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Parameterizing Conjugacy Classes of Maximal Unramified Tori via Bruhat–Tits Theory

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0. Introduction

The main result of this paper is a uniform parameterization of the set of conjugacy classes of maximal unramified tori in a reductive p-adic group. This classification matches conjugacy classes of maximal unramified tori with certain equivalence classes that arise naturally from Bruhat–Tits theory. The motivation for this result comes from harmonic analysis; specifically, from J.-L. Waldspurger's papers [16; 17]. Using the parameterization scheme discussed in this paper, David Kazhdan and I [6] have been able to generalize some of the results of [17] in a uniform manner.

The Main Result. Let k denote a field with nontrivial discrete valuation ν. We assume that k is complete with perfect residue field $\mathfrak{f}$. Let $\bar{k}$ denote a fixed algebraic closure of k and let K denote the maximal unramified extension of k in $\bar{k}$. Let G denote the group of k-rational points of a reductive linear algebraic k-group $\mathbf{G}$ and let G° denote the group of k-rational points of the identity component $\mathbf{G}^\circ$ of $\mathbf{G}$. Let $\mathcal{B}(G)$ denote the (enlarged) Bruhat–Tits building of G°.

A subgroup of G is called an *unramified torus* when it is the group of k-rational points of a k-torus in $\mathbf{G}^\circ$ that splits over an unramified extension of k. In this paper we classify G-conjugacy classes of maximal unramified tori in G in terms of equivalence classes of pairs $(\mathsf{G}_F, \mathsf{T})$. Here F is a facet in the building, G_F is the connected reductive $\mathfrak{f}$-group associated to F, and T is an $\mathfrak{f}$-minisotropic maximal torus in G_F. (The torus T is called $\mathfrak{f}$-*minisotropic* when the maximal $\mathfrak{f}$-split torus in T coincides with the maximal $\mathfrak{f}$-split torus in the center of G_F.)

In more detail: Let I^t denote the set of pairs (F, T) where F is a facet in $\mathcal{B}(G)$ and T is a maximal $\mathfrak{f}$-torus in G_F. In Section 3.2 we define on I^t an equivalence relation, denoted $\sim$. In Section 3.3 we associate to each element $(F, \mathsf{T}) \in I^t$ a G-conjugacy class $\mathcal{C}(F, \mathsf{T})$ of maximal unramified tori in G. The set I^t is too

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large, so we restrict our attention to the subset I^m of minisotropic pairs in I^t. A pair $(F, \mathsf{T}) \in I^t$ is said to be *minisotropic* when T is an $\mathfrak{f}$-minisotropic maximal torus of G.

We now state Theorem 3.4.1, the main result of this paper. Let $\mathcal{C}^T$ denote the set of G-conjugacy classes of maximal unramified tori in G.

THEOREM. *There is a bijective correspondence between $I^m/\sim$ and $\mathcal{C}^T$ given by the map that sends (F, T) to $\mathcal{C}(F, \mathsf{T})$.*

If k is p-adic and if $\mathbf{G}$ is connected and k-split, then this result can be derived from some work of Gérardin [8]. If k is p-adic and $\mathbf{G}$ is unramified, then Waldspurger [16] stated a variant of this result as a hypothesis.

We remark that if $\mathfrak{f}$ is algebraically closed, then $\mathcal{C}^T$ and $I^m/\sim$ both have one element. In this case, I^m consists of those pairs (F, T) where F is an alcove in $\mathcal{B}(G)$ and T is a maximal torus in G_F, and the element of $\mathcal{C}^T$ is the G-conjugacy class of the group of k-rational points of a maximal k-split torus in $\mathbf{G}$.

By Lemma 2.1.1, a maximal unramified torus in G is the group of k-rational points of a maximal K-split k-torus in $\mathbf{G}$. From a theorem of Steinberg (see e.g. [14, Chap. II, Sec. 3.3 and Chap. III, Sec. 2.3]), $\mathbf{G}^\circ$ is quasi-split over K. Thus, the centralizer in G° of a maximal unramified torus of G is the group of k-rational points of a maximal k-torus in $\mathbf{G}$. Since this correspondence is one-to-one, our theorem also provides a classification of the G-conjugacy classes of maximal tori of G that arise in this way.

ADDITIONAL RESULTS. In Section 4 we use the foregoing result to give an explicit description of the set of G-conjugacy classes in certain stable conjugacy classes. More precisely: Suppose that $\mathfrak{f}$ is quasi-finite and that $\mathbf{G}$ is connected and K-split. If $(F, \mathsf{T}) \in I^m$ and $T \in \mathcal{C}(F, \mathsf{T})$, then for $\gamma \in T$ with $C_G(\gamma) = T$ we describe the set of G-conjugacy classes in

$$^{\mathbf{G}(\bar{k})}\gamma \cap G.$$

Finally, in Section 5 we present a generalization of the main result. Let $\mathcal{C}$ denote the set of G-conjugacy classes of pairs (H, x), where H is a maximal-rank unramified subgroup (see Section 1.3) in G and x is a hyperspecial point in $\mathcal{B}^{\text{red}}(H)$, the reduced Bruhat–Tits building of H. When $\mathbf{G}$ is K-split, we classify the elements of $\mathcal{C}$ in terms of equivalence classes of pairs (F, H). Here F is a facet in $\mathcal{B}(G)$, and H is an $\mathfrak{f}$-cuspidal (see Section 5.1) maximal-rank connected reductive subgroup in G_F.

ADDITIONAL COMMENTS. The main result (Theorem 3.4.1) was circulated as a preprint in 2001. I later realized that the main result could be generalized to include Theorem 5.3.6. The next version of the paper (not circulated) gave a proof of Theorem 5.3.6 and presented Theorem 3.4.1 as a corollary; unfortunately, some generality and transparency were lost. Consequently, in this version I have chosen to separate the proofs (and another result, Theorem 4.5.1, has been added).

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1. Notation

In addition to the notation discussed in the introduction, we shall require the following.

1.1. Basic Notation

Let $\mathfrak{F}$ denote the residue field of K. Note that $\mathfrak{F}$ is an algebraic closure of $\mathfrak{f}$. Let $\Gamma = \mathrm{Gal}(K/k)$, which we also identify with $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$.

We denote by $\mathcal{D}G^\circ$ the group of k-rational points of the derived group $\boldsymbol{\mathcal{D}}\mathbf{G}^\circ$ of $\mathbf{G}^\circ$. When we talk about a torus in G, we mean the group of k-rational points of a k-torus in $\mathbf{G}^\circ$.

In order to avoid a proliferation of superscripts, we adopt the following convention. We shall call a subgroup of $\mathbf{G}$ a *parabolic subgroup of* $\mathbf{G}$ when it is a parabolic subgroup of $\mathbf{G}^\circ$; we adopt a similar convention with respect to tori and Levi subgroups.

If $g, h \in G$, then ${}^g h = ghg^{-1}$. If $S \subset G$, then ${}^g S = \{{}^g h \mid h \in S\}$. If a group L acts on a set S, then S^L denotes the set of L-fixed points of S.

1.2. Apartments, Buildings, and Associated Notation

Let $\mathcal{B}(G) = \mathcal{B}(\mathbf{G}, k)$ denote the (enlarged) Bruhat–Tits building of G°. We identify $\mathcal{B}(G)$ with the Γ-fixed points of $\mathcal{B}(\mathbf{G}, K)$, the Bruhat–Tits building of $\mathbf{G}^\circ(K)$. Let $\mathcal{B}^{\mathrm{red}}(G) = \mathcal{B}^{\mathrm{red}}(\mathbf{G}, k) = \mathcal{B}(\boldsymbol{\mathcal{D}}\mathbf{G}^\circ, k)$ denote the reduced Bruhat–Tits building of G°. According to [12] (see also [3, 4.2.16]) we have a decomposition $\mathcal{B}(\mathbf{G}, k) = \mathcal{B}^{\mathrm{red}}(\mathbf{G}, k) \times \mathcal{B}(\mathbf{Z_G}, k)$, where $\mathbf{Z_G}$ denotes the center of $\mathbf{G}$.

For a Levi k-subgroup $\mathbf{M}$ of a parabolic k-subgroup of $\mathbf{G}$, we identify $\mathcal{B}(\mathbf{M}, k)$ in $\mathcal{B}(\mathbf{G}, k)$. There is not a canonical way to do this, but every natural embedding of $\mathcal{B}(\mathbf{M}, k)$ in $\mathcal{B}(\mathbf{G}, k)$ has the same image [3, 4.2.18].

For $\Omega \subset \mathcal{B}(G)$, we let $\mathrm{stab}_G(\Omega)$ denote the stabilizer of Ω in G and let $\mathrm{Fix}_G(\Omega)$ denote the pointwise stabilizer of Ω.

Given a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$, we have the torus $S = \mathbf{S}(k)$ in G and the corresponding apartment $\mathcal{A}(S) = \mathcal{A}(\mathbf{S}, k)$ in $\mathcal{B}(G)$. Let $\mathbf{T}$ be a maximal K-split k-torus of $\mathbf{G}$ containing $\mathbf{S}$ [3, Cor. 5.1.12]. We identify $\mathcal{A}(\mathbf{S}, k)$ with $\mathcal{A}(\mathbf{T}, K)^\Gamma$. For $\Omega \subset \mathcal{A}(S)$, let $A(\mathcal{A}(S), \Omega)$ denote the smallest affine subspace of $\mathcal{A}(S)$ containing Ω.

If ψ is an affine root of $\mathbf{G}$ with respect to k, $\mathbf{S}$, and ν, then $\dot{\psi}$ denotes the root of $\mathbf{G}$ with respect to k and $\mathbf{S}$ that is the gradient of ψ. We let $U_{\dot{\psi}}$ denote the corresponding root subgroup of G.

Suppose $x \in \mathcal{B}(G)$. We will denote the parahoric subgroup of G° attached to x by G_x and denote its pro-unipotent radical by G_x^+. Note that both G_x and G_x^+ depend only on the facet of $\mathcal{B}(G)$ to which x belongs. If F is a facet in $\mathcal{B}(G)$ and if $x \in F$, then we define $G_F = G_x$ and $G_F^+ = G_x^+$. Recall that G_x is a subgroup of $\mathrm{stab}_{G^\circ}(x)$. For a facet F in $\mathcal{B}(G)$, the quotient G_F/G_F^+ is the group of $\mathfrak{f}$-rational points of a connected reductive $\mathfrak{f}$-group G_F.

We denote the parahoric subgroup of $\mathbf{G}^\circ(K)$ corresponding to $x \in \mathcal{B}(\mathbf{G}, K)$ by $\mathbf{G}(K)_x$, and we denote the pro-unipotent radical of $\mathbf{G}(K)_x$ by $\mathbf{G}(K)_x^+$. The subgroups $\mathbf{G}(K)_x$ and $\mathbf{G}(K)_x^+$ depend only on the facet of $\mathcal{B}(\mathbf{G}, K)$ to which x belongs. If F is a facet in $\mathcal{B}(\mathbf{G}, K)$ and if $x \in F$, then we define $\mathbf{G}(K)_F = \mathbf{G}(K)_x$ and $\mathbf{G}(K)_F^+ = \mathbf{G}(K)_x^+$. For a facet F in $\mathcal{B}(\mathbf{G}, K)$, the quotient $\mathbf{G}(K)_F/\mathbf{G}(K)_F^+$ is the group of $\mathfrak{F}$-rational points of a connected reductive $\mathfrak{F}$-group G_F.

Suppose F is a Γ-invariant facet in $\mathcal{B}(\mathbf{G}, K)$. In this case, $F' = F^\Gamma$ is a facet in $\mathcal{B}(G)$. Moreover, we have $G_{F'} = (\mathbf{G}(K)_F)^\Gamma$, $G_{F'}^+ = (\mathbf{G}(K)_F^+)^\Gamma$, and $\mathsf{G}_F = \mathsf{G}_{F'} \times_{\mathfrak{f}} \mathfrak{F}$. Sometimes, we will abuse notation and denote by G_F (resp. G_F^+, $\mathbf{G}(K)_{F'}$, and $\mathbf{G}(K)_{F'}^+$) the group $G_{F'}$ (resp. $G_{F'}^+$, $\mathbf{G}(K)_F$, and $\mathbf{G}(K)_F^+$).

1.3. Unramified Groups

An algebraic group $\mathbf{H}$ is called an *unramified group* if $\mathbf{H}$ is a connected reductive k-group and there exists a hyperspecial vertex in $\mathcal{B}^{\mathrm{red}}(\mathbf{H}(k))$. We recall that a vertex $x \in \mathcal{B}^{\mathrm{red}}(\mathbf{H}(k))$ is said to be *hyperspecial* provided that $\mathbf{H}$ is K-split (i.e., if $\mathbf{H}$ contains a K-split maximal torus) and x is a Γ-invariant special vertex in $\mathcal{B}^{\mathrm{red}}(\mathbf{H}, K)$.

In this paragraph, suppose $\mathfrak{f}$ is finite. Then there is a standard definition of the term *unramified group,* namely: the group $\mathbf{H}$ is unramified provided that $\mathbf{H}$ is connected, reductive, k-quasi-split, and K-split. By [15, 1.10.2], if $\mathbf{H}$ is unramified in this sense, then it is unramified in our sense. We show the converse as follows: Since $\mathbf{H}$ is K-split, we may choose a K-split maximal k-torus $\mathbf{T}$ of $\mathbf{H}$ that contains a maximal k-split torus of $\mathbf{H}$. If x is a hyperspecial vertex (in the apartment corresponding to $\mathbf{T}$) in $\mathcal{B}^{\mathrm{red}}(\mathbf{H}, k)$, then, since $\mathfrak{f}$ is finite, we may choose a Borel $\mathfrak{f}$-subgroup in H_x whose group of $\mathfrak{F}$-rational points contains the image of $\mathbf{T}(K) \cap \mathbf{H}(K)_x$ in $\mathsf{H}_x(\mathfrak{F})$. Since x is hyperspecial and $\mathbf{H}$ is K-split, this will determine a Γ-stable set of simple roots for $\mathbf{H}$ with respect to $\mathbf{T}$. Thus, $\mathbf{H}$ is k-quasi-split.

We shall also call $\mathbf{H}(k)$, the group of k-rational points of $\mathbf{H}$, an unramified group whenever $\mathbf{H}$ is unramified.

2. Tori over k and $\mathfrak{f}$

In this section we show how to move between tori over $\mathfrak{f}$ and tori over k.

2.1. Maximal Unramified Tori

We recall that a subgroup T of G is an *unramified torus* when T is the group of k-rational points of a k-torus $\mathbf{T}$ of $\mathbf{G}$ that splits over an unramified extension of k. We shall call a torus $\mathbf{T}$ as above an unramified torus of $\mathbf{G}$. The following result will be used throughout the remainder of the paper.

LEMMA 2.1.1. *Suppose* $\mathbf{T}$ *is a torus in* $\mathbf{G}$. *Then the following statements are equivalent.*

(1) $\mathbf{T}$ *is defined over* k *and* $\mathbf{T}(k)$ *is a maximal unramified torus of* G.
(2) $\mathbf{T}$ *is a maximal* K*-split* k*-torus of* $\mathbf{G}$.
(3) $\mathbf{T}$ *is a maximal* K*-split torus of* $\mathbf{G}$ *and* $\mathbf{T}$ *is defined over* k.

Proof. By definition, (1) and (2) are equivalent. Moreover, (3) implies (2).

We now show that (2) implies (3). Suppose $\mathbf{T}$ is a maximal K-split k-torus of $\mathbf{G}$ and let $\mathbf{M} = C_{\mathbf{G}^\circ}(\mathbf{T})$. Then $\mathbf{M}$ is a Levi k-subgroup of a parabolic K-subgroup of $\mathbf{G}$. Let $\mathbf{S}'$ be a maximal k-split torus in $\mathbf{M}$. From [3, Cor. 5.1.12], there exists a maximal K-split torus $\mathbf{S}$ of $\mathbf{M}$ such that $\mathbf{S}' \subset \mathbf{S}$ and $\mathbf{S}$ is defined over k. Note that $\mathbf{S}$ is also a maximal K-split torus in $\mathbf{G}$. Since $\mathbf{S} \subset \mathbf{M}$, we have $\mathbf{T} \subset \mathbf{S}$. Since $\mathbf{S}$ and $\mathbf{T}$ are K-split k-tori and $\mathbf{T}$ is maximal in $\mathbf{G}$ with respect to this property, we must have $\mathbf{T} = \mathbf{S}$. □

2.2. *From Maximal Unramified Tori over* k *to Tori over* $\mathfrak{f}$

Suppose T is a maximal unramified torus in G. Let $\mathbf{T}$ denote the maximal K-split k-torus in $\mathbf{G}$ such that $T = \mathbf{T}(k)$. Define

$$\mathbf{T}(K)_c := \{t \in \mathbf{T}(K) \mid \nu(\chi(t)) = 0 \text{ for all } \chi \in \mathbf{X}^*(\mathbf{T})\}.$$

By [15, Sec. 3.6.1] there is a natural embedding of $\mathcal{B}(T)$ in $\mathcal{B}(G)$; namely,

$$\begin{aligned}\mathcal{B}(T) &= \mathcal{B}(\mathbf{T}, K)^\Gamma = \mathcal{A}(\mathbf{T}, K)^\Gamma \\ &= (\mathcal{B}(\mathbf{G}, K)^{\mathbf{T}(K)_c})^\Gamma = \mathcal{B}(\mathbf{G}, K)^{\mathbf{T}(K)_c \rtimes \Gamma} \\ &\subset \mathcal{B}(G).\end{aligned}$$

We shall always think of $\mathcal{B}(T)$ as being embedded in $\mathcal{B}(G)$ in this way. We now collect some facts about $\mathcal{B}(T)$.

LEMMA 2.2.1. *Suppose* $\mathbf{T}$ *is a maximal* K*-split* k*-torus of* $\mathbf{G}$. *Let* T *denote the group of* k*-rational points of* $\mathbf{T}$.

(1) $\mathcal{B}(T)$ *is a nonempty, closed, convex subset of* $\mathcal{B}(G)$. *Moreover,* $\mathcal{B}(T)$ *is the union of the facets in* $\mathcal{B}(G)$ *that meet it.*
(2) *There is a maximal* k*-split torus* $\mathbf{S}$ *in* $\mathbf{G}$ *such that* $\mathcal{B}(T)$ *is an affine subspace of* $\mathcal{A}(\mathbf{S}, k)$.
(3) *For all* G*-facets* F *in* $\mathcal{B}(T)$, *there exists* $(F, \mathsf{T}) \in I^t$ *such that the image of* $\mathbf{T}(K) \cap \mathbf{G}(K)_F$ *in* $\mathsf{G}_F(\mathfrak{F})$ *is* $\mathsf{T}(\mathfrak{F})$. *Moreover, if* F *is a maximal* G*-facet in* $\mathcal{B}(T)$, *then* $(F, \mathsf{T}) \in I^m$. (*We order facets with respect to closure.*)
(4) *If* F_1 *and* F_2 *are maximal* G*-facets in* $\mathcal{B}(T)$, *then, for all apartments* $\mathcal{A}$ *in* $\mathcal{B}(G)$ *containing* F_1 *and* F_2, *we have* $A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$.

Proof. (1) Since $\mathcal{B}(T)$ is the Bruhat–Tits building of T, the first half of the statement follows from the work of Bruhat and Tits [2; 3].

For any Γ-invariant facet F of $\mathcal{B}(\mathbf{G}, K)$, we have that $F^\Gamma = F \cap \mathcal{B}(G)$ is a facet of $\mathcal{B}(G)$. Consequently, for any Γ-invariant $\mathbf{G}(K)$-facet F of $\mathcal{B}(\mathbf{T}, K) \subset \mathcal{B}(\mathbf{G}, K)$, we have that F^Γ is a G-facet of $\mathcal{B}(G)$ that is contained in $\mathcal{B}(T)$.

(2) By [1, Prop. 8.15] we can write $\mathbf{T} = \mathbf{T}_s \cdot \mathbf{T}_a$, where $\mathbf{T}_s$ denotes the maximal k-split torus in $\mathbf{T}$ and where $\mathbf{T}_a$ is the maximal k-anisotropic subtorus of $\mathbf{T}$. Let $\mathbf{M} = C_{\mathbf{G}^\circ}(\mathbf{T}_s)$. Then $\mathbf{T} \subset \mathbf{M}$ and $\mathbf{M}$ is a Levi k-subgroup of a parabolic k-subgroup of $\mathbf{G}$. Let M denote the group of k-rational points of $\mathbf{M}$. We have that the image of $\mathcal{B}(T)$ in $\mathcal{B}^{\mathrm{red}}(M)$ is a point, call it x_T. Let $\mathbf{S}$ be a maximal k-split torus in $\mathbf{M}$, and hence in $\mathbf{G}$, such that the apartment of $\mathbf{S}$ in $\mathcal{B}^{\mathrm{red}}(M)$ contains x_T. Since $\mathbf{T}_s \subset \mathbf{S}$, it follows that $\mathcal{B}(T) \subset \mathcal{A}(\mathbf{S}, k)$.

(3) Suppose that F is a G-facet in $\mathcal{B}(T)$. Let T be the maximal $\mathfrak{f}$-torus in G_F whose group of $\mathfrak{F}$-points is the image of $\mathbf{T}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$. We have $(F, \mathsf{T}) \in I^t$.

Now suppose that F is a maximal G-facet in $\mathcal{B}(T)$. Choose $\mathsf{T} \leq \mathsf{G}_F$ as in the previous paragraph. Let $\mathbf{T}_s$ be the maximal k-split torus in $\mathbf{T}$ and let T_s denote the $\mathfrak{f}$-split torus in G_F whose group of $\mathfrak{F}$-rational points is the image of $\mathbf{T}_s(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$. We have that T_s is the maximal $\mathfrak{f}$-split torus in T. If we embed $\mathcal{B}(\mathbf{T}_s, K)$ in $\mathcal{B}(\mathbf{T}, K) \subset \mathcal{B}(\mathbf{G}, K)$ in the natural way, then $\mathcal{B}(\mathbf{T}_s(k)) = \mathcal{B}(T)$. As in the proof of part (2), we may choose a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ such that $\mathcal{B}(T) \subset \mathcal{A}(\mathbf{S}, k)$ and $\mathbf{T}_s \subset \mathbf{S}$. Since F is a maximal G-facet in $\mathcal{B}(T)$, it follows that an affine root of $\mathbf{G}$ with respect to $\mathbf{S}$, k, and ν is zero on $\mathcal{B}(\mathbf{T}_s(k)) = \mathcal{B}(T)$ if and only if it is zero on F. Hence T_s is the maximal $\mathfrak{f}$-split torus in the center of G_F. Thus $(F, \mathsf{T}) \in I^m$.

(4) Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing F_1 and F_2. Since F_1 is maximal in $\mathcal{B}(T)$ and since $\mathcal{B}(T)$ is convex, we conclude that $F_2 \subset A(\mathcal{A}, F_1)$. Similarly, we have $F_1 \subset A(\mathcal{A}, F_2)$. Thus $A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$. □

Lemma 2.2.1 gives us a way to associate to a maximal unramified torus in G a pair $(F, \mathsf{T}) \in I^m$. We now address the question: If two maximal unramified tori in G can give rise to the same pair (F, T), what can we say about the two tori?

LEMMA 2.2.2. *Suppose $\mathbf{T}_1$ and $\mathbf{T}_2$ are maximal K-split k-tori of $\mathbf{G}$. If F is a Γ-invariant $\mathbf{G}(K)$-facet in $\mathcal{A}(\mathbf{T}_1, K) \cap \mathcal{A}(\mathbf{T}_2, K)$ and if the images of $\mathbf{T}_1(K) \cap \mathbf{G}(K)_F$ and $\mathbf{T}_2(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ coincide, then $\mathbf{T}_1$ and $\mathbf{T}_2$ are G_F^+-conjugate.*

Proof. Let T denote the maximal torus in G_F whose group of $\mathfrak{F}$-rational points is the image of $\mathbf{T}_1(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$. Note that T is defined over $\mathfrak{f}$.

Let $\mathbf{Z}_1$ denote the centralizer of $\mathbf{T}_1$ in $\mathbf{G}^\circ$. The group $\mathbf{Z}_1$ is a Levi k-subgroup of a parabolic K-subgroup (it is also a maximal torus) of $\mathbf{G}$. Note that $\mathcal{B}(\mathbf{Z}_1, K) = \mathcal{A}(\mathbf{T}_1, K)$, and so, for all $\mathbf{G}(K)$-facets F in $\mathcal{A}(\mathbf{T}_1, K)$, we have $\mathbf{Z}_1(K)_F^+ = \mathbf{Z}_1(K) \cap \mathbf{G}(K)_F^+$.

There exists an $h \in \mathbf{G}(K)_F$ such that ${}^h\mathbf{T}_1 = \mathbf{T}_2$. Let $\bar{h}$ denote the image of h in $\mathsf{G}_F(\mathfrak{F})$. By hypothesis, ${}^{\bar{h}}\mathsf{T} = \mathsf{T}$; thus, $\bar{h} \in (N_{\mathsf{G}_F}\mathsf{T})(\mathfrak{F})$. Consequently, by looking at the affine Bruhat decomposition, we see that there exist an $n \in (N_{\mathbf{G}^\circ}\mathbf{T}_1)(K) \cap \mathbf{G}(K)_F$ and a $g \in \mathbf{G}(K)_F^+$ such that $h = gn$. We have $\mathbf{T}_2 = {}^h\mathbf{T}_1 = {}^g\mathbf{T}_1$.

For $\gamma \in \Gamma$, let $c_g(\gamma) := g^{-1}\gamma(g)$; c_g is a 1-cocycle. We will show that $c_g(\gamma) \in \mathbf{Z}_1(K)_F^+$ for all $\gamma \in \Gamma$. Fix $\gamma \in \Gamma$. Since F is Γ-stable and $g \in \mathbf{G}(K)_F^+$, we have $c_g(\gamma) \in \mathbf{G}(K)_F^+$. Since ${}^{c_g(\gamma)}\mathbf{T}_1 = \mathbf{T}_1$, we have $c_g(\gamma) \in N_{\mathbf{G}^\circ}(\mathbf{T}_1)(K)$. Thus

$\mathcal{A}(\mathbf{T}_1, K)$ is $c_g(\gamma)$-stable. If C is an alcove in $\mathcal{A}(\mathbf{T}_1, K)$ such that $F \subset \bar{C}$, then $c_g(\gamma)$ fixes C pointwise and therefore $c_g(\gamma)$ fixes $\mathcal{A}(\mathbf{T}_1, K)$. Thus, we conclude that $c_g(\gamma) \in \mathbf{Z}_1(K)_F^+$.

Since $\mathrm{H}^1(\Gamma, \mathbf{Z}_1(K)_F^+)$ is trivial [18, Prop. 2.2], there exists a $z \in \mathbf{Z}_1(K)_F^+$ such that gz is fixed by Γ. We have ${}^{gz}\mathbf{T}_1 = \mathbf{T}_2$ and $gz \in (\mathbf{G}(K)_F^+)^\Gamma = G_F^+$. □

2.3. From Tori over $\mathfrak{f}$ to Tori over k

Suppose $(F, \mathsf{T}) \in I^t$. Let F' be the $\mathbf{G}(K)$-facet in $\mathcal{B}(\mathbf{G}, K)$ whose set of Γ-fixed points is F. In the final paragraph of the proof of [3, Prop. 5.1.10], Bruhat and Tits use [7, Exp. XI, Cor. 4.2] to prove the following.

LEMMA 2.3.1. *If $(F, \mathsf{T}) \in I^t$, then there exists a maximal K-split torus $\mathbf{T}$ of $\mathbf{G}$ such that $\mathbf{T}$ is defined over k, the apartment $\mathcal{A}(\mathbf{T}, K)$ contains F, and the image of $\mathbf{T}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ is $\mathsf{T}(\mathfrak{F})$.*

2.4. An Aside on k-Minisotropic Maximal Tori

When $\mathfrak{f}$ is finite, it is of some interest to know that a k-minisotropic torus exists in $\mathbf{G}$. (A k-torus $\mathbf{T}$ is said to be *k-minisotropic* when the maximal k-split torus in $\mathbf{T}$ coincides with the maximal k-split torus in the center of $\mathbf{G}$.)

We begin with a lemma that establishes the existence of $\mathfrak{f}$-minisotropic tori when $\mathfrak{f}$ is finite.

LEMMA 2.4.1. *Suppose $\mathfrak{f}$ is a finite field. If G is a connected reductive $\mathfrak{f}$-group, then G contains an $\mathfrak{f}$-minisotropic maximal torus.*

Proof (Gopal Prasad). Without loss of generality, we may assume that G is absolutely almost simple. Let S denote a maximal $\mathfrak{f}$-split torus of G.

Suppose first that G is $\mathfrak{f}$-split, that is, S is a maximal torus in G. Choose $n \in N_{\mathsf{G}}(\mathsf{S})(\mathfrak{f})$ such that the image of n in the Weyl group $N_{\mathsf{G}}(\mathsf{S})/\mathsf{S}$ is a Coxeter element. By Lang's theorem, there exists a $g \in \mathsf{G}(\mathfrak{F})$ such that $n = g^{-1}\sigma(g)$. (Here σ denotes the action of the Frobenius automorphism of $\mathfrak{F}$ over $\mathfrak{f}$ on $\mathsf{G}(\mathfrak{F})$.) The torus ${}^g\mathsf{S}$ is an $\mathfrak{f}$-anisotropic maximal torus in G.

Now suppose that G is not $\mathfrak{f}$-split. Since $\mathfrak{f}$ is finite, G is $\mathfrak{f}$-quasi-split and so $\mathsf{Z} = C_{\mathsf{G}}(\mathsf{S})$ is a maximal $\mathfrak{f}$-torus. Let A denote the maximal $\mathfrak{f}$-anisotropic subtorus of Z and let $\mathsf{H} = C_{\mathsf{G}}(\mathsf{A})$. Note that $\mathsf{S} \leq \mathsf{H}$. We shall show that $\mathsf{H}^{\mathrm{der}}$, the derived group of H, is $\mathfrak{f}$-split. Thus, by the previous paragraph, the group $\mathsf{H}^{\mathrm{der}}$ contains an $\mathfrak{f}$-anisotropic maximal torus A'. Since the dimension of $\mathsf{A}\mathsf{A}'$ is equal to that of Z, the $\mathfrak{f}$-torus $\mathsf{A}\mathsf{A}'$ is an $\mathfrak{f}$-anisotropic maximal torus in G.

In order to show that $\mathsf{H}^{\mathrm{der}}$ is $\mathfrak{f}$-split, we first show that S is a subgroup of $\mathsf{H}^{\mathrm{der}}$. If α is a long root in $\Phi(\mathsf{G}, \mathsf{S})$ (the set of roots of G with respect to S), then the associated root group U_α in G is one-dimensional and so commutes with the $\mathfrak{f}$-anisotropic torus A. Thus, $\mathsf{U}_\alpha \leq \mathsf{H}^{\mathrm{der}}$. Since the $\mathbb{Q}$-vector space spanned by the set of long roots in $\Phi(\mathsf{G}, \mathsf{S})$ coincides with the $\mathbb{Q}$-vector space spanned by the set $\Phi(\mathsf{G}, \mathsf{S})$, we conclude that $\mathsf{S} \leq \mathsf{H}^{\mathrm{der}}$.

Since $\mathsf{AS} = \mathsf{Z}$ is a maximal torus of G and since A is contained in the center of H, it follows that S is a maximal torus of $\mathsf{H}^{\mathrm{der}}$. Hence, $\mathsf{H}^{\mathrm{der}}$ is $\mathfrak{f}$-split. $\square$

Suppose $(F, \mathsf{T}) \in I^m$ with F a minimal facet in $\mathcal{B}(G)$; by Lemma 2.4.1, such a pair must exist when $\mathfrak{f}$ is finite. Let $\mathbf{T}$ be a maximal K-split k-torus associated to (F, T) as in Lemma 2.3.1. Since F is a minimal facet, the maximal $\mathfrak{f}$-split torus in the center of G_F has the same dimension as the maximal k-split torus in the center of $\mathbf{G}$. Hence, since the maximal k-split torus in $\mathbf{T}$ has the same dimension as the maximal $\mathfrak{f}$-split torus in T, we conclude that $\mathbf{T}$ is k-minisotropic. Since $\mathbf{G}$ is K-quasi-split, $\mathbf{Z} = C_{\mathbf{G}}(\mathbf{T})$ is a maximal k-torus. Since $\mathbf{Z}/\mathbf{T}$ is K-anisotropic, it is k-anisotropic. Thus $\mathbf{Z}$ is a k-minisotropic maximal torus. Consequently, when $\mathfrak{f}$ is finite, we have established that a k-minisotropic maximal torus of $\mathbf{G}$ exists.

3. A Parameterization of Conjugacy Classes of Maximal Unramified Tori

In this section, we present a parameterization of $\mathcal{C}^T$ via Bruhat–Tits theory.

3.1. Strong Associativity

Following [10; 11], in [5, Sec. 3.3] the concept of strong associativity is developed. We recall the definition and some of its consequences.

DEFINITION 3.1.1. Two G-facets F_1 and F_2 of $\mathcal{B}(G)$ are *strongly associated* if, for all apartments $\mathcal{A}$ containing F_1 and F_2,

$$A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2).$$

REMARK 3.1.2. Two facets F_1, F_2 of $\mathcal{B}(G)$ are strongly associated if and only if there exists an apartment $\mathcal{A}$ containing F_1 and F_2 such that $A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$; see [5, Lemma 3.3.3].

REMARK 3.1.3. Suppose F_1 and F_2 are strongly associated facets in $\mathcal{B}(G)$. Then there is an identification of G_{F_1} with G_{F_2} (see e.g. [5, Lemma 3.5.1]). Namely, the natural Γ-equivariant map

$$\mathbf{G}(K)_{F_1} \cap \mathbf{G}(K)_{F_2} \to \mathsf{G}_{F_i}(\mathfrak{F})$$

is surjective with kernel

$$\mathbf{G}(K)^+_{F_1} \cap \mathbf{G}(K)_{F_2} = \mathbf{G}(K)_{F_1} \cap \mathbf{G}(K)^+_{F_2} = \mathbf{G}(K)^+_{F_1} \cap \mathbf{G}(K)^+_{F_2}.$$

Since the kernel of the map has trivial Galois cohomology, the map induces a surjective map on Γ-fixed points.

DEFINITION 3.1.4. If F_1 and F_2 are strongly associated facets in $\mathcal{B}(G)$, then we denote the natural identification of G_{F_1} and G_{F_2} introduced in the preceding remark by $\mathsf{G}_{F_1} \overset{\mathrm{id}}{=} \mathsf{G}_{F_2}$.

3.2. *An Equivalence Relation on* I^t

We first consider the action of G on I^t. Suppose $g \in G$ and $(F, \mathsf{T}) \in I^t$. By Lemma 2.3.1 there exists a maximal K-split torus $\mathbf{T}$ of $\mathbf{G}$ such that $\mathbf{T}$ is defined over k, the apartment $\mathcal{A}(\mathbf{T}, K)$ contains F, and the image of $\mathbf{T}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ is $\mathsf{T}(\mathfrak{F})$. Define

$$g(F, \mathsf{T}) := (gF, {}^g\mathsf{T}),$$

where ${}^g\mathsf{T}$ is the maximal $\mathfrak{f}$-torus in G_{gF} whose group of $\mathfrak{F}$-rational points coincides with the image of ${}^g\mathbf{T}(K) \cap \mathbf{G}(K)_{gF}$ in $\mathsf{G}_{gF}(\mathfrak{F})$. By Lemma 2.2.2, this definition is independent of the torus $\mathbf{T}$ we choose to represent T.

We are now prepared to introduce a relation on I^t.

DEFINITION 3.2.1. Suppose (F_1, T_1) and (F_2, T_2) are two elements of I^t. We will write $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$ provided that there exist an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ and $g \in G$ such that

- $\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, gF_2)$ and
- $\mathsf{T}_1 \overset{\text{id}}{=} {}^g\mathsf{T}_2$ in $\mathsf{G}_{F_1} \overset{\text{id}}{=} \mathsf{G}_{gF_2}$.

LEMMA 3.2.2. *The relation* $\sim$ *on* I^t *is an equivalence relation.*

Proof. We will verify that the relation is transitive. The proofs that the relation is reflexive and symmetric are easier and are left to the reader.

Suppose $(F_i, \mathsf{T}_i) \in I^t$ for $i = 1, 2, 3$. Suppose $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$ and $(F_2, \mathsf{T}_2) \sim (F_3, \mathsf{T}_3)$. We want to show $(F_1, \mathsf{T}_1) \sim (F_3, \mathsf{T}_3)$.

There exist $g_2, g_3 \in G$ and apartments $\mathcal{A}_{12}$ and $\mathcal{A}_{23}$ in $\mathcal{B}(G)$ such that

- $\emptyset \neq A(\mathcal{A}_{12}, F_1) = A(\mathcal{A}_{12}, g_2F_2)$,
- $\emptyset \neq A(\mathcal{A}_{23}, F_2) = A(\mathcal{A}_{23}, g_3F_3)$

and

- $\mathsf{T}_1 \overset{\text{id}}{=} {}^{g_2}\mathsf{T}_2$ in $\mathsf{G}_{F_1} \overset{\text{id}}{=} \mathsf{G}_{g_2F_2}$,
- $\mathsf{T}_2 \overset{\text{id}}{=} {}^{g_3}\mathsf{T}_3$ in $\mathsf{G}_{F_2} \overset{\text{id}}{=} \mathsf{G}_{g_3F_3}$.

Since $g_2F_2 \subset \mathcal{A}_{12} \cap g_2\mathcal{A}_{23}$, there exists an element $h \in G_{g_2F_2}$ such that $hg_2\mathcal{A}_{23} = \mathcal{A}_{12}$. We have

$$\begin{aligned}\emptyset \neq A(\mathcal{A}_{12}, F_1) &= A(\mathcal{A}_{12}, g_2F_2) = A(hg_2\mathcal{A}_{23}, hg_2F_2)\\ &= hg_2A(\mathcal{A}_{23}, F_2) = hg_2A(\mathcal{A}_{23}, g_3F_3)\\ &= A(\mathcal{A}_{12}, hg_2g_3F_3).\end{aligned}$$

Moreover, $G_{F_1} \cap G_{g_2F_2} \cap G_{hg_2g_3F_3}$ surjects, under the natural map, onto $\mathsf{G}_{F_1}(\mathfrak{f})$ (resp., onto $\mathsf{G}_{g_2F_2}(\mathfrak{f})$ and $\mathsf{G}_{hg_2g_3F_3}(\mathfrak{f})$). Hence there exists an $h' \in G_{F_1} \cap G_{g_2F_2} \cap G_{hg_2g_3F_3}$ such that

$$\mathsf{T}_1 \overset{\text{id}}{=} {}^{g_2}\mathsf{T}_2 \overset{\text{id}}{=} {}^{h'hg_2}\mathsf{T}_2 \overset{\text{id}}{=} {}^{h'hg_2g_3}\mathsf{T}_3 \text{ in } \mathsf{G}_{F_1} \overset{\text{id}}{=} \mathsf{G}_{g_2F_2} \overset{\text{id}}{=} \mathsf{G}_{g_2F_2} \overset{\text{id}}{=} \mathsf{G}_{hg_2g_3F_3}. \qquad \square$$

3.3. A Map from $I^t/\!\sim$ to $\mathcal{C}^T$

By Lemmas 2.2.2 and 2.3.1, the following definition makes sense.

DEFINITION 3.3.1. Suppose $(F, \mathsf{T}) \in I^t$. Let $\mathbf{T}$ be any maximal K-split torus in $\mathbf{G}$ such that $\mathbf{T}$ is defined over k, the apartment $\mathcal{A}(\mathbf{T}, K)$ contains F, and the image of $\mathbf{T}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ is $\mathsf{T}(\mathfrak{F})$. Define $\mathcal{C}(F, \mathsf{T}) \in \mathcal{C}^T$ by setting $\mathcal{C}(F, \mathsf{T})$ equal to the G-conjugacy class of $\mathbf{T}(k)$.

REMARK 3.3.2. If $g \in G$ and $(F, \mathsf{T}) \in I^t$, then $\mathcal{C}(F, \mathsf{T}) = \mathcal{C}(gF, {}^g\mathsf{T})$.

LEMMA 3.3.3. *The map from I^t to $\mathcal{C}^T$ that sends $(F, \mathsf{T}) \in I^t$ to $\mathcal{C}(F, \mathsf{T})$ induces a well-defined map from $I^t/\!\sim$ to $\mathcal{C}^T$.*

Proof. Suppose (F_1, T_1) and (F_2, T_2) are two elements of I^t. We need to show that if $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$, then $\mathcal{C}(F_1, \mathsf{T}_1) = \mathcal{C}(F_2, \mathsf{T}_2)$.

Since $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$, there exist a $g \in G$ and an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that

$$\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, gF_2)$$

and

$$\mathsf{T}_1 \overset{\text{id}}{=} {}^g\mathsf{T}_2 \ \text{ in } \mathsf{G}_{F_1} \overset{\text{id}}{=} \mathsf{G}_{gF_2}.$$

By Remark 3.3.2, we can assume that $g = 1$.

By Lemma 2.3.1 there exists a maximal K-split k-torus $\mathbf{T}_2$ of $\mathbf{G}$ such that $F_2 \subset \mathcal{A}(\mathbf{T}_2, K)$ and the image of $\mathbf{T}_2(K) \cap \mathbf{G}(K)_{F_2}$ in $\mathsf{G}_{F_2}(\mathfrak{F})$ coincides with $\mathsf{T}_2(\mathfrak{F})$. Note that $\mathcal{C}(F_2, \mathsf{T}_2)$ is the G-conjugacy class of $\mathbf{T}_2(k)$. It follows from Lemma 2.2.1(2) that we can choose $h \in G_{F_2}$ such that $\mathcal{B}({}^h\mathbf{T}_2, k) \subset \mathcal{A}$. Since $\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2) \subset \mathcal{B}({}^h\mathbf{T}_2, k)$, we conclude that $F_1 \subset \mathcal{B}({}^h\mathbf{T}_2, k)$.

Let T' denote the maximal $\mathfrak{f}$-torus in G_{F_1} such that the image of ${}^h\mathbf{T}_2(K) \cap \mathbf{G}(K)_{F_1}$ in $\mathsf{G}_{F_1}(\mathfrak{F})$ coincides with $\mathsf{T}'(\mathfrak{F})$. We have

$$\mathsf{T}' \overset{\text{id}}{=} {}^h\mathsf{T}_2 \ \text{ in } \mathsf{G}_{F_1} \overset{\text{id}}{=} \mathsf{G}_{F_2}$$

and

$$\mathsf{T}_1 \overset{\text{id}}{=} \mathsf{T}_2 \ \text{ in } \mathsf{G}_{F_1} \overset{\text{id}}{=} \mathsf{G}_{F_2}.$$

Hence there exists an $h' \in G_{F_1} \cap G_{F_2}$ such that

$${}^{h'}\mathsf{T}_1 \overset{\text{id}}{=} {}^{h'}\mathsf{T}_2 \overset{\text{id}}{=} {}^h\mathsf{T}_2 \overset{\text{id}}{=} \mathsf{T}' \ \text{ in } \mathsf{G}_{F_1} \overset{\text{id}}{=} \mathsf{G}_{F_2} \overset{\text{id}}{=} \mathsf{G}_{F_2} \overset{\text{id}}{=} \mathsf{G}_{F_1}.$$

In other words, ${}^{h'}\mathsf{T}_1 = \mathsf{T}'$ in G_{F_1}. We conclude from Lemma 2.2.2 that $\mathcal{C}(F_1, \mathsf{T}_1)$ is the G-conjugacy class of ${}^{(h')^{-1}h}\mathbf{T}_2(k)$; that is, $\mathcal{C}(F_1, \mathsf{T}_1) = \mathcal{C}(F_2, \mathsf{T}_2)$. $\square$

3.4. A Bijective Correspondence

We now prove the main result of this paper.

THEOREM 3.4.1. *There is a bijective correspondence between $I^m/\!\sim$ and $\mathcal{C}^T$ given by the map sending (F, T) to $\mathcal{C}(F, \mathsf{T})$.*

Proof. By Lemma 3.3.3, this map is well-defined; by Lemma 2.2.1(3) and Lemma 2.2.2, the map is surjective. It remains to show that the map is injective.

Suppose (F_1, T_1) and (F_2, T_2) are pairs in I^m such that $\mathcal{C}(F_1, \mathsf{T}_1) = \mathcal{C}(F_2, \mathsf{T}_2)$. We need to show that $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$.

For $i = 1, 2$, by Lemma 2.3.1 we can choose a maximal K-split k-torus $\mathbf{T}_i$ of $\mathbf{G}$ such that the G-conjugacy class of $\mathbf{T}_i(k)$ is $\mathcal{C}(F_i, \mathsf{T}_i)$, the apartment $\mathcal{A}(\mathbf{T}_i, K)$ contains F_i, and the image of $\mathbf{T}_i(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_{F_i}(\mathfrak{F})$ is $\mathsf{T}_i(\mathfrak{F})$. Since $\mathcal{C}(F_1, \mathsf{T}_1) = \mathcal{C}(F_2, \mathsf{T}_2)$, there exists a $g \in G$ such that ${}^g\mathbf{T}_2 = \mathbf{T}_1$. Let $\mathbf{T} = {}^g\mathbf{T}_2 = \mathbf{T}_1$ and let $T = \mathbf{T}(k)$.

Observe that both F_1 and gF_2 lie in $\mathcal{A}(\mathbf{T}, K)^\Gamma = \mathcal{B}(T)$. Because (F_1, T_1) is a minisotropic pair, F_1 is a maximal G-facet in $\mathcal{B}(T)$. Similarly, gF_2 is a maximal G-facet in $\mathcal{B}(T)$. By Lemma 2.2.1(4), the facets F_1 and gF_2 are strongly associated. Since the image of $\mathbf{T}(K) \cap \mathbf{G}(K)_{F_1} \cap \mathbf{G}(K)_{gF_2}$ in $\mathsf{G}_{F_1}(\mathfrak{F})$ (resp., in $\mathsf{G}_{gF_2}(\mathfrak{F})$) is $\mathsf{T}_1(\mathfrak{F})$ (resp., ${}^g\mathsf{T}_2(\mathfrak{F})$), it follows that

$$\mathsf{T}_1 \overset{\mathrm{id}}{=} {}^g\mathsf{T}_2 \text{ in } \mathsf{G}_{F_1} \overset{\mathrm{id}}{=} \mathsf{G}_{gF_2}. \qquad \square$$

4. Conjugacy Classes in Stable Conjugacy Classes of Unramified Strongly Regular Elements

In this section we assume that $\mathfrak{f}$ is quasi-finite and that $\mathbf{G}$ is K-split and connected. We recall that $\mathfrak{f}$ is quasi-finite provided that $\mathfrak{f}$ is perfect and Γ is isomorphic to $\hat{\mathbb{Z}}$.

Suppose that γ is a strongly regular semisimple element of G (i.e., a regular semisimple element of G whose centralizer is connected) such that $C_G(\gamma)$ is a maximal unramified torus of G. In this section, we provide an explicit description of the set of G-conjugacy classes in the stable conjugacy class of γ. When $\mathfrak{f}$ is finite, this description is useful for harmonic analysis.

4.1. Some Fixed Notation for Section 4

Fix a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ and let $\mathbf{Z}$ be a maximal K-split k-torus of $\mathbf{G}$ containing $\mathbf{S}$ [3, Cor. 5.1.12]. Let W denote the Weyl group $N_{\mathbf{G}}(\mathbf{Z})(K)/\mathbf{Z}(K)$.

Choose a topological generator $\sigma \in \Gamma$ for Γ. Since σ preserves $\mathbf{Z}$, it acts on W. Two elements $w_1, w_2 \in W$ are *σ-conjugate* if there exists a $w' \in W$ such that $w'^{-1} w_1 \sigma(w') = w_2$. This defines an equivalence relation on W, and the partitions associated to this equivalence relation are called *σ-conjugacy classes.*

4.2. Conjugacy Classes of Maximal Tori over Quasi-finite Fields

For finite fields, Carter [4, Sec. 3.3] establishes a natural bijection between the set of conjugacy classes of maximal tori in a finite group of Lie type and the set of Frobenius conjugacy classes in its absolute Weyl group. For the field $\mathbb{C}((t))$ of Laurent series over the complex numbers in an indeterminate t, Kazhdan and Lusztig [9, Sec. 1] show that the analogue of Carter's result holds for $\mathbb{C}((t))$-split groups. Since finite fields and Laurent series over an algebraically closed field of characteristic 0 are what Serre [14, XIII, Sec. 2] describes as "the only 'non-pathological'" examples of quasi-finite fields, it is natural to ask if there is a uniform proof of this result. In this section, we answer this question in the affirmative.

Let G denote a connected reductive $\mathfrak{f}$-group. By [14, XIII, Props. 3 and 5] we have that $\mathfrak{f}$ is of dimension ≤ 1. Thus, by Steinberg's theorem (see e.g. [13, III, Sec. 2.3]), $\mathrm{H}^1(\mathfrak{f}, \mathsf{G})$ is trivial and G is $\mathfrak{f}$-quasi-split.

Fix a maximal $\mathfrak{f}$-split torus T of G. Since G is $\mathfrak{f}$-quasi-split, it follows that Z, the centralizer of T in G, is a maximal $\mathfrak{f}$-torus. We let N denote the normalizer of T in G and identify its absolute Weyl group with $\mathsf{W} := \mathsf{N}/\mathsf{Z}$. As before, we partition W into σ-conjugacy classes.

LEMMA 4.2.1. *Suppose that $\mathfrak{f}$ is quasi-finite. Then there is a natural bijective correspondence between the set of $\mathsf{G}(\mathfrak{f})$-conjugacy classes of maximal $\mathfrak{f}$-tori in G and the set of σ-conjugacy classes in W.*

Proof (Mark Reeder). Let X denote the $\mathfrak{f}$-variety consisting of all maximal tori in G. By the conjugacy of maximal tori we have the exact sequence

$$0 \to \mathsf{N} \to \mathsf{G} \to \mathsf{X} \to 0,$$

where the second-to-last map sends g to ${}^{g}\mathsf{Z}$. By [13, I, Prop. 36] we have the exact sequence (as pointed sets)

$$0 \to \mathsf{N}^{\Gamma} \to \mathsf{G}(\mathfrak{f}) \to \mathsf{X}^{\Gamma} \to \mathrm{H}^1(\mathfrak{f}, \mathsf{N}) \to \mathrm{H}^1(\mathfrak{f}, \mathsf{G}).$$

Since $\mathrm{H}^1(\mathfrak{f}, \mathsf{G})$ is trivial, we conclude that the set of $\mathsf{G}(\mathfrak{f})$-conjugacy classes of maximal $\mathfrak{f}$-tori in G is parameterized by $\mathrm{H}^1(\mathfrak{f}, \mathsf{N})$. From [13, III, Sec. 2.4, Cor. 3] it follows that the canonical map from $\mathrm{H}^1(\mathfrak{f}, \mathsf{N})$ to $\mathrm{H}^1(\mathfrak{f}, \mathsf{W})$ is bijective.

Because W is finite and σ is a topological generator for Γ, for each $w \in \mathsf{W}$ the map

$$\sigma^m \mapsto w \cdot \sigma(w) \cdots \sigma^{(m-1)}(w)$$

defines a 1-cocycle c_w. Moreover, for w, w' in W we have that c_w is cohomologous to $c_{w'}$ if and only if w and w' lie in the same σ-conjugacy class. We conclude that the map $w \mapsto c_w$ from the set of σ-conjugacy classes in W to $\mathrm{H}^1(\mathfrak{f}, \mathsf{W})$ is bijective. □

REMARK 4.2.2. Suppose $g_1, g_2 \in \mathsf{G}(\mathfrak{F})$ such that ${}^{g_1}\mathsf{Z}$ and ${}^{g_2}\mathsf{Z}$ are maximal $\mathfrak{f}$-tori. The foregoing argument shows us that the maximal $\mathfrak{f}$-tori ${}^{g_1}\mathsf{Z}$ and ${}^{g_2}\mathsf{Z}$ are $\mathsf{G}(\mathfrak{f})$-conjugate if and only if the projections of $g_1^{-1}\sigma(g_1) \in \mathsf{N}$ and $g_2^{-1}\sigma(g_2) \in \mathsf{N}$ into W lie in the same σ-conjugacy class. Moreover, for each σ-conjugacy class in W, there is a $g \in \mathsf{G}(\mathfrak{F})$ for which the image in W of $g^{-1}\sigma(g) \in \mathsf{N}$ lies in the class and ${}^{g}\mathsf{Z}$ is a maximal $\mathfrak{f}$-torus.

4.3. A Parameterization of Stable Conjugacy Classes of Maximal Unramified Tori

We say that two maximal k-tori $\mathbf{T}_1$ and $\mathbf{T}_2$ of $\mathbf{G}$ are *stably conjugate* when there exists a $g \in \mathbf{G}(\bar{k})$ such that $\mathbf{T}_1(k) = {}^{g}(\mathbf{T}_2(k))$. This is equivalent to saying that there exist a $g \in \mathbf{G}(\bar{k})$ and a strongly regular $t \in \mathbf{T}_2(k)$ such that ${}^{g}t \in \mathbf{T}_1(k)$.

The statement of the following lemma was suggested to me by Robert Kottwitz, and its proof proceeds very much like Carter's proof [4, Sec. 3.3] of the classification of the set of conjugacy classes of maximal tori in a finite group of Lie type.

LEMMA 4.3.1. *Suppose that $\mathfrak{f}$ is quasi-finite and that* $\mathbf{G}$ *is K-split and connected. Then there is a natural injective map from the set of stable conjugacy classes of maximal unramified tori in G to the set of σ-conjugacy classes in W. If* $\mathbf{G}$ *is also k-quasi-split, then this map is surjective.*

Proof. Suppose that $\mathbf{T}_i$ $(i = 1, 2)$ is a maximal unramified torus in $\mathbf{G}$. By Hilbert's Theorem 90, if $t_1 \in \mathbf{T}_1(k)$ and $t_2 \in \mathbf{T}_2(k)$ are strongly regular elements that are conjugate by an element of $\mathbf{G}(\bar{k})$, then t_1 and t_2 are conjugate by an element of $\mathbf{G}(K)$. Consequently, two maximal unramified tori $\mathbf{T}_1$ and $\mathbf{T}_2$ of $\mathbf{G}$ are stably conjugate if and only if there exists a $g \in \mathbf{G}(K)$ such that $\mathbf{T}_1(k) = {}^g(\mathbf{T}_2(k))$.

Since there is a single $\mathbf{G}(K)$-conjugacy class of maximal K-split tori in $\mathbf{G}$, all maximal unramified tori of $\mathbf{G}$ are $\mathbf{G}(K)$-conjugate to $\mathbf{Z}$. If $g \in \mathbf{G}(K)$ and ${}^g\mathbf{Z}$ is a maximal unramified torus in $\mathbf{G}$, then ${}^g\mathbf{Z} = \sigma({}^g\mathbf{Z}) = {}^{\sigma(g)}\mathbf{Z}$. Consequently, $g^{-1}\sigma(g) \in N_{\mathbf{G}}(\mathbf{Z})(K)$. Note that if $g, g' \in \mathbf{G}(K)$ such that ${}^g\mathbf{Z} = {}^{g'}\mathbf{Z}$ and ${}^g\mathbf{Z}$ is defined over k, then there exists an $n \in N_{\mathbf{G}}(\mathbf{Z})(K)$ such that $g = g'n$ and so $g^{-1}\sigma(g) = n^{-1}g'^{-1}\sigma(g')\sigma(n)$. In this way, we obtain a well-defined map ω from maximal unramified tori in $\mathbf{G}$ to the set of σ-conjugacy classes in W: $\omega({}^g\mathbf{Z})$ is the σ-conjugacy class of $g^{-1}\sigma(g)\mathbf{Z}(K)$.

Suppose $g, g' \in \mathbf{G}(K)$. We first show that if ${}^g\mathbf{Z}$ and ${}^{g'}\mathbf{Z}$ are two stably conjugate maximal unramified tori in $\mathbf{G}$, then $\omega({}^g\mathbf{Z}) = \omega({}^{g'}\mathbf{Z})$. Since ${}^g\mathbf{Z}$ and ${}^{g'}\mathbf{Z}$ are stably conjugate, there exist an $h \in \mathbf{G}(K)$ and a strongly regular $t \in \mathbf{Z}(K)$ such that ${}^gt \in ({}^g\mathbf{Z})(k)$ and ${}^{hg}t \in ({}^{g'}\mathbf{Z})(k)$. This implies that $h^{-1}\sigma(h) \in ({}^g\mathbf{Z})(K) = ({}^{\sigma(g)}\mathbf{Z})(K)$. Since

$$\begin{aligned}(hg)^{-1}\sigma(hg)\mathbf{Z}(K) &= g^{-1} \cdot (h^{-1}\sigma(h)) \cdot \sigma(g)\mathbf{Z}(K)\\ &= (g^{-1}\sigma(g)) \cdot \sigma(g)^{-1} \cdot (h^{-1}\sigma(h)) \cdot \sigma(g)\mathbf{Z}(K)\\ &= g^{-1}\sigma(g)\mathbf{Z}(K),\end{aligned}$$

we conclude that $\omega({}^g\mathbf{Z}) = \omega({}^{hg}\mathbf{Z})$. Since ${}^{hg}\mathbf{Z} = {}^{g'}\mathbf{Z}$, from the previous paragraph we have $\omega({}^{hg}\mathbf{Z}) = \omega({}^{g'}\mathbf{Z})$. Consequently, $\omega({}^g\mathbf{Z}) = \omega({}^{g'}\mathbf{Z})$.

Therefore, we have a map from the set of stable conjugacy classes of maximal unramified tori of $\mathbf{G}$ to the set of σ-conjugacy classes in W. We now show that this map is injective. Suppose $g, g' \in \mathbf{G}(K)$ such that ${}^g\mathbf{Z}$ and ${}^{g'}\mathbf{Z}$ are maximal unramified tori in $\mathbf{G}$ and $\omega({}^g\mathbf{Z}) = \omega({}^{g'}\mathbf{Z})$. By replacing g' with $g'n$ for some $n \in N_{\mathbf{G}}(\mathbf{Z})(K)$, we may assume that

$$g^{-1}\sigma(g) \in g'^{-1}\sigma(g')\mathbf{Z}(K). \tag{4.1}$$

Fix a strongly regular element $t \in ({}^g\mathbf{Z})(k)$. It will be enough to show that ${}^{g'g^{-1}}t \in ({}^{g'}\mathbf{Z})(k)$. Thus, it is enough to show

$$ {}^{g'g^{-1}}t = {}^{\sigma(g')\sigma(g)^{-1}}t. \tag{4.2}$$

However, (4.2) is valid if and only if

$$^{g^{-1}}t = {}^{(g'^{-1}\sigma(g'))\sigma(g)^{-1}}t.$$

Since ${}^{\sigma(g)^{-1}}t \in \mathbf{Z}(K)$, it follows from (4.1) that the map is injective.

Finally, we must show that the map is surjective when G is an unramified group. Let x be a hyperspecial point in the apartment corresponding to $\mathbf{S}$ in $\mathcal{B}^{\mathrm{red}}(G)$ and let Z be the maximal $\mathfrak{F}$-split $\mathfrak{f}$-torus in G_x corresponding to $\mathbf{Z}$. Note that W is naturally isomorphic to $N_{\mathsf{G}_x}(\mathsf{Z})(\mathfrak{F})/\mathsf{Z}(\mathfrak{F})$. By Remark 4.2.2, for every σ-conjugacy class $\mathcal{O}$ in W there exists a $\tilde{g} \in \mathsf{G}_x(\mathfrak{F})$ such that ${}^{\tilde{g}}\mathsf{Z}$ is a maximal $\mathfrak{f}$-torus in G_x and the image of $\tilde{g}^{-1}\sigma(\tilde{g})$ in W lies in $\mathcal{O}$. As in Lemma 2.3.1, we can lift ${}^{\tilde{g}}\mathsf{Z}$ to a maximal unramified torus $\mathbf{T}'$ in $\mathbf{G}$ with $x \in \mathcal{B}(\mathbf{T}', k) = \mathcal{A}(\mathbf{T}', K)^{\Gamma}$. Since $x \in \mathcal{A}(\mathbf{T}', K) \cap \mathcal{A}(\mathbf{Z}, K)$, there exists a $g \in \mathbf{G}(K)_x$ such that $\mathbf{T}' = {}^{g}\mathbf{Z}$. Let $\bar{g}$ denote the image of g in $\mathsf{G}_x(\mathfrak{F})$. We have ${}^{\bar{g}}\mathsf{Z} = {}^{\tilde{g}}\mathsf{Z}$ and so (by Remark 4.2.2) the image of $\bar{g}^{-1}\sigma(\bar{g})$, and hence that of $g^{-1}\sigma(g)$, lies in $\mathcal{O}$. □

For $(F_i, \mathsf{T}_i) \in I^m$ $(i = 1, 2)$, the proof of Lemma 4.3.1 yields a simple criterion for determining whether $\mathcal{C}(F_1, \mathsf{T}_1)$ and $\mathcal{C}(F_2, \mathsf{T}_2)$ lie in the same stable conjugacy class. Without loss of generality, we assume that $F_1, F_2 \subset \mathcal{A}(\mathbf{S}, k)$. Let Z_i denote the maximal $\mathfrak{f}$-torus in G_{F_i} corresponding to $\mathbf{Z}$. Let $\mathsf{W}_{F_i} := N_{\mathsf{G}_{F_i}}(\mathsf{Z}_i)(\mathfrak{F})/\mathsf{Z}_i(\mathfrak{F})$ and let $\mathcal{O}_{F_i}(\mathsf{T}_i)$ be the σ-conjugacy class in W_{F_i} parameterizing the $\mathsf{G}_{F_i}(\mathfrak{f})$-conjugacy class of $\mathsf{T}_i(\mathfrak{f})$. Set $\mathcal{O}(\mathsf{T}_i)$ equal to the σ-conjugacy class in W that contains the image under the projection from $N_{\mathbf{G}}(\mathbf{Z})(K)$ onto W of the lift of $\mathcal{O}_{F_i}(\mathsf{T}_i)$ into $N_{\mathbf{G}}(\mathbf{Z})(K)$. Observe that we have a natural embedding of W_{F_i} in W and of $\mathcal{O}_{F_i}(\mathsf{T}_i)$ in $\mathcal{O}(\mathsf{T}_i)$.

COROLLARY 4.3.2. *Suppose that $\mathfrak{f}$ is quasi-finite and that $\mathbf{G}$ is K-split and connected. In the notation introduced previously, we then have: $\mathcal{C}(F_1, \mathsf{T}_1)$ and $\mathcal{C}(F_2, \mathsf{T}_2)$ lie in the same stable conjugacy class if and only if $\mathcal{O}(\mathsf{T}_1) = \mathcal{O}(\mathsf{T}_2)$.*

4.4. Stable Conjugacy Classes in an Unramified Maximal Torus

Suppose $\gamma \in G$ is a strongly regular semisimple element such that $T = C_G(\gamma)$ is a maximal unramified torus of G. Set

$$S_\gamma^T := {}^{\mathbf{G}(\bar{k})}\gamma \cap T;$$

this is a finite set. For $s, s' \in S_\gamma^T$, we write $s \approx s'$ when there is a $g \in G$ such that ${}^{g}s = s'$; this defines an equivalence relation on S_γ^T. In this section we give an explicit description of

$$S_\gamma^T/\approx.$$

DEFINITION 4.4.1. Suppose that F is a G-facet in $\mathcal{A}(\mathbf{S}, k)$. Let $W(F)$ denote the image in W of the stabilizer (not the fixator) in $N_{\mathbf{G}}(\mathbf{Z})(K)$ of $A(\mathcal{A}(\mathbf{Z}, K), F)$. For $w \in W$, we define the subgroups

$$W_{w\circ\sigma} := \{w' \in W \mid {}^{w}(\sigma(w')) = w'\}$$

and

$$W(F)_{w\circ\sigma} := W(F) \cap W_{w\circ\sigma}$$

of W.

LEMMA 4.4.2. *Suppose that $\mathfrak{f}$ is quasi-finite and that $\mathbf{G}$ is K-split and connected. Choose $(F, \mathsf{T}) \in I^m$ such that $F \subset \mathcal{A}(\mathbf{S}, k)$ and fix $T \in \mathcal{C}(F, \mathsf{T})$. For a strongly regular element $\gamma \in T$, we have that $S_\gamma^T/\approx$ is in (natural) bijective correspondence with the coset space*

$$W_{w_\mathsf{T}\circ\sigma}/W(F)_{w_\mathsf{T}\circ\sigma}.$$

Here w_T is any element of $\mathcal{O}_F(\mathsf{T}) \subset \mathcal{O}(\mathsf{T}) \subset W$ (notation introduced prior to Corollary 4.3.2).

REMARK 4.4.3. If the image of F in $\mathcal{B}^{\mathrm{red}}(G)$ is a hyperspecial vertex, then it follows that $W(F) = W$ and so

$$^{\mathbf{G}(\bar{k})}\gamma \cap T = {}^G\gamma \cap T.$$

Proof of Lemma 4.4.2. If we replace $w_\mathsf{T} \in \mathcal{O}_F(\mathsf{T})$ with a σ-conjugate, say $w'_\mathsf{T} = w'^{-1}w_\mathsf{T}\sigma(w')$ for some $w' \in W_F \leq W$, then

$$^{w'}W_{w'_\mathsf{T}\circ\sigma} = W_{w_\mathsf{T}\circ\sigma}$$

and similarly

$$^{w'}W(F)_{w'_\mathsf{T}\circ\sigma} = W(F)_{w_\mathsf{T}\circ\sigma}.$$

Thus, it is enough to show that the bijection works for a single element of $\mathcal{O}_F(\mathsf{T})$.

Let $\mathbf{T}$ denote the maximal unramified torus of $\mathbf{G}$ for which $T = \mathbf{T}(k)$. From Hilbert's Theorem 90, we have

$$^{\mathbf{G}(\bar{k})}\gamma \cap T = {}^{\mathbf{G}(K)}\gamma \cap T.$$

Therefore, the map $n \mapsto {}^{n^{-1}}\gamma$ from $N_\mathbf{G}(\mathbf{T})(K)$ to $\mathbf{T}(K)$ induces a bijective correspondence between the coset space

$$(N_\mathbf{G}(\mathbf{T})(K)/\mathbf{T}(K))^\Gamma/(N_G(T)/T)$$

and $S_\gamma^T/\approx$. (Here we think of $N_G(T)/T$ as a subgroup of $(N_\mathbf{G}(\mathbf{T})(K)/\mathbf{T}(K))^\Gamma$ via the injection induced from the natural embedding of $N_G(T)$ in $N_\mathbf{G}(\mathbf{T})(K)$.)

Let M_F be the Levi subgroup of G generated by $\mathbf{Z}(k)$ and the root groups U_ψ for affine roots ψ of $\mathbf{G}$ (with respect to $\mathbf{S}$, k, and ν) that are constant on F. Let $\mathbf{M}_F$ denote the Levi k-subgroup of a parabolic k-subgroup for which $M_F = \mathbf{M}_F(k)$. We have $\mathbf{M}_F(K)_F/\mathbf{M}_F(K)_F^+ = \mathbf{G}(K)_F/\mathbf{G}(K)_F^+$. Consequently, the pair (F, T) occurs in the analogue of I^m for M_F and we may assume that $\mathbf{T} \subset \mathbf{M}_F$. By Lemma 2.2.1(2) and (3) we may also assume that $\mathcal{B}(T) = A(\mathcal{A}(\mathbf{S}, k), F) \subset \mathcal{A}(\mathbf{S}, k) = \mathcal{A}(\mathbf{Z}, K)^\Gamma \subset \mathcal{A}(\mathbf{Z}, K)$. Since

$$\mathcal{B}(T) = \mathcal{A}(\mathbf{T}, K)^\Gamma \subset \mathcal{A}(\mathbf{T}, K) \cap \mathcal{A}(\mathbf{Z}, K),$$

there exists an $m_\mathsf{T} \in \mathbf{M}_F(K)_F$ such that $\mathbf{T} = {}^{m_\mathsf{T}}\mathbf{Z}$. Set $n_\mathsf{T} := (m_\mathsf{T}^{-1}\sigma(m_\mathsf{T})) \in N_\mathbf{G}(\mathbf{Z})(K)$ and let w_T denote the image of n_T in W. In the notation introduced prior to Corollary 4.3.2, we have $w_{\mathsf{T}} \in \mathcal{O}_F(\mathsf{T})$. Note that both m_{T} and n_{T} lie in $\mathbf{G}(K)_y$ for all $y \in \mathcal{B}(T) = A(\mathcal{A}(\mathbf{Z}, K), F)$.

To finish, we show that the group isomorphism $n \to {}^{m_{\mathsf{T}}^{-1}}n$ from $N_{\mathbf{G}}(\mathbf{T})(K)$ to $N_{\mathbf{G}}(\mathbf{Z})(K)$ induces group isomorphisms

$$(N_{\mathbf{G}}(\mathbf{T})(K)/\mathbf{T}(K))^{\Gamma} \cong W_{w_{\mathsf{T}}\circ\sigma}$$

and

$$N_G(T)/T \cong W(F)_{w_{\mathsf{T}}\circ\sigma}.$$

We first show that $(N_{\mathbf{G}}(\mathbf{T})(K)/\mathbf{T}(K))^{\Gamma}$ is isomorphic to $W_{w_{\mathsf{T}}\circ\sigma}$. Choose $w \in W$ and let $n \in N_{\mathbf{G}}(\mathbf{Z})(K)$ be a representative for w. It is enough to show that

$$w = {}^{w_{\mathsf{T}}}\sigma(w) \text{ if and only if } ({}^{m_{\mathsf{T}}}n)^{-1}\sigma({}^{m_{\mathsf{T}}}n) \in \mathbf{T}(K). \tag{4.3}$$

Observe that $w = {}^{w_{\mathsf{T}}}\sigma(w)$ if and only if $\sigma(w)^{-1}({}^{w_{\mathsf{T}}^{-1}}w) = 1$. This last equality is equivalent to $\sigma(n)^{-1}({}^{n_{\mathsf{T}}^{-1}}n) \in \mathbf{Z}(K)$, which happens if and only if $({}^{m_{\mathsf{T}}}n)^{-1}\sigma({}^{m_{\mathsf{T}}}n) \in \mathbf{T}(K)$.

We now show that $N_G(T)/T$ is isomorphic to $W(F)_{w_{\mathsf{T}}\circ\sigma}$. Because the map induced from m_{T}^{-1}-conjugation carries $N_G(T)/T$ into $W(F)_{w_{\mathsf{T}}\circ\sigma}$, it is enough to show that, for each $w \in W(F)_{w_{\mathsf{T}}\circ\sigma}$, there is an $n' \in N_G(T)$ such that

$$({}^{m_{\mathsf{T}}^{-1}}n')\mathbf{Z}(K) = w.$$

Fix $w \in W(F)_{w_{\mathsf{T}}\circ\sigma}$. Since $w \in W(F)$, there is an $n \in N_{\mathbf{G}}(\mathbf{Z})(K)$ such that the image of n in W is w and such that n, and hence ${}^{m_{\mathsf{T}}}n \in N_{\mathbf{G}}(\mathbf{T})(K)$, stabilizes $\mathcal{B}(T)$. Fix $y \in \mathcal{B}(T)$. We have

$$({}^{m_{\mathsf{T}}}n) \cdot y = \sigma(({}^{m_{\mathsf{T}}}n) \cdot y) = \sigma({}^{m_{\mathsf{T}}}n) \cdot y.$$

Hence $({}^{m_{\mathsf{T}}}n)^{-1}\sigma({}^{m_{\mathsf{T}}}n) \in \mathrm{Fix}_{\mathbf{G}(K)}(y)$. Since $w \in W_{w_{\mathsf{T}}\circ\sigma}$, from (4.3) it follows that $({}^{m_{\mathsf{T}}}n)^{-1}\sigma({}^{m_{\mathsf{T}}}n) \in \mathbf{T}(K)$. Therefore,

$$({}^{m_{\mathsf{T}}}n)^{-1}\sigma({}^{m_{\mathsf{T}}}n) \in \mathrm{Fix}_{\mathbf{G}(K)}(y) \cap \mathbf{T}(K) = \mathbf{T}(K)_y.$$

Since $\mathrm{H}^1(\Gamma, \mathbf{T}(K)_y^+)$ and $\mathrm{H}^1(\Gamma, \mathsf{T})$ are both trivial, by [13, I, Prop. 43] we conclude that $\mathrm{H}^1(\Gamma, \mathbf{T}(K)_y)$ is trivial. Consequently, there exists a $t \in \mathbf{T}(K)_y$ such that $({}^{m_{\mathsf{T}}}n)t \in N_G(T)$. Let $n' = ({}^{m_{\mathsf{T}}}n)t$. $\square$

4.5. The Result

The proof of the following theorem is immediate from the results of Sections 4.3 and 4.4.

THEOREM 4.5.1. *Suppose that $\mathfrak{f}$ is quasi-finite and that $\mathbf{G}$ is K-split and connected. Choose $(F', \mathsf{T}') \in I^m$ such that $F' \subset \mathcal{A}(\mathbf{S}, k)$ and fix $T' \in \mathcal{C}(F', \mathsf{T}')$. For a strongly regular element $\gamma \in T'$, the set of G-conjugacy classes in*

$$^{\mathbf{G}(\bar{k})}\gamma \cap G$$

is parameterized by the set of pairs

$$\{\bar{E}, \bar{w}\}.$$

Here $\bar{E} \in I^m/{\sim}$ is represented by some $(F, \mathsf{T}) \in I^m$ with $F \subset \mathcal{A}(\mathbf{S}, k)$ and $\mathcal{O}(\mathsf{T}) = \mathcal{O}(\mathsf{T}')$. If we fix $w_{\mathsf{T}} \in \mathcal{O}_F(\mathsf{T})$, then $\bar{w}$ runs over the cosets in $W_{w_{\mathsf{T}}\circ\sigma}/W(F)_{w_{\mathsf{T}}\circ\sigma}$.

5. A Parameterization of Maximal-Rank Unramified Subgroups of *G*

We return to our original assumptions on $\mathfrak{f}$ and $\mathbf{G}$ (i.e., those of Sections 1–3).

This section presents a generalization of the material in Sections 2–3. As a by-product of the way in which this paper was written, the material presented next borrows heavily from the presentation there.

5.1. Definitions

We recall that an algebraic group $\mathbf{H}$ is called an *unramified group* provided that $\mathbf{H}$ is a connected reductive K-split k-group and $\mathcal{B}^{\mathrm{red}}(\mathbf{H}, k)$ contains a hyperspecial vertex. We also call $H = \mathbf{H}(k)$ an *unramified group.*

Let I denote the set of pairs (F, H) where F is a facet in $\mathcal{B}(G)$ and H is a maximal-rank connected reductive $\mathfrak{f}$-subgroup in G_F. Note that $I^m \subset I^t \subset I$. We also consider the subset $I^c \subset I$ of cuspidal pairs in I; a pair $(F, \mathsf{H}) \in I$ is said to be *cuspidal* when the maximal $\mathfrak{f}$-split torus in the center of H coincides with the maximal $\mathfrak{f}$-split torus in the center of G_F. (Equivalently, (F, H) is a cuspidal pair if and only if H lies in no proper parabolic $\mathfrak{f}$-subgroup of G_F.) Observe that $I^m = I^c \cap I^t$.

5.2. Maximal-Rank Subgroups over k and $\mathfrak{f}$

In this section we show how to move between maximal-rank unramified k-subgroups of $\mathbf{G}$ and maximal-rank connected reductive subgroups over $\mathfrak{f}$. The following lemma is an immediate consequence of Lemma 2.1.1.

LEMMA 5.2.1. *Suppose that* $\mathbf{H}$ *is a maximal-rank connected reductive K-split k-subgroup of* $\mathbf{G}$. *Then every maximal K-split k-torus in* $\mathbf{H}$ *is a maximal K-split torus in* $\mathbf{G}$.

5.2.A. From Maximal-Rank Unramified Subgroups of $\mathbf{G}$ *to Connected Reductive* $\mathfrak{f}$*-Groups*

Suppose that $\mathbf{H}$ is a maximal-rank unramified subgroup of $\mathbf{G}$. We identify $\mathcal{B}(\mathbf{H}, K)$ with its image in $\mathcal{B}(\mathbf{G}, K)$. (As usual, there does not exist a canonical embedding of $\mathcal{B}(\mathbf{H}, K)$ in $\mathcal{B}(\mathbf{G}, K)$, but the image of any natural embedding is independent of the embedding [3, 4.2.18].) We therefore have

$$\mathcal{B}(H) = \mathcal{B}(\mathbf{H}, K)^{\Gamma} \subset \mathcal{B}(\mathbf{G}, K)^{\Gamma} = \mathcal{B}(G).$$

We now collect some facts about $\mathcal{B}(H)$.

LEMMA 5.2.2. *Suppose that* $\mathbf{H}$ *is a maximal-rank unramified subgroup of* $\mathbf{G}$. *Let H denote the group of k-rational points of* $\mathbf{H}$.

(1) *$\mathcal{B}(H)$ is a nonempty, closed, convex subset of $\mathcal{B}(G)$. Moreover, $\mathcal{B}(H)$ is the union of the G-facets in $\mathcal{B}(G)$ that meet it.*

(2) *For all G-facets F in $\mathcal{B}(H)$, there exists $(F, \mathsf{H}) \in I$ such that the image of $\mathbf{H}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ is $\mathsf{H}(\mathfrak{F})$. Moreover, if F is a maximal G-facet in the preimage in $\mathcal{B}(H)$ of a facet in $\mathcal{B}^{\mathrm{red}}(H)$, then $(F, \mathsf{H}) \in I^c$.*

(3) *If F_1 and F_2 are maximal G-facets in the preimage in $\mathcal{B}(H)$ of a facet in $\mathcal{B}^{\mathrm{red}}(H)$, then, for all apartments $\mathcal{A}$ in $\mathcal{B}(G)$ containing F_1 and F_2, we have $A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$.*

Proof. (1) The proof here is identical to that of Lemma 2.2.1(1).

(2) Suppose that F is a G-facet in $\mathcal{B}(H)$. Let H be the maximal-rank connected reductive $\mathfrak{f}$-subgroup in G_F whose group of $\mathfrak{F}$-rational points is the image of $\mathbf{H}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$. We have $(F, \mathsf{H}) \in I$.

Now suppose that F is a maximal G-facet in the preimage in $\mathcal{B}(H)$ of a facet in $\mathcal{B}^{\mathrm{red}}(H)$. Choose a subgroup H in G_F as in the previous paragraph.

Let $\mathbf{S}'$ be a maximal k-split torus in $\mathbf{H}$ such that $F \subset \mathcal{B}(\mathbf{S}', k) \subset \mathcal{B}(\mathbf{H}, k)$ and let $\mathbf{S}$ be a maximal k-split torus in $\mathbf{G}$ such that $\mathbf{S}' \subset \mathbf{S}$. Let $\mathbf{Z_H}$ denote the maximal k-split torus in the center of $\mathbf{H}$ and let $\mathsf{Z}_{\mathbf{H}} \subset \mathsf{G}_F$ denote the $\mathfrak{f}$-split torus whose group of $\mathfrak{F}$-rational points coincides with the image of $\mathbf{Z_H}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$. We have $\mathbf{Z_H} \subset \mathbf{S}' \subset \mathbf{S}$. Since F is a maximal G-facet in the preimage in $\mathcal{B}(H) = \mathcal{B}^{\mathrm{red}}(H) \times \mathcal{B}(\mathbf{Z_H}, k)$ of a facet in $\mathcal{B}^{\mathrm{red}}(H)$, it follows that, for all affine roots ψ of $\mathbf{G}$ with respect to $\mathbf{S}$, k, and ν, if ψ is constant on F, then ψ is constant on $\mathcal{B}(\mathbf{Z_H}, k)$. Therefore, $\mathsf{Z}_{\mathbf{H}}$ is contained in the center of G_F and so H cannot lie in a proper parabolic $\mathfrak{f}$-subgroup of G_F.

(3) This is clear. □

REMARK 5.2.3. In the notation used in the proof of part (2), we have that the torus $\mathsf{Z}_{\mathbf{H}}$ is the maximal $\mathfrak{f}$-split torus in the center of G_F exactly when F is a maximal G-facet in the preimage in $\mathcal{B}(H)$ of a vertex in $\mathcal{B}^{\mathrm{red}}(H)$.

Given H a maximal-rank unramified subgroup in G and $x \in \mathcal{B}^{\mathrm{red}}(H)$ a hyperspecial vertex, Lemma 5.2.2 gives us a way to associate to the pair (H, x) a pair (F, H) in I^c.

5.2.B. From Subgroups over $\mathfrak{f}$ to Unramified Subgroups over k

LEMMA 5.2.4. *Suppose $\mathbf{G}$ is K-split and suppose that $\mathbf{H}_1$ and $\mathbf{H}_2$ are maximal-rank unramified subgroups of $\mathbf{G}$. Suppose F is a Γ-invariant $\mathbf{G}(K)$-facet in $\mathcal{B}(\mathbf{H}_1, K) \cap \mathcal{B}(\mathbf{H}_2, K)$ that projects to a Γ-fixed special vertex in $\mathcal{B}^{\mathrm{red}}(\mathbf{H}_i, K)$ for $i \in \{1, 2\}$. If the images of $\mathbf{H}_1(K) \cap \mathbf{G}(K)_F$ and $\mathbf{H}_2(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ coincide, then $\mathbf{H}_1$ and $\mathbf{H}_2$ are G_F^+-conjugate.*

Proof. Let H denote the maximal-rank reductive subgroup in G_F whose group of $\mathfrak{F}$-rational points is the image of $\mathbf{H}_1(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$. Note that H is defined over $\mathfrak{f}$. Let T be a maximal $\mathfrak{f}$-torus in H that contains a maximal $\mathfrak{f}$-split torus of H. Let $\Phi(\mathsf{H}, \mathsf{T})$ denote the $\mathfrak{F}$-root system of H with respect to T. As in Lemma 2.3.1, we choose a maximal K-split k-torus $\mathbf{T}_i$ in $\mathbf{H}_i$ lifting T. Observe that, since $\mathbf{G}$ is K-split and the image of F in $\mathcal{B}^{\mathrm{red}}(\mathbf{H}_i, k)$ is hyperspecial, it follows that $\mathbf{H}_i$ is completely determined by $\Phi(\mathsf{H},\mathsf{T})$ and $\mathbf{T}_i$. By Lemma 2.2.2 we now conclude that $\mathbf{H}_1$ and $\mathbf{H}_2$ are G_F^+-conjugate. □

LEMMA 5.2.5. *Suppose* $\mathbf{G}$ *is* K*-split. Suppose that* F *is a facet in* $\mathcal{B}(G)$ *and that* H *is a maximal-rank connected reductive* $\mathfrak{f}$*-subgroup of* G_F*. Then there exists a maximal-rank unramified subgroup* $\mathbf{H}$ *in* $\mathbf{G}$ *such that*:

(1) *the facet* F *belongs to* $\mathcal{B}(\mathbf{H},k)$;
(2) *the image of* $\mathbf{H}(K)\cap\mathbf{G}(K)_F$ *in* $\mathsf{G}_F(\mathfrak{F})$ *is the group of* $\mathfrak{F}$*-rational points of* H; *and*
(3) *the image of* F *in* $\mathcal{B}^{\mathrm{red}}(\mathbf{H},k)$ *is a hyperspecial vertex,* x_F.

Proof. Let T be a maximal $\mathfrak{f}$-torus in H (and hence in G_F) that contains a maximal $\mathfrak{f}$-split torus of H. Let $\Phi(\mathsf{H},\mathsf{T})$ denote the $\mathfrak{F}$-root system of H with respect to T. As in Lemma 2.3.1, let $\mathbf{T}$ be a lift of T to a maximal K-split k-torus $\mathbf{T}$ in $\mathbf{G}$. We think of $\Phi(\mathsf{H},\mathsf{T})$ as a subset of $\Phi(\mathbf{G},\mathbf{T})$, the root system of $\mathbf{G}$ with respect to $\mathbf{T}$; note that $\Phi(\mathsf{H},\mathsf{T})$ is a closed root system of $\Phi(\mathbf{G},\mathbf{T})$. Let $\mathbf{H}$ be the K-split full-rank subgroup of $\mathbf{G}$ whose group of K-rational points is generated by $\mathbf{T}(K)$ and the root groups in $\mathbf{G}(K)$ corresponding to elements of $\Phi(\mathsf{H},\mathsf{T})$. Observe that $\mathbf{H}$ is defined over k and that (by construction) the image of $F\subset\mathcal{B}(\mathbf{H},K)$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{H},K)$ is a Γ-fixed special vertex. The lemma follows. □

REMARK 5.2.6. If, in the statement of Lemma 5.2.5, H is also assumed to be $\mathfrak{f}$-cuspidal, then F is a maximal G-facet in the preimage in $\mathcal{B}(\mathbf{H},k)$ of x_F.

5.3. A Parameterization of Maximal-Rank Unramified Subgroups

Suppose $\mathbf{G}$ is K-split. Recall that $\mathcal{C}$ denotes the set of G-conjugacy classes of pairs (H,x), where H is a maximal-rank unramified subgroup in G and x is a hyperspecial point in $\mathcal{B}^{\mathrm{red}}(H)$. In this section we present a parameterization of $\mathcal{C}$ via Bruhat–Tits theory.

5.3.A. An Equivalence Relation on I

Suppose $g\in G$ and $(F,\mathsf{H})\in I$. By Lemma 5.2.5 there exists a maximal-rank unramified subgroup $\mathbf{H}$ of $\mathbf{G}$ such that the building of $\mathcal{B}(H)$ contains F, the image of $\mathbf{H}(K)\cap\mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ is $\mathsf{H}(\mathfrak{F})$, and the image of F in $\mathcal{B}^{\mathrm{red}}(H)$ is a hyperspecial vertex. Define

$$g(F,\mathsf{H}) := (gF, {}^{g}\mathsf{H}),$$

where ${}^{g}\mathsf{H}$ is the maximal-rank connected reductive $\mathfrak{f}$-group in G_{gF} whose group of $\mathfrak{F}$-rational points coincides with the image of ${}^{g}\mathbf{H}(K)\cap\mathbf{G}(K)_{gF}$ in $\mathsf{G}_{gF}(\mathfrak{F})$. By Lemma 5.2.4, the definition of ${}^{g}\mathsf{H}$ is independent of the unramified subgroup $\mathbf{H}$ of $\mathbf{G}$ that we choose to represent H.

DEFINITION 5.3.1. Suppose (F_1,H_1) and (F_2,H_2) are two elements of I. We will write $(F_1,\mathsf{H}_1)\sim(F_2,\mathsf{H}_2)$ provided that there exist an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ and a $g\in G$ such that

- $\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, gF_2)$ and
- $\mathsf{H}_1 \stackrel{\text{id}}{=} {}^g\mathsf{H}_2$ in $\mathsf{G}_{F_1} \stackrel{\text{id}}{=} \mathsf{G}_{gF_2}$.

The proof of the following lemma is nearly identical to the proof of Lemma 3.2.2.

LEMMA 5.3.2. *Suppose* $\mathbf{G}$ *is* K*-split. Then the relation* $\sim$ *on* I *is an equivalence relation.*

5.3.B. A Map from $I/\!\sim$ *to* $\mathcal{C}$

By Lemmas 5.2.4 and 5.2.5, the following definition makes sense.

DEFINITION 5.3.3. Suppose $(F, \mathsf{H}) \in I$. Let $\mathbf{H}$ be any maximal-rank unramified subgroup of $\mathbf{G}$ such that the building $\mathcal{B}(\mathbf{H}, K)$ contains F, the image of $\mathbf{H}(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_F(\mathfrak{F})$ is $\mathsf{H}(\mathfrak{F})$, and the image of F in $\mathcal{B}^{\text{red}}(\mathbf{H}, k)$ is a hyperspecial vertex x_F. Define $\mathcal{C}(F, \mathsf{H}) \in \mathcal{C}$ by setting $\mathcal{C}(F, \mathsf{H})$ equal to the G-conjugacy class of the pair $(\mathbf{H}(k), x_F)$.

REMARK 5.3.4. If $g \in G$ and $(F, \mathsf{H}) \in I$, then $\mathcal{C}(F, \mathsf{H}) = \mathcal{C}(gF, {}^g\mathsf{H})$.

LEMMA 5.3.5. *Suppose* $\mathbf{G}$ *is* K*-split. Then the map from* I *to* $\mathcal{C}$ *that sends* $(F, \mathsf{H}) \in I$ *to* $\mathcal{C}(F, \mathsf{H})$ *induces a well-defined map from* $I/\!\sim$ *to* $\mathcal{C}$.

Proof. Suppose (F_1, H_1) and (F_2, H_2) are two elements of I. We need to show that if $(F_1, \mathsf{H}_1) \sim (F_2, \mathsf{H}_2)$, then $\mathcal{C}(F_1, \mathsf{H}_1) = \mathcal{C}(F_2, \mathsf{H}_2)$.

Since $(F_1, \mathsf{H}_1) \sim (F_2, \mathsf{H}_2)$, there exist a $g \in G$ and an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that

$$\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, gF_2)$$

and

$$\mathsf{H}_1 \stackrel{\text{id}}{=} {}^g\mathsf{H}_2 \text{ in } \mathsf{G}_{F_1} \stackrel{\text{id}}{=} \mathsf{G}_{gF_2}.$$

By Remark 5.3.4, we can assume that $g = 1$.

By Lemma 5.2.5, there exists a maximal-rank unramified subgroup $\mathbf{H}_2$ of $\mathbf{G}$ such that $F_2 \subset \mathcal{B}(\mathbf{H}_2, K)$, the image of $\mathbf{H}_2(K) \cap \mathbf{G}(K)_{F_2}$ in $\mathsf{G}_{F_2}(\mathfrak{F})$ coincides with $\mathsf{H}_2(\mathfrak{F})$, and the image of F_2 in $\mathcal{B}^{\text{red}}(\mathbf{H}, k)$ is hyperspecial. Note that $\mathcal{C}(F_2, \mathsf{H}_2)$ is the G-conjugacy class of $(\mathbf{H}_2(k), x_{F_2})$, where x_{F_2} is the image of F_2 in $\mathcal{B}^{\text{red}}(H)$. Let T_2 be a maximal $\mathfrak{f}$-torus in H_2. By Lemma 2.3.1, we can choose a maximal K-split k-torus $\mathbf{T}_2$ in $\mathbf{H}_2$ lifting T_2 such that $F_2 \subset \mathcal{B}(\mathbf{T}_2, k)$. It follows from Lemma 2.2.1(2) that we can choose $h \in G_{F_2}$ such that $\mathcal{B}({}^h\mathbf{T}_2, k) \subset \mathcal{A}$. Since $\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2) \subset \mathcal{B}({}^h\mathbf{T}_2, k)$, we conclude that $F_1 \subset \mathcal{B}({}^h\mathbf{T}_2, k) \subset \mathcal{B}({}^h\mathbf{H}_2, k)$.

Let H' denote the maximal-rank connected reductive $\mathfrak{f}$-subgroup in G_{F_1} such that the image of ${}^h\mathbf{H}_2(K) \cap \mathbf{G}(K)_{F_1}$ in $\mathsf{G}_{F_1}(\mathfrak{F})$ coincides with $\mathsf{H}'(\mathfrak{F})$. We have

$$\mathsf{H}' \stackrel{\text{id}}{=} {}^h\mathsf{H}_2 \text{ in } \mathsf{G}_{F_1} \stackrel{\text{id}}{=} \mathsf{G}_{F_2}$$

and

$$\mathsf{H}_1 \stackrel{\text{id}}{=} \mathsf{H}_2 \text{ in } \mathsf{G}_{F_1} \stackrel{\text{id}}{=} \mathsf{G}_{F_2}.$$

Hence there exists an $h' \in G_{F_1} \cap G_{F_2}$ such that

$$ {}^{h'}\mathsf{H}_1 \overset{\mathrm{id}}{=} {}^{h'}\mathsf{H}_2 \overset{\mathrm{id}}{=} {}^{h}\mathsf{H}_2 \overset{\mathrm{id}}{=} \mathsf{H}' \text{ in } \mathsf{G}_{F_1} \overset{\mathrm{id}}{=} \mathsf{G}_{F_2} \overset{\mathrm{id}}{=} \mathsf{G}_{F_2} \overset{\mathrm{id}}{=} \mathsf{G}_{F_1}. $$

In other words, ${}^{h'}\mathsf{H}_1 = \mathsf{H}'$ in G_{F_1}. We conclude from Lemma 5.2.4 that $\mathcal{C}(F_1, \mathsf{H}_1)$ is the G-conjugacy class of ${}^{(h')^{-1}h}\mathbf{H}_2(k)$; that is, $\mathcal{C}(F_1, \mathsf{H}_1) = \mathcal{C}(F_2, \mathsf{H}_2)$.

5.3.C. A Bijective Correspondence

We now prove the final result of this paper.

THEOREM 5.3.6. *Suppose* $\mathbf{G}$ *is K-split. Then there is a bijective correspondence between* $I^c/\!\sim$ *and* $\mathcal{C}$ *given by the map sending* (F, H) *to* $\mathcal{C}(F, \mathsf{H})$.

Proof. By Lemma 5.3.5, this map is well-defined; by Lemma 5.2.2(2), the map is surjective. It remains to show that the map is injective.

Suppose (F_1, H_1) and (F_2, H_2) are pairs in I^c such that $\mathcal{C}(F_1, \mathsf{H}_1) = \mathcal{C}(F_2, \mathsf{H}_2)$. We need to show that $(F_1, \mathsf{H}_1) \sim (F_2, \mathsf{H}_2)$.

For $i = 1, 2$, by Lemma 5.2.5 we can choose a maximal unramified subgroup $\mathbf{H}_i$ in $\mathbf{G}$ such that the building $\mathcal{B}(\mathbf{H}_i, K)$ contains F_i, the image of $\mathbf{H}_i(K) \cap \mathbf{G}(K)_F$ in $\mathsf{G}_{F_i}(\mathfrak{F})$ is $\mathsf{H}_i(\mathfrak{F})$, the image of F_i in $\mathcal{B}^{\mathrm{red}}(H)$ is a hyperspecial vertex x_{F_i}, and the G-conjugacy class of the pair $(\mathbf{H}_i(k), x_{F_i})$ is $\mathcal{C}(F_i, \mathsf{H}_i)$. Because $\mathcal{C}(F_1, \mathsf{H}_1) = \mathcal{C}(F_2, \mathsf{H}_2)$, there exists a $g \in G$ such that ${}^g\mathbf{H}_2 = \mathbf{H}_1$ and $gx_{F_2} = x_{F_1}$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{H}_1, k)$. Let $\mathbf{H} = {}^g\mathbf{H}_2 = \mathbf{H}_1$ and let $H = \mathbf{H}(k)$.

Note that both F_1 and gF_2 lie in $\mathcal{B}(H)$. Moreover, both F_1 and gF_2 lie in the preimage in $\mathcal{B}(H)$ of $x_{F_1} \in \mathcal{B}^{\mathrm{red}}(H)$. Since (F_1, H_1) is an $\mathfrak{f}$-cuspidal pair, by Remark 5.2.6 we have that F_1 is a maximal G-facet in the preimage in $\mathcal{B}(H)$ of $x_{F_1} \in \mathcal{B}^{\mathrm{red}}(H)$. Similarly, gF_2 is a maximal G-facet in the preimage in $\mathcal{B}(H)$ of $x_{F_1} \in \mathcal{B}^{\mathrm{red}}(H)$. By Lemma 5.2.2(3), the G-facets F_1 and gF_2 are strongly associated. Since the image of $\mathbf{H}(K) \cap \mathbf{G}(K)_{F_1} \cap \mathbf{G}(K)_{gF_2}$ in $\mathsf{G}_{F_1}(\mathfrak{F})$ (resp., in $\mathsf{G}_{gF_2}(\mathfrak{F})$) is $\mathsf{H}_1(\mathfrak{F})$ (resp., ${}^g\mathsf{H}_2(\mathfrak{F})$), it follows that

$$ \mathsf{H}_1 \overset{\mathrm{id}}{=} {}^g\mathsf{H}_2 \text{ in } \mathsf{G}_{F_1} \overset{\mathrm{id}}{=} \mathsf{G}_{gF_2}. \qquad \square $$

REMARK 5.3.7. I believe that in Theorem 5.3.6 the requirement that $\mathbf{G}$ be K-split can be removed. However, I was unable to show this. In fact, it took a rather long computation on the part of Jeff Adler and myself just to show that the result is true for the group SU_3 splitting over a totally ramified quadratic extension of k. One difficulty is that the set $\Phi(\mathsf{H}, \mathsf{T})$ in the proofs of Lemmas 5.2.4 and 5.2.5 need not be a closed root subsystem of $\Phi(\mathbf{G}, \mathbf{T})$.

References

[1] A. Borel, *Linear algebraic groups,* 2nd ed., Grad. Texts in Math., 126, Springer-Verlag, New York, 1991.

[2] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. I. Données radicielles valuées,* Inst. Hautes Études Sci. Publ. Math. 41 (1972), 5–251.

[3] ———, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée,* Inst. Hautes Études Sci. Publ. Math. 60 (1984), 197–376.

[4] R. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, Wiley, Chichester, 1993 [1985].

[5] S. DeBacker, *Parameterizing nilpotent orbits via Bruhat–Tits theory,* Ann. of Math. (2) 156 (2002), 295–332.

[6] S. DeBacker and D. Kazhdan, *Murnaghan–Kirillov theory for depth zero supercuspidal representations: Reduction to cuspidal local systems,* preprint.

[7] M. Demazure and A. Grothendieck, *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux,* Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Lecture Notes in Math., 152, Springer-Verlag, Berlin, 1962/1964.

[8] P. Gérardin, *Construction de séries discrètes p-adiques,* Lecture Notes in Math., 462, Springer-Verlag, Berlin, 1975.

[9] D. Kazhdan and G. Lusztig, *Fixed point varieties on affine flag manifolds,* Israel J. Math. 62 (1988), 129–168.

[10] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups,* Invent. Math. 116 (1994), 393–408.

[11] ———, *Jacquet functors and unrefined minimal K-types,* Comment. Math. Helv. 71 (1996), 98–121.

[12] G. Rousseau, *Immeubles des groupes réductifs sur les corps locaux,* Thèse, Paris XI, 1977.

[13] J.-P. Serre, *Local fields* (M. J. Greenberg, trans.), Grad. Texts in Math., 67, Springer-Verlag, Berlin, 1979.

[14] ———, *Galois cohomology* (P. Ion, trans.), Springer-Verlag, Berlin, 1997.

[15] J. Tits, *Reductive groups over local fields,* Automorphic forms, representations, and L-functions (A. Borel, W. Casselman, eds.), Proc. Sympos. Pure Math., 33, pp. 29–69, Amer. Math. Soc., Providence, RI, 1979.

[16] J.-L. Waldspurger, *Quelques questions sur les intégrales orbitales unipotentes et les algèbres de Hecke,* Bull. Soc. Math. France 124 (1996), 1–34.

[17] ———, *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés,* Astérisque 269 (2001).

[18] J.-K. Yu, *Construction of tame supercuspidal representations,* J. Amer. Math. Soc. 14 (2001), 579–622.

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