{Fl}^{\ell}$

and the relations

- $\left[x_i, \sum_{j \in F} x_j\right] = 0$ *for each flat* $F \in L_2(\mathcal{A})$ *and each* $i \in F$,
- $\sum_{i \in F_{\ell+1}} [x_i, y_{\mathbf{F}-i}] = 0$ *for each flag* $\mathbf{F} = (F_1, \dots, F_{\ell+2}) \in \mathrm{Fl}^{\ell+1}$, *and*
- $\left[\sum_{i=1}^n x_i, y_{\mathbf{F}}\right] = 0$ *for each* $\mathbf{F} \in \mathrm{Fl}^{\ell}$.

We illustrate this theorem with an example.

EXAMPLE 5.2. Consider the arrangement $\mathcal{A}$ defined by the polynomial

$$Q_{\mathcal{A}} = xyz(x - z)(y - z)(2x - y - 4z)(2x - y - 5z)(x + 5y + 2z)(x + 5y + z).$$

This is a generic slice of the supersolvable arrangement $\mathcal{B}$, the cone over the arrangement defined by the polynomial $Q_{\mathcal{B}'} = vwxy(x-1)(y-1)(v-1)(w-1)$. The Poincaré polynomials of the deconed arrangements are given by

$$\pi(\mathcal{A}', t) = 1 + 8t + 24t^2 \quad \text{and}$$
$$\pi(\mathcal{B}', t) = (1+2t)^4 = 1 + 8t + 24t^2 + 32t^3 + 16t^4.$$

Thus the homotopy module M' has 32 generators and 16 relations, which can be described as follows.

Label the hyperplanes of $\mathcal{B}'$ as $0_0, 1_0, 2_0, 3_0, 0_1, 1_1, 2_1, 3_1$, in the order of the factors of $Q_{\mathcal{B}'}$. A basis of 32 flags of length 3 can be constructed by choosing three intersecting hyperplanes i_a, j_b, k_c with $0 \le i < j < k \le 3$ and $a, b, c \in \{0, 1\}$ and then forming a flag by successively intersecting the hyperplanes from right to left; we call this flag $\mathbf{F}_{i_a j_b k_c}$. Likewise, a basis of 16 flags of length 4 in $\mathcal{B}'$ is constructed by choosing four intersecting hyperplanes $0_a, 1_b, 2_c, 3_d$ for all choices of $a, b, c, d \in \{0, 1\}$ and again forming a flag by successive intersection.

Let $\mathfrak{g}_A$ be the holonomy Lie algebra of $\mathcal{A}$. Then $\mathfrak{g}_A$ has one generator x_H for each hyperplane H together with 32 additional generators $y_{i_a j_b k_c}$ in degree $(2, 1)$—as well as the relations

$$[x_{0_a}, y_{1_b 2_c 3_d}] - [x_{1_b}, y_{0_a 2_c 3_d}] + [x_{2_c}, y_{0_a 1_b 3_d}] - [x_{3_d}, y_{0_a 1_b 2_c}]$$

for each $a, b, c, d \in \{0, 1\}$—in addition to the holonomy relations (8) and the relations

$$\left[\sum_{H \in \mathcal{A}} x_H, y_{i_a j_b k_c} \right]$$

for each choice of i, j, k and a, b, c.

6. 2-Generic Arrangements of Rank 4

We now present a method for computing the Hilbert series of the homotopy Lie algebra of a particularly nice class of arrangements: rank-4 arrangements for which no three hyperplanes contain a common plane.

For any rank-ℓ arrangement $\mathcal{A}$ with n hyperplanes, let $E = \bigwedge_{\Bbbk}(e_1, \ldots, e_n)$ be the exterior algebra, $A = E/I$ the Orlik–Solomon algebra, and $S = \Bbbk[x_1, \ldots, x_n]$ the polynomial algebra. We recall the following.

THEOREM 6.1 (Eisenbud–Popescu–Yuzvinsky [8]). *The complex of S-modules*

$$0 \longleftarrow F(\mathcal{A}) \longleftarrow A^{\ell} \otimes S \longleftarrow \cdots \longleftarrow A^1 \otimes S \longleftarrow A^0 \otimes S \longleftarrow 0$$

is exact, where boundary maps are induced via multiplication by $\sum_{i=1}^{n} e_i \otimes x_i$ *and where the S-module* $F(\mathcal{A})$ *is taken as the cokernel of the map* $A^{\ell-1} \otimes S \to A^{\ell} \otimes S$.

It follows from Bernstein–Gelfand–Gelfand duality that, for each $p \ge 0$, there is a graded isomorphism of S-modules:

$$\mathrm{Ext}^p_E(A, \Bbbk)_q = \mathrm{Ext}^{\ell-q}_S(F(\mathcal{A}), S)_{p+q}. \tag{43}$$

We refer to [28] for the case of the smallest $q > 0$ for which this is nonzero. Details will appear in further work.

Now let $\mathcal{A}$ be a 2-generic arrangement. Notice that $B = E$ and $U(\mathfrak{h}) = B^! = S$. Then, applying Lemma 2.1 to (43), we obtain

$$M^p_q = \mathrm{Ext}^{\ell-q+1}_S(F(\mathcal{A}), S)_{p+q} \tag{44}$$

for $p \geq 0$ and $0 \leq q \leq \ell$. As a result, presentations for the S-modules M_q can be obtained computationally for specific examples using formula (44).

We recall from Example 4.8 that, if the rank of the arrangement $\ell = 4$, then $\mathcal{A}$ satisfies hypothesis (i) of Theorem 1.8: $M_q = 0$ unless $q = 3$ or $q = 4$; that is, the singular range of $\mathcal{A}$ is $(3, 4)$.

EXAMPLE 6.2. Consider arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ defined by the polynomials

$$Q_1 = xyzw(x + y + z)(y + z + w)(x - y + z + w),$$
$$Q_2 = xyzw(x + y + z)(y + z + w)(x - y + z - w).$$

Both arrangements have 7 hyperplanes and 5 lines that each contain 4 hyperplanes, so the characteristic polynomials are $\pi(\mathcal{A}_1, t) = \pi(\mathcal{A}_2, t) = 1 + 7t + 21t^2 + 30t^3 + 15t^4$. Since there are no nontrivial intersections in codimension 2, it follows that the fundamental group of both complements is $\mathbb{Z}^7$ and that $R = U(\mathfrak{h})$ is a polynomial ring.

We now use (44) to compute the Hilbert series of the graded modules M_3 and M_4 (recalling $M_q = 0$ for $q \neq 3, 4$). With the help of Macaulay 2, we find for $\mathcal{A}_1$ that

$$h(M_3, t) = (5 + 2t)/(1 - t)^3 = 5 + 17t + 36t^2 + 62t^3 + \cdots \quad \text{and}$$
$$h(M_4, t) = (2 - t)(1 + 2t + 2t^2)/(1 - t)^6 = 2 + 15t + 62t^2 + 185t^3 + \cdots;$$

for $\mathcal{A}_2$,

$$h(M_3, t) = (5 + t)/(1 - t)^3 = 5 + 16t + 33t^2 + 56t^3 + \cdots \quad \text{and}$$
$$h(M_4, t) = (1 + 6t - t^2 - t^3)/(1 - t)^6 = 1 + 12t + 56t^2 + 175t^3 + \cdots.$$

Using formula (34), this yields expressions for the Hilbert series of $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$. Comparing these Hilbert series shows that $U(\mathfrak{g}_1) \not\cong U(\mathfrak{g}_2)$ and so the two arrangements must have nonisomorphic homotopy Lie algebras.

EXAMPLE 6.3. In 1946, Nandi [21] showed that there are exactly three inequivalent block designs with parameters $(10, 15, 6, 4, 2)$. Each block is described in Figure 1. Each block design gives rise to a rank-4 matroid on ten points by taking the dependent sets to be those subsets that either contain one of the blocks or contain at least five elements.

By construction, there are no nontrivial dependent sets of size 3, so each arrangement is 2-generic.

D_1	$\{abcd, abef, aceg, adhi, bchi, bdgj, cdfj, afhj, agij, behj, bfgi, ceij, cfgh, defi, degh\}$
D_2	$\{abcd, abef, aceg, adhi, bcij, bdgh, cdfj, afhj, agij, aehj, bfgi, cehi, cfgh, defi, degj\}$
D_3	$\{abcd, abef, acgh, adij, bcij, bdgh, cdef, aegi, afhj, behj, bfgi, cehi, cfgj, degj, dfhi\}$

Figure 1

If we label the corresponding OS algebras A_1, A_2, and A_3, then it is straightforward to calculate that $h(A_i, t) = 1 + 10t + 45t^2 + 105t^3 + 69t^4$ for $i = 1, 2, 3$. In each case, the singular range is $(3, 4)$. The ideals J_1, J_2, J_3 have differing resolutions, however, from which it follows that $\mathfrak{g}_{A_1}, \mathfrak{g}_{A_2}, \mathfrak{g}_{A_3}$ are pairwise nonisomorphic.

7. Topological Interpretations

7.1. Generic Slices

A particularly simple situation, analyzed in detail by Dimca and Papadima [7], is when $\mathcal{A}$ is a generic slice of rank $\ell > 2$ of a supersolvable arrangement $\mathcal{B}$. Let $\mathcal{A}'$ and $\mathcal{B}'$ be the respective decones with complements $X = X(\mathcal{A}')$ and $Y = X(\mathcal{B}')$. The two spaces share the same fundamental group, π, and the same integral holonomy Lie algebra, $\mathfrak{h}$.

In Theorems 18(ii) and 23 of [7], Dimca and Papadima establish the following facts. The universal enveloping algebra $U(\mathfrak{h})$ is isomorphic (as a Hopf algebra) to the associated graded algebra $\mathrm{gr}_{I\pi}(\mathbb{Z}\pi)$, where $\mathbb{Z}\pi$ is the group ring of π with filtration determined by the powers of the augmentation ideal $I\pi$. The first nonvanishing higher homotopy group of X is $\pi_{\ell-1}(X)$; when viewed as a module over $\mathbb{Z}\pi$, it has resolution of the form

$$0 \longrightarrow H_m(Y) \otimes \mathbb{Z}\pi \longrightarrow \cdots \longrightarrow H_\ell(Y) \otimes \mathbb{Z}\pi \longrightarrow \pi_{\ell-1}(X) \longrightarrow 0. \qquad (45)$$

Finally, the associated graded module of $\pi_{\ell-1}(X)$, with respect to the filtration by powers of $I\pi$, has Hilbert series

$$h(\mathrm{gr}^{\bullet}_{I\pi}\, \pi_{\ell-1}(X), t) = (-1/t)^{\ell} \left(1 - \frac{\sum_{j=0}^{\ell-1} (-1)^j \beta_j t^j}{\sum_{j=0}^{m} (-1)^j \beta_j t^j}\right), \qquad (46)$$

where β_j are the Betti numbers of Y.

Consider the integral cohomology rings $A = H^{\bullet}(X, \mathbb{Z})$ and $B = H^{\bullet}(Y, \mathbb{Z})$. We have $(B^i)^* = H_i(Y, \mathbb{Z})$, since the homology of an arrangement complement is torsion-free. Thus, tensoring with $\Bbbk$ and passing to the associated graded modules in resolution (45) recovers resolution (32). As a consequence, we obtain the following.

Proposition 7.1. *Let $\mathcal{A}$ be a generic slice of rank $\ell > 2$ of a supersolvable arrangement, and let $X = X(\mathcal{A}')$ be the complement of its decone. The homotopy module of the algebra $A = H^\bullet(X, \Bbbk)$ is isomorphic to the graded module associated to the the first nonvanishing higher homotopy group of X:*

$$M_A \cong \operatorname{gr}_I^\bullet \pi_{\ell-1}(X) \otimes \Bbbk. \tag{47}$$

7.2. Rescaling

Fix an integer $q \geq 1$. The q-rescaling of a graded algebra A is the graded algebra $A^{[q]}$, with $A^{[q]}_{i(2q+1)} = A_i$ and $A^{[q]}_j = 0$ if $(2q+1) \nmid j$ and with multiplication rescaled accordingly. When taking the Yoneda algebra of $A^{[q]}$, the internal degree of the Yoneda algebra of A gets rescaled while the resolution degree stays unchanged:

$$\operatorname{Ext}_{A^{[q]}}(\Bbbk, \Bbbk) = \operatorname{Ext}_A(\Bbbk, \Bbbk)^{[q]}. \tag{48}$$

Similarly, the q-rescaling of a graded Lie algebra L is the graded Lie algebra $L^{[q]}$, with $L^{[q]}_{2iq} = L_i$ and $L^{[q]}_j = 0$ if $2q \nmid j$ and with Lie bracket rescaled accordingly. Rescaling works well with the holonomy and homotopy Lie algebras:

$$\mathfrak{h}_{A^{[q]}} = \mathfrak{h}_A^{[q]}, \qquad \mathfrak{g}_{A^{[q]}} = \mathfrak{g}_A^{[q]}. \tag{49}$$

The Hilbert series of the enveloping algebras of $\mathfrak{g}_A^{[q]}$ and $\mathfrak{g}_A$ are related as follows:

$$h(U(\mathfrak{g}_A^{[q]}), t, u) = h(U(\mathfrak{g}_A), tu^{2q}, u^{2q+1}). \tag{50}$$

Now let X be a connected finite-type CW-complex. A simply-connected finite-type CW-complex Y is called a q-rescaling of X (over a field $\Bbbk$) if the cohomology algebra $H^\bullet(Y, \Bbbk)$ is the q-rescaling of $H^\bullet(X, \Bbbk)$; that is,

$$H^\bullet(Y, \Bbbk) = H^\bullet(X, \Bbbk)^{[q]}. \tag{51}$$

Rational rescalings always exist: take a Sullivan minimal model for the 1-connected finite-type differential graded algebra $(H^\bullet(X, \mathbb{Q})^{[q]}, d = 0)$, and use [31] to realize it by a finite-type 1-connected CW-complex Y. The space that is so constructed is the desired rescaling. Moreover, Y is formal—that is, its rational homotopy type is a formal consequence of its rational cohomology algebra. Hence Y is uniquely determined, up to rational homotopy equivalence, among spaces with the same cohomology ring (though there may be other, nonformal rescalings of X; see [24]).

Proposition 7.2. *Let X be a finite-type CW-complex with cohomology algebra $A = H^\bullet(X; \mathbb{Q})$, and let Y be a finite-type simply connected CW-complex with $H^\bullet(Y; \mathbb{Q}) \cong A^{[q]}$. If Y is formal, then*

$$\pi_\bullet(\Omega Y) \otimes \mathbb{Q} \cong \mathfrak{g}_A^{[q]}. \tag{52}$$

Proof. Since Y is formal, the Eilenberg–Moore spectral sequence of the path fibration $\Omega Y \to PY \to Y$ collapses, yielding an isomorphism of Hopf algebras between the Yoneda algebra of $H^\bullet(Y;\mathbb{Q})$ and the Pontryagin algebra $H_\bullet(\Omega Y;\mathbb{Q})$. From the rescaling assumption we obtain

$$\operatorname{Ext}_{A^{[q]}}(\mathbb{Q},\mathbb{Q}) \cong H_\bullet(\Omega Y;\mathbb{Q}), \tag{53}$$

and by Milnor–Moore [20] we find that $\mathfrak{g}_{A^{[q]}} \cong \pi_\bullet(\Omega Y)\otimes\mathbb{Q}$ as Lie algebras. Using (49) finishes the proof. □

As a consequence, we obtain a quick proof of a special case of Theorem A from [24].

COROLLARY 7.3 (Papadima–Suciu [24]). *Suppose X and Y are spaces as described in Proposition 7.2. If both X and Y are formal and A is Koszul, then*

$$\pi_\bullet(\Omega Y)\otimes\mathbb{Q} \cong (\operatorname{gr}_\bullet(\pi_1 X)\otimes\mathbb{Q})^{[q]}. \tag{54}$$

Proof. Since A is Koszul, $\mathfrak{g}_A = \mathfrak{h}_A$; since X is formal, $\operatorname{gr}_\bullet(\pi_1 X)\otimes\mathbb{Q}\cong\mathfrak{h}_A$ (cf. [31]). The conclusion follows from (52). □

REMARK 7.4. When X is formal (but not necessarily simply connected), a theorem of Papadima and Yuzvinsky [25] states that the cohomology algebra $A = H^\bullet(X;\mathbb{Q})$ is Koszul if and only if the Bousfield–Kan rationalization $X_\mathbb{Q}$ is aspherical. Now, by a classical result of Quillen [26], $U(\mathfrak{h}_A)\cong\operatorname{gr}_{I\pi}\mathbb{Q}\pi_1(X_\mathbb{Q})$. More generally, it seems likely that

$$U(\mathfrak{g}_A)\cong U(\pi_\bullet(\Omega\tilde{X}_\mathbb{Q}))\,\widehat{\otimes}\,\operatorname{gr}_{I\pi}\mathbb{Q}\pi_1(X_\mathbb{Q}) \tag{55}$$

in view of a result due to Félix and Thomas [12]. (Here again, $\mathbb{Q}\pi_1(X_\mathbb{Q})$ acts on the left-hand factor by the action induced from $\pi_1(X_\mathbb{Q})$ on the universal cover $\tilde{X}_\mathbb{Q}$.)

However, if X is a hyperplane arrangement complement, then X is not in general a nilpotent space. This means that we can expect to find such spaces X for which $\pi_i(X_\mathbb{Q})\ncong\pi_i(X)\otimes\mathbb{Q}$. The first such example was found by Falk [9], who noted that the complement X of the D_4 reflection arrangement is aspherical whereas its Bousfield–Kan rationalization $X_\mathbb{Q}$ is not. In general, then, we know of no way to relate $\mathfrak{g}_A$ with the topological homotopy Lie algebra $\pi_\bullet(\Omega X)\otimes\mathbb{Q}$.

7.3. Redundant Subspace Arrangements

Let $\mathcal{A}=\{H_1,\dots,H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. If q is a positive integer, then $\mathcal{A}^{(q)}=\{H_1^{\times q},\dots,H_n^{\times q}\}$ is an arrangement of codimension-q subspaces in $\mathbb{C}^{q\ell}$. For example, if $\mathcal{A}$ is the braid arrangement in $\mathbb{C}^\ell$ with complement equal to the configuration space of ℓ distinct points in $\mathbb{C}$, then the complement of $\mathcal{A}^{(q)}$ is the configuration space of ℓ distinct points in $\mathbb{C}^q$.

PROPOSITION 7.5. *Let $\mathcal{A}$ be a hyperplane arrangement with Orlik–Solomon algebra $A = H^\bullet(X;\mathbb{Q})$. Fix $q\geq 1$, and let $Y = X(\mathcal{A}^{(q+1)})$ be the complement of the corresponding subspace arrangement. Then*

$$\pi_\bullet(\Omega Y)\otimes\mathbb{Q}\cong\mathfrak{g}_A^{[q]}. \tag{56}$$

Proof. Clearly, Y is simply connected. As shown in [5], $H^\bullet(Y;\mathbb{Q})$ is the q-rescaling of $H^\bullet(X;\mathbb{Q})$. Since $\mathcal{A}^{(q+1)}$ has geometric intersection lattice, its complement Y is formal (see [32, Prop. 7.2]). The conclusion then follows from Proposition 7.2. □

COROLLARY 7.6. *Let $\mathcal{A}$ be a hypersolvable arrangement satisfying either of the hypotheses of Theorem 1.8. Then*

$$\pi_\bullet(\Omega Y)\otimes\mathbb{Q}\cong(\mathrm{Lie}(M_A[-2])\rtimes\mathfrak{h}_A)^{[q]}.$$

EXAMPLE 7.7. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the hyperplane arrangements from Example 6.2. Denote by $\mathfrak{g}_i=\mathfrak{g}_{A_i}$ the respective homotopy Lie algebras, $i=1,2$. Consider the redundant subspace arrangements $\mathcal{A}_1^{(2)}$ and $\mathcal{A}_2^{(2)}$; both are arrangements of seven codimension-2 complex subspaces of $\mathbb{C}^8$. Denoting their complements by Y_1 and Y_2 (respectively), we have $\pi_1(Y_1)=\pi_1(Y_2)=0$ and $H_*(Y_1)\cong H_*(Y_2)$ as graded abelian groups.

Let $\pi_\bullet(\Omega Y_i)\otimes\mathbb{Q}$ be the respective (topological) homotopy Lie algebras. By Proposition 7.2, $\pi_\bullet(\Omega Y_i)\otimes\mathbb{Q}\cong\mathfrak{g}_i^{[1]}$. Making use of our previous calculations for the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ together with formula (50) yields that, for $i=1,2$ both, $U(\mathfrak{g}_i^{[1]})_p$ has rank $1,0,7,0,28,0,84,5,210$ (respectively) for $0\le p\le 8$. It follows that, for $p\le 9$, the group $\pi_p(Y_i)\otimes\mathbb{Q}=0$ except for $\pi_3(Y_i)\otimes\mathbb{Q}\cong\mathbb{Q}^7$ and $\pi_8(Y_i)\otimes\mathbb{Q}\cong\mathbb{Q}^5$.

For $p=9$, however, the rank of $U(\mathfrak{g}_i^{[1]})_p$ is 52 and 51 for $i=1,2$, respectively. Hence

$$\pi_{10}(Y_1)\otimes\mathbb{Q}\cong\mathbb{Q}^{17}\quad\text{and}\quad\pi_{10}(Y_2)\otimes\mathbb{Q}\cong\mathbb{Q}^{16},$$

so $Y_1\not\simeq Y_2$.

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