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On the Restriction Conjecture

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1. Introduction

The restriction conjecture is a challenging open problem in Fourier analysis. Denoting by

$$\hat{f}(\zeta) = \int_{\mathbf{R}^d} f(x)e^{-2\pi i\langle x,\zeta\rangle}dx$$

the Fourier transform of a C_0^∞ function on $\mathbf{R}^d$ and by $\mathbf{S}^{d-1} = \{x \in \mathbf{R}^d : \|x\| = 1\}$ the unit sphere in $\mathbf{R}^d$, the restriction conjecture (RC henceforth) states that, for every $1 \le p < \frac{2d}{d+1}$ and $q \ge \frac{d-1}{d+1}p'$, the following inequality holds:

$$\sup_{F\in C_0^\infty(\mathbf{R}^d)} \frac{\|\hat{F}\|_{L^q(\mathbf{S}^{d-1},d\sigma)}}{\|F\|_{L^p(\mathbf{R}^d)}} \le C, \tag{1.1}$$

where $d\sigma(\zeta)$ denotes surface measure on $\mathbf{S}^{d-1}$ and $\mathbf{R}^+ = (0,\infty)$. Here C is a constant that depends only on p, q, and d, and p' is the dual exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

The conditions on p and q are optimal (see [10]). The RC has been proved in the case $d = 2$ by Fefferman and Stein (see [6]) and is still open in higher dimensions. When $d > 2$ only partial results are known; one of these results is the Stein–Tomas theorem [9; 13], which asserts that the RC holds whenever $1 \le p < \frac{2(d+1)}{d+3}$ and every $q \ge \frac{d-1}{d+1}p'$. See also [10]. When $p = \frac{2(d+1)}{d+3}$ we have $\frac{d-1}{d+1}p' = 2$, and the exponent $q = 2$ plays a crucial role as it allows a reduction of (1.1) to the equivalent "dual" inequality

$$\left\| \int_{\mathbf{S}^{d-1}} \hat{F}(\zeta)e^{2\pi i\langle x,\zeta\rangle}\, d\sigma(\zeta)\right\|_{L^{p'}(\mathbf{R}^n)} \le C\|F\|_{L^p(\mathbf{R}^n)}$$

via a TT^* technique. The case $q < 2$ cannot be handled with the same technique and requires more delicate work.

When $\frac{2(d+1)}{d+3} < p < \frac{2d}{d+1}$ we can prove that the ratio in (1.1) is uniformly bounded on special subspaces of $L^p(\mathbf{R}^d)$. For example, it is easy to see that (1.1) holds for every $q \le 2$ and every $p \le \frac{2d}{d+1}$ if $L^p(\mathbf{R}^d)$ is replaced by the Sobolev space $W^{s,p_0}(\mathbf{R}^d)$, where $p_0 = \frac{2(d+1)}{d+3}$ and $s \ge \frac{d-1}{d(d+1)}$. By the Sobolev embedding theorem, the latter embeds in $L^p(\mathbf{R}^d)$ for every $p \le \frac{2d}{d+1}$.

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Another class of functions for which the conjecture is valid is the class of radial functions. Let $x = r\omega$, with $r = |x|$ and $\omega \in \mathbf{S}^{d-1}$. Let $F(x) = f(|x|) \in C_0^\infty(\mathbf{R}^d)$. The Fourier transform of $F(x)$ is

$$\hat{F}(\xi) = |\xi|^{-\frac{d}{2}+1} \int_0^{+\infty} f(r) J_{\frac{d}{2}-1}(r|\xi|) r^{\frac{d}{2}}\, dr = \tilde{\mathcal{H}}_{\frac{d}{2}-1} f(|\xi|),$$

where $J_\nu(r)$ is the usual Bessel function of the first kind (see [11]) and $\tilde{\mathcal{H}}_\alpha f(\rho)$ is the Hankel–Fourier–Bessel transform of $f(r)$.

To see the validity of the RC for radial functions, we note that the $L^p(\mathbf{R}^d)$ norm of a radial function $F(x) = f(|x|) \in C_0^\infty(\mathbf{R}^d)$ is

$$\|F\|_{L^p(\mathbf{R}^d)} = |\mathbf{S}^{d-1}|^{\frac{1}{p}} \left\| f(r) r^{\frac{d-1}{p}} \right\|_{L^p(\mathbf{R}^+)},$$

where $|\mathbf{S}^{d-1}| = 2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$ denotes the measure of the surface of $\mathbf{S}^{d-1}$. We also have

$$\left(\int_{\mathbf{S}^{d-1}} |\hat{F}(\xi)|^q \, d\sigma(\xi) \right)^{\frac{1}{q}} = |\mathbf{S}^{d-1}|^{\frac{1}{q}} \left| \int_0^{+\infty} f(r) J_{\frac{d}{2}-1}(r) r^{\frac{d}{2}}\, dr \right|,$$

and by applying Hölder's inequality we obtain

$$\begin{aligned} \left(\int_{\mathbf{S}^{d-1}} |\hat{F}(\xi)|^q \, d\sigma(\xi) \right)^{\frac{1}{q}} &= |\mathbf{S}^{d-1}|^{\frac{1}{q}} \left\| f(r) r^{\frac{d-1}{p}} \right\|_{L^p(\mathbf{R}^+)} \left\| r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r) \right\|_{L^{p'}(\mathbf{R}^+)} \\ &= |\mathbf{S}^{d-1}|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(\mathbf{R}^d)} \left\| r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r) \right\|_{L^{p'}(\mathbf{R}^+)}. \end{aligned}$$

Since $J_{\frac{d}{2}-1}(r)$ is $\mathcal{O}\left(r^{-\frac{1}{2}}\right)$ when $r \to +\infty$ and is $\mathcal{O}\left(r^{\frac{d}{2}-1}\right)$ when $r \to 0$, we can easily check that $r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r) \in L^{p'}(\mathbf{R}^+)$ if and only if $p < \frac{2d}{d+1}$.

Note that, in this special case, $|\mathbf{S}^{d-1}|^{\frac{1}{q}-\frac{1}{p}} \left\| r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1} \right\|_{L^{p'}(\mathbf{R}^+)}$ is the best constant for the restriction inequality (1.1); that is,

$$\sup_{F \text{ radial}} \frac{\|\hat{F}\|_{L^q(\mathbf{S}^{d-1})}}{\|F\|_p} = |\mathbf{S}^{d-1}|^{\frac{1}{q}-\frac{1}{p}} \left\| r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{d}{2}-1}(r) \right\|_{L^{p'}(\mathbf{R}^+)}.$$

We also observe that in this case (1.1) holds for every $q < \infty$.

More generally, let $\mathcal{H}_m$ be the subspace of $L^2(\mathbf{S}^{d-1})$ spanned by the products of spherical harmonics of degree $m \geq 0$ and radial functions in $C_0^\infty(\mathbf{R}^d)$. If $F(x) = F(r\omega) = r^m f_m(r) Y(\omega) \in \mathcal{H}_m$, where Y is a spherical harmonic, then

$$\hat{F}(\zeta) = \hat{F}(\rho\sigma) = \rho^m \widehat{f_m}(\rho) Y(\sigma), \tag{1.2}$$

where

$$\widehat{f_m}(\rho) = i^m \rho^{-\frac{n}{2}+1} \int_0^{+\infty} f_m(r) J_{\frac{d}{2}-1+m}(r\rho) r^{\frac{d}{2}+m}\, dr = i^m \tilde{\mathcal{H}}_{\frac{d}{2}-1+m} f_m(\rho).$$

Let n be a nonnegative integer and let $s > -\frac{1}{2}$. We denote by $P_n^{(s)}$ the ultraspherical polynomial of degree n and order s. This is defined by

$$P_n^{(s)}(t) = C_n^s P_n^{s-\frac{1}{2}, s-\frac{1}{2}}(t),$$

where $P_n^{(\alpha,\beta)}(t)$ is the usual Jacobi polynomial of degree n on $[-1,1]$ and C_n^s is a constant of normalization. We refer the reader to Section 8 for the value of the constant C_n^s and for the definition of Jacobi polynomials.

The spherical harmonics have an explicit expression in terms of the Jacobi (or ultraspherical) polynomials. Indeed, let $m_0 \geq m_1 \geq \cdots \geq m_{d-2} \geq 0$ be integers and let

$$Y_{(m_k)}(z) = e^{\pm i m_{d-2} z_{d-1}} \prod_{k=0}^{d-3} (\sin z_{k+1})^{m_{k+1}} P_{m_k - m_{k+1}}^{(m_{k+1} + \frac{d-1-k}{2})}(\cos z_{k+1}). \tag{1.3}$$

Then every spherical harmonic $Y_m(\omega)$ of degree $m = m_0 \geq 0$ can be written as a finite linear combination of the $Y_{(m_k)}$ (see [5]). This may be proved using a dimension comparison with space of the spherical harmonics of degree m that has dimension

$$\delta_{m,d} = (2m + d - 2) \frac{\Gamma(m + d - 2)}{\Gamma(m+1)\Gamma(d-1)}.$$

In this paper we consider the following class of functions: products of radial functions in $C_0^\infty(\mathbf{R}^d)$ and spherical harmonics that, in polar coordinates, can be expressed as products of factors of the form of $(\sin z)^{s-j} P_n^{(s)}(\cos z)$. We denote this class of functions by $\mathcal{L}$. It is easy to verify that the space $\mathcal{L}$ is invariant under the action of the Fourier transform. Moreover, one can easily see that the space

$$\operatorname{span}(\mathcal{L}) = \left\{ \sum_{i=1}^{N} r^{m_i} f_i(r) Y_{m_i}(\omega) : N > 0,\ f_i(r) \in C_0^\infty(\mathbf{R}^+),\ Y_{m_i}(\omega) \text{ as in (1.3)} \right\}$$

is dense in $L^p(\mathbf{R}^d)$ for every $p \leq 2$. Therefore, the RC is equivalent to the estimate

$$\sup_{\operatorname{span}(\mathcal{L})} \frac{\left\| \sum_{i=1}^{N} \widehat{f_i}(1) Y_{m_i} \right\|_{L^q(\mathbf{S}^{d-1})}}{\left\| \sum_{i=1}^{N} r^{m_i} f_i Y_{m_i} \right\|_{L^p(\mathbf{R}^d)}} \leq C, \tag{1.4}$$

where C depends only on p, q, and d (and in particular is independent of N). This provides a strong motivation for the consideration of the class $\mathcal{L}$.

Our main result, Theorem 1.1, states that RC holds for the space $\mathcal{L}$; that is, (1.4) is valid when $N = 1$.

THEOREM 1.1. *Let $1 \leq p < \frac{2d}{d+1}$ and let $q = \frac{d-1}{d+1} p'$. Then we have*

$$\sup_{F \in \mathcal{L}} \frac{\|\hat{F}\|_{L^q(\mathbf{S}^{d-1})}}{\|F\|_{L^p(\mathbf{R}^d)}} \leq C, \tag{1.5}$$

where C depends only on p, q, and d.

The basic strategy in proving Theorem 1.1 is as follows. Let $F(x) \in \mathcal{L}$. Since $F(r\omega) = r^m f_m(r) Y(\omega)$ and $\hat{F}(\zeta)$ is as in (1.2), we have

$$\frac{\|\hat{F}\|_{L^q(\mathbf{S}^{d-1})}}{\|F\|_{L^p(\mathbf{R}^d)}} = \frac{\|\widehat{f_m}(1) Y\|_{L^q(\mathbf{S}^{d-1})}}{\|r^m f_m Y\|_{L^p(\mathbf{R}^d)}}. \tag{1.6}$$

We can therefore reduce matters to estimating the ratios of the radial parts and the angular parts separately. Our main task is to obtain the appropriate estimates for these parts. Finally, we show that the combined estimates for

$$\frac{|\widehat{f_m}(1)|}{\|r^m f_m\|_{L^p(\mathbf{R}^+,r^{d-1}dr)}} \quad \text{and} \quad \frac{\|Y\|_{L^q(\mathbf{S}^{d-1})}}{\|Y\|_{L^p(\mathbf{S}^{d-1})}}$$

yield (1.5).

2. Four Useful Propositions

In what follows we will often denote by C a generic constant that is not necessarily the same at each occurrence. The following results are ingredients of the proof of Theorem 1.1.

PROPOSITION 1. *Let $J_\nu(x)$ be the usual Bessel function of the first kind with $\nu \geq 0$. Then $x^\alpha J_\nu(x) \in L^q(\mathbf{R}^+)$ if and only if*

$$-\frac{1}{q} - \nu < \alpha < \frac{1}{2} - \frac{1}{q}, \tag{2.1}$$

and for $\frac{1}{4} - \frac{1}{q} < \alpha < \frac{1}{2} - \frac{1}{q}$, for $2 \leq q < \infty$, and for ν sufficiently large we have

$$\|x^\alpha J_\nu(x)\|_q \leq A\nu^{\alpha - \frac{1}{2} + \frac{1}{q}}, \tag{2.2}$$

where A depends only on α and q.

PROPOSITION 2. *Let $s \geq j \geq 0$. Then*

$$\sup_{0 \leq z \leq \frac{\pi}{2}} |(\sin z)^{s-j} P_n^{(s)}(\cos z)| \leq (P_n^{(s)}(1))^{\frac{j}{s}} (c_{n,s})^{1-\frac{j}{s}}, \tag{2.3}$$

where $P_n^{(s)}(1) = \frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}$, and

$$c_{n,s} = \begin{cases} \dfrac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma\left(\frac{n}{2} + 1\right)\Gamma(s)} & \text{if } n \text{ is even,} \\[2ex] \dfrac{(1+n)\frac{\Gamma\left(\frac{n+1}{2}+s\right)}{\Gamma\left(\frac{n+3}{2}\right)\Gamma(s)}}{\sqrt{(1-s)s + (n+s)^2}} & \text{if } n \text{ is odd.} \end{cases} \tag{2.4}$$

Moreover,

$$\sup_{0 \leq z \leq \frac{\pi}{2}} |(\sin z)^{s-j} P_n^{(s)}(\cos z)| \leq e^j \left(1 + \frac{n+1}{2s}\right)^j c_{n,s}. \tag{2.5}$$

Proposition 2 is a generalization of Theorem 7.33.2 in [12], where the same result is proved for $j = 0$ and $0 < s < 1$. Note that the inequality (2.3) is sharp in the case $j = s$. Indeed, $P_n^{(s)}(t) \leq P_n^{(s)}(1)$ for every $-1 \leq t \leq 1$ (see (8.7)).

PROPOSITION 3. *Let $n \geq 0$, and let $j \leq s$ with $s \geq 0$. Then*

$$\frac{\sup_{t\in[0,1]} |(1-t^2)^{\frac{1}{2}(s-j)} P_n^{(s)}(t)|}{\left(\int_0^1 |P_n^{(s)}(t)|^2 (1-t^2)^{s+\frac{1}{2}}\, dt\right)^{\frac{1}{2}}} \leq C(s+n)^{\frac{1}{4}}, \tag{2.6}$$

where C is a constant that depends only on j.

The following proposition is an easy consequence of Proposition 3 and complex interpolation.

PROPOSITION 4. *Let $2 \leq r \leq q$, and let $\eta(x)$ be an analytic function on $[2,\infty) \times i\mathbf{R}$ that is bounded on $[2,\infty]$ and satisfies $\eta(2) \geq -\frac{1}{2}$. Then*

$$\frac{\left(\int_0^1 |P_n^{(s)}(t)|^q (1-t^2)^{q(\frac{s}{2}-\eta(q))}\, dt\right)^{\frac{1}{q}}}{\left(\int_0^1 |P_n^{(s)}(t)|^r (1-t^2)^{r(\frac{s}{2}-\eta(r))}\, dt\right)^{\frac{1}{r}}} \leq C(s+n)^{\frac{1}{2r}-\frac{1}{2q}}, \tag{2.7}$$

where C is a constant that depends only on r, q, and $\sup_{x\geq 2}|\eta(x)|$.

It is worthwhile comparing Proposition 4 with Theorem 3 in the article of Carbery and Wright [3]. They prove that the following inequality is satisfied for all $0 \leq p \leq q \leq \infty$, all $j \in \mathbf{N}$ and $\lambda \geq 1$, and every polynomial on $\mathbf{R}$ of degree at most n:

$$\left(\frac{\int_0^1 |p(t)|^{\frac{q}{n}} (\lambda-t)^{j-1}\, dt}{\int_0^1 (\lambda-t)^{n-1}\, dt}\right)^{\frac{1}{q}} \leq \sigma \frac{(jB(j,q+1))^{\frac{1}{q}}}{(jB(j,r+1))^{\frac{1}{r}}} \left(\frac{\int_0^1 |p(t)|^{\frac{r}{n}} (\lambda-t)^{j-1}\, dt}{\int_0^1 (\lambda-t)^{n-1}\, dt}\right)^{\frac{1}{r}}, \tag{2.8}$$

where σ is independent of the listed parameters and $B(a,b)$ is the Beta function. If we let $\bar{q} = nq$, $\bar{r} = nr$, and $\lambda = 1$, from (2.8) we obtain

$$\left(\int_0^1 |p(t)|^{\bar{q}} (1-t)^{j-1}\, dt\right)^{\frac{1}{\bar{q}}} \leq \sigma^n \frac{(B(j,n\bar{q}+1))^{\frac{1}{\bar{q}}}}{(B(j,n\bar{r}+1))^{\frac{1}{\bar{r}}}} \left(\int_0^1 |p(t)|^{\bar{r}} (1-t)^{j-1}\, dt\right)^{\frac{1}{\bar{r}}}.$$

It is not difficult to show (see also Lemma 5) that

$$\frac{(B(j,n\bar{q}+1))^{\frac{1}{\bar{q}}}}{(B(j,n\bar{r}+1))^{\frac{1}{\bar{r}}}} \approx C\Gamma(j)^{\frac{1}{\bar{q}}-\frac{1}{\bar{r}}} (n+1)^{\frac{j}{\bar{r}}-\frac{j}{\bar{q}}}$$

as $n \to \infty$ with the other parameters fixed. Hence (2.8) is equivalent to

$$\left(\int_0^1 |p(t)|^{\bar{q}}(1-t)^{j-1}\,dt\right)^{\frac{1}{\bar{q}}} \le C\sigma^n\Gamma(j)^{\frac{1}{\bar{q}}-\frac{1}{\bar{r}}}(n+1)^{\frac{j}{\bar{r}}-\frac{j}{\bar{q}}}\left(\int_0^1 |p(t)|^{\bar{r}}(1-t)^{j-1}\,dt\right)^{\frac{1}{\bar{r}}},$$

which is weaker than (2.7); moreover, the constant σ is not explicit.

3. Proof of Proposition 1

In this section we prove Proposition 1 and also state some facts that we shall need in the proof of Theorem 1.1. To prove Proposition 1 we make use of the following precise asymptotics of the Bessel functions for large values of the argument that J. A. Barceló proved in his thesis (see also [2]).

THEOREM B. *There exists a universal constant $C > 0$ such that, for all $\nu > \frac{1}{2}$ and all $r > \nu + \nu^{\frac{1}{3}}$,*

$$J_\nu(r) = \sqrt{\frac{2}{\pi}}\frac{\cos\theta(r)}{(r^2-\nu^2)^{\frac{1}{4}}} + h_\nu(r),$$

where

$$\theta(r) = (r^2-\nu^2)^{\frac{1}{2}} - \nu\arccos\left(\frac{\nu}{r}\right) - \frac{\pi}{4}$$

and

$$|h_\nu(r)| \le \begin{cases} C\left(\dfrac{\nu^2}{(r^2-\nu^2)^{\frac{7}{4}}} + \dfrac{1}{r}\right) & \text{if } \nu+\nu^{\frac{1}{3}} \le r \le 2\nu, \\ \dfrac{C}{r} & \text{if } r > 2\nu. \end{cases}$$

Proof of Proposition 1. The conditions (2.1) on α are necessary because $J_\nu(x) = \mathcal{O}(x^\nu)$ when $x \to 0$ and is $\mathcal{O}\big(x^{-\frac{1}{2}}\big)$ when $x \to \infty$.

By Theorem B we have

$$\|J_\nu(x)x^\alpha\|_{L^q(2\nu,\infty)} \le C\left(\int_{2\nu}^\infty \left(r^{(\alpha-\frac{1}{2})} + r^{(\alpha-1)}\right)^q\,dr\right)^{\frac{1}{q}}.$$

Condition (2.1) on α guarantees that this integral converges. Thus,

$$\|J_\nu(x)x^\alpha\|_{L^q(2\nu,\infty)} \le C\nu^{\alpha-\frac{1}{2}+\frac{1}{q}},$$

which is the required estimate.

We use again Theorem B in the interval $\big(\nu+\nu^{\frac{1}{3}}, 2\nu\big)$. We obtain

$$\begin{aligned}
&\|J_\nu(x)x^\alpha\|_{L^q(\nu+\nu^{1/3},2\nu)} \\
&\quad\le C\Big(\big\|r^\alpha(r^2-\nu^2)^{-\frac{1}{4}}\big\|_{L^q(\nu+\nu^{1/3},2\nu)} + \nu^2\big\|r^\alpha(r^2-\nu^2)^{-\frac{7}{4}}\big\|_{L^q(\nu+\nu^{1/3},2\nu)} \\
&\qquad\quad + \|r^{\alpha-1}\|_{L^q(\nu+\nu^{1/3},2\nu)}\Big) \\
&\quad\le C\nu^{\alpha+\frac{1}{q}-\frac{1}{2}}\Big(\big\|s^\alpha(s^2-1)^{-\frac{1}{4}}\big\|_{L^q(1+\nu^{-2/3},2)} \\
&\qquad\qquad + \big\|s^\alpha(s^2-1)^{-\frac{7}{4}}\big\|_{L^q(1+\nu^{-2/3},2)} + \nu^{-\frac{1}{2}}\Big) \\
&\quad\le C\nu^{\alpha+\frac{1}{q}-\frac{1}{2}}.
\end{aligned}$$

We are left with estimating the norm of $r^\alpha J_\nu(r)$ in the interval $\big(0, \nu+\nu^{\frac{1}{3}}\big)$.

It is a well-known fact (see e.g. [7]) that there is a constant $C > 0$ such that, for all $\nu \geq 0$ and all $r \geq 0$, we have $|J_\nu(r)| < C\nu^{-\frac{1}{3}}$. Furthermore, $J_\nu(r)$ is increasing, and $|J_\nu(r)| < C\nu^{-\frac{1}{2}}$ in the interval $\left[0, \nu - \nu^{\frac{1}{3}}\right]$. The latter can be easily proved using the estimate

$$J_\nu(\nu x) \leq \frac{e^{-\nu f(x)}}{(1-x^2)^{\frac{1}{4}}\sqrt{2\pi\nu}}, \quad 0 \leq x < 1$$

(see [14, p. 255]) together with

$$f(x) = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}.$$

Therefore,

$$\begin{aligned}\|J_\nu(x)x^\alpha\|_{L^q(\nu-\nu^{1/3},\nu+\nu^{1/3})} &\leq C\nu^{-\frac{1}{3}} \frac{\left(\nu+\nu^{\frac{1}{3}}\right)^{\alpha+\frac{1}{q}} - \left(\nu-\nu^{\frac{1}{3}}\right)^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}} \\ &\leq C\nu^{\frac{1}{3}\left(\alpha+\frac{1}{q}-1\right)},\end{aligned}$$

which is better than what we need. Indeed,

$$\frac{1}{3}\left(\alpha+\frac{1}{q}-1\right) \leq \alpha + \frac{1}{q} - \frac{1}{2} \iff \alpha \geq \frac{1}{4} - \frac{1}{q},$$

as required.

Since $|J_\nu(r)| < C\nu^{-\frac{1}{2}}$ for all $r \leq \nu - \nu^{\frac{1}{3}}$, the estimate claimed in Proposition 1 easily follows. □

Remark. Proposition 1 can also be proved as a corollary of [4, Prop. 1].

We now let $F(x) = F(r\omega) = r^m f_m(r) Y_m(\omega) \in \mathcal{L}$, and we recall that $\hat{F}(\zeta) = \hat{F}(\rho\sigma) = \rho^m \widehat{f_m}(\rho) Y_m(\sigma)$, where

$$\widehat{f_m}(\rho) = i^m \rho^{-\frac{n}{2}+1} \int_0^{+\infty} f_m(r) J_{\frac{d}{2}-1+m}(r\rho) r^{\frac{d}{2}+m}\, dr.$$

In order to prove (1.1) for a function F in $\mathcal{L}$, we shall prove that, for every $1 \leq p < \frac{2d}{d+1}$ and $q \geq \frac{d-1}{d+1}p'$, the ratio

$$\frac{\|\hat{F}\|_{L^q(\mathbf{S}^{d-1},d\sigma)}}{\|F\|_{L^p(\mathbf{R}^d)}} = \frac{\left|\int_0^{+\infty} f(r) J_{\frac{d}{2}-1+m}(r) r^{\frac{d}{2}+m}\, dr\right|}{\left(\int_0^{+\infty} |f(r)|^p r^{d-1+mp}\, dr\right)^{\frac{1}{p}}} \frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1},d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},d\sigma)}} \tag{3.1}$$

is bounded by a constant that depends only on p, q, and d. Then (3.1) will be a consequence of the following lemmas.

LEMMA 1. *Let* $1 \le p < \frac{2d}{d+1}$ *and let* $f(r) \in C_0^\infty(0,+\infty)$. *Then*

$$\frac{\left|\int_0^{+\infty} f_m(r) J_{\frac{d}{2}-1+m}(r) r^{\frac{d}{2}+m}\, dr\right|}{\left(\int_0^{+\infty} |f_m(r)|^p r^{d-1+mp}\, dr\right)^{\frac{1}{p}}} \le C m^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{1}{p'}}. \tag{3.2}$$

LEMMA 2. *Let* $p \le q \le 2$. *Let* $Y_m(\omega)$ *be a spherical harmonics which, in polar coordinates, can be expressed as the product of factors of the form of* $(\sin z)^{s-j} P_n^{(s)}(\cos z)$ (*see Section* 2). *Then*

$$\frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1},d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},d\sigma)}} \le C m^{(d-2)\left(\frac{1}{2p}-\frac{1}{2q}\right)}. \tag{3.3}$$

When Lemmas 1 and 2 are proved, then Theorem 1.1 easily follows. Indeed, let $p \le \frac{2d}{d-1}$ and $q = \frac{d-1}{d+1} p'$. By (3.2) and (3.3), the right-hand side of (3.1) is at most

$$C m^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{1}{p'}+(d-1)\left(\frac{1}{2p}-\frac{1}{2q}\right)},$$

and the conditions on p and q guarantee that the exponent of m is here equal to $-\frac{d-1}{d+1}\frac{1}{q} = -\frac{1}{p'}$, which is nonpositive.

Proof of Lemma 1. By Hölder's inequality,

$$\begin{aligned} &\left|\int_0^{+\infty} f(r) J_{\frac{d}{2}-1+m}(r) r^{\frac{d}{2}+m}\, dr\right| \\ &\qquad \le \left(\int_0^{+\infty} |f(r)|^p r^{mp+d-1}\, dr\right)^{\frac{1}{p}} \left\| J_{\frac{d}{2}-1+m} r^{\frac{d}{2}-\frac{d-1}{p}} \right\|_{L^{p'}(\mathbf{R}^+)}. \end{aligned} \tag{3.4}$$

By Proposition 1, the $L^{p'}$ norm in (3.4) is finite if and only if $p < \frac{2d}{d+1}$ and is at most a constant multiple of the quantity $m^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{1}{p'}}$. □

The proof of Lemma 2 utilizes Proposition 2 and will be given in Section 7.

4. Some More Lemmas

The proof of Proposition 2 relies on Lemmas 3, 4, and 5.

LEMMA 3. *Let* $0 < j \le s$. *The relative extrema of* $(\sin z)^{s-j} P_n^{(s)}(\cos z)$ *in the interval* $[0,\pi/2]$ *are increasing whenever*

$$z \le z_{j,n}^s = \frac{1}{2}\arccos\left(\frac{j - 3j^2 + j^3 + jn^2 - s + 2js + 2jns + s^2 - js^2}{j(n^2 - j^2 + 2ns + s^2)}\right) \tag{4.1}$$

and are decreasing otherwise. The relative extrema of $(\sin z)^s P_n^{(s)}(\cos z)$ *in the interval* $[0,\pi/2]$ *are increasing whenever* $0 \le s \le 1$ *and are decreasing otherwise.*

Proof. Let
$$\psi_j(z) = (n+s)^2 + j^2 + \frac{j(j-1)+s(s-1)}{\sin^2 z}.$$
Since $y^s_{0,n}(z) = (\sin z)^s P_n^{(s)}(\cos z)$ satisfies the differential equation
$$u'' + \psi_0(z)u = 0$$
(see Section 8), it is not difficult to prove that $y^s_{j,n}(z)$ satisfies the differential equation
$$v'' + 2jv' \cot z + \psi_j(z)v = 0. \tag{4.2}$$
Let
$$f(z) = (y^s_{j,n}(z))^2 + \frac{\left(\frac{d}{dz} y^s_{j,n}(z)\right)^2}{\psi_j(z)}.$$
Then
$$f'(z) = 2\frac{d}{dz} y^s_{j,n}(z)\left(y^s_{j,n}(z) + \frac{\frac{d^2}{d^2z} y^s_{j,n}(z)\psi_j(z) - \frac{1}{2}\frac{d}{dz} y^s_{j,n}(z)\psi_j'(z)}{\psi_j^2(z)}\right)$$
and, by (4.2),
$$f'(z) = -2g(z)\frac{(\cot z)(\csc z)^2\left(\frac{d}{dz} y^s_{j,n}(z)\right)^2}{(n^2 - j^2 + 2ns + s^2 + (j-s)(-1+j+s)(\csc z)^2)^2},$$
where
$$\begin{aligned} g(z) = {} & j^3 - 3j^2 + s(s-1) + j(1+n^2+2s+2ns-s^2) \\ & + j(j^2 - (n+s)^2)\cos(2z). \end{aligned}$$
Observe that $f(z)$ is increasing if and only if $g(z) \le 0$. If $j = 0$ then $g(z) = s(s-1)$; therefore, the sequence of the relative extrema of $y^{(s)}_{0,n}(z)$ is increasing if $s \ge 1$ and is decreasing if $0 \le s \le 1$.

If $j \ne 0$ then $f(z)$ is increasing if and only if $z \le z^s_{j,n}$, where $z^s_{j,n}$ is defined as in (4.1). Since $f(z) = (y^s_{j,n}(z))^2$ at the critical points of $y^s_{j,n}(z)$, the theorem is proved. □

The next ingredient of our proof is a theorem of Sturm type.

LEMMA 4. *Let $H(z)$ be continuous on (z_1, z_2). Suppose that $u(z)$ satisfies $u'' + H(z)u = 0$ and that $H(z) \ge N > 0$ on (z_1, z_2). Then $u(z)$ has a zero on every subinterval of (z_1, z_2) of length $\pi/\sqrt{N}$.*

Proof. This is an easy consequence of Theorem 1.82.1 in [12] (see also [8]). □

We will also need the following easy lemma.

LEMMA 5. *Let $-x < y < \infty$ with $x > 0$. The function*
$$x \to \frac{\Gamma(x)x^y}{\Gamma(x+y)}$$
is an increasing function of x.

Proof. Let $f(x) = \frac{\Gamma(x)x^y}{\Gamma(x+y)}$. To prove that $f(x)$ is increasing we prove that

$$\ln f(x) = y \ln x + \ln(\Gamma(x)) - \ln(\Gamma(x+y))$$

is increasing, that is, its derivative is positive.

We recall that the logarithmic derivative of $\Gamma(z)$ is

$$\frac{\Gamma'(z)}{\Gamma(z)} = \gamma - \frac{1}{z} - \sum_{m=1}^{\infty}\left(\frac{1}{z+m} - \frac{1}{m}\right),$$

where γ is Euler's constant. Therefore,

$$(\ln f(x))' = \frac{y}{x} - \sum_{m=0}^{\infty} \frac{1}{x+m} - \frac{1}{x+y+m}.$$

The sum above is

$$\leq \int_0^{\infty}\left(\frac{1}{x+\zeta} - \frac{1}{x+y+\zeta}\right) d\zeta = \ln\left(\frac{x+y}{x}\right),$$

so

$$(\ln f(x))' \geq \frac{y}{x} - \ln\left(1 + \frac{y}{x}\right) > 0$$

as required. $\square$

An immediate consequence of Lemma 5 is that $\frac{\Gamma(x)x^y}{\Gamma(x+y)} \leq \lim_{x\to\infty} f(x) = 1$ (by Stirling's formula), while if $x \geq x_0$ then $f(x) \geq f(x_0)$. Therefore,

$$\frac{\Gamma(x+y)}{\Gamma(x)} \leq x^y x_0^{-y} \frac{\Gamma(x_0+y)}{\Gamma(x_0)}. \tag{4.3}$$

5. Proof of Proposition 2

Let $y^s_{j,n}(z) = (\sin z)^{s-j} P_n^{(s)}(\cos z)$, and let $c_{n,s}$ be as defined in (2.4).

In what follows we will assume that n is even, since the proof in the other case is similar. We first consider the case $j = 0$. By complex interpolation we can extend the result to the general case. Indeed, the function $y^s_{j,n}(z)$ depends analytically on j. If $j = s$ then $\|y^s_{s,n}\|_\infty = P_n^{(s)}(1) = \frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}$ (see Section 8). If we prove that $\|y^s_{0,n}\|_\infty = c_{n,s}$, then

$$\|y^s_{j,n}\|_\infty \leq (c^s_n)^{1-\frac{j}{s}} (P_n^{(s)}(1))^{\frac{j}{s}},$$

which is (2.3). We now prove (2.5). From the inequality just displayed it follows that

$$\|y^s_{j,n}\|_\infty \leq c_{n,s}\left(\frac{P_n^{(s)}(1)}{c_{n,s}}\right)^{\frac{j}{s}} = c_{n,s}\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{1+n}{2}+s\right)}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1}{2}+s\right)}\right)^{\frac{j}{s}}.$$

Let $t = \frac{1+n}{2}$ for the sake of simplicity. We prove that

$$\left(\frac{\sqrt{\pi}\,\Gamma(t+s)}{\Gamma(t)\Gamma\left(\frac{1}{2}+s\right)}\right)^{\frac{1}{s}} \leq e\left(1+\frac{t}{s}\right). \tag{5.1}$$

Let

$$g(t,s) = \frac{\sqrt{\pi}\, s^s \Gamma(t+s)}{e^s (t+s)^s \Gamma(t) \Gamma\left(\frac{1}{2}+s\right)}.$$

We aim to prove that $g(t,s) \le 1$ for every $t \ge \frac{1}{2}$ and $s \ge 0$.

By Lemma 5, $t \to g(t,s)$ is increasing. This can be easily seen if we let $x = s+t$ and $s = y$. Therefore,

$$g(t,s) \le \sqrt{\pi} \frac{s^s}{e^s \Gamma\left(s+\frac{1}{2}\right)} \lim_{t\to\infty} \frac{\Gamma(t+s)}{(t+s)^s \Gamma(t)}.$$

By Stirling's formula,

$$\frac{\Gamma(t+s)}{(t+s)^s \Gamma(t)} \sim \frac{\left(\frac{s+t}{e}\right)^{t+s-\frac{1}{2}}}{(t+s)^s \left(\frac{t}{e}\right)^{t-\frac{1}{2}}} = e^{-s}\left(1+\frac{s}{t}\right)^{t-\frac{1}{2}}$$

and thus

$$\lim_{t\to\infty} \frac{\Gamma(t+s)}{(t+s)^s \Gamma(t)} = 1.$$

Therefore,

$$g(t,s) \le \sqrt{\pi} \frac{s^s}{e^s \Gamma\left(s+\frac{1}{2}\right)}. \tag{5.2}$$

Let $h(s)$ be the function on the right-hand side of (5.2). We prove that $h(s)$ is decreasing and therefore that $g(t,s) \le h(0) = 1$ as required.

It is enough to prove that, for every $s > 0$, $h(s+1) \ge h(s)$ or (equivalently) that

$$\frac{h(s+1)}{h(s)} = \frac{(s+1)^{s+1}}{e s^s \left(s+\frac{1}{2}\right)} = \frac{1}{e}\left(1+\frac{1}{s}\right)^s \frac{s+1}{s+\frac{1}{2}} \le 1,$$

which is easily seen to be true.

To prove Proposition 2 in the case $j = 0$ we use induction on n. Assume $s > 1$, since the case $s < 1$ is known (see [12]). The case $n = 0$ is easy to check. Indeed, $P_0^{(s)}(t) \equiv 1$, and the right-hand side of (2.5) is also equal to unity. We now assume that the result is true for $n-1$ and prove that it is also true for n.

Recall that we have set $y_{j,n}^s(t) = (\sin t)^{s-j} P_n^{(s)}(t)$ and also that $(P_n^{(s)}(t))' = 2sP_{n-1}^{(s+1)}(t)$ (see Section 8). Thus,

$$(y_{j,n}^s)'(z) = (s-j)(\cos z) P_n^{(s)}(\cos z)(\sin z)^{s-1-j} - 2sP_{n-1}^{(s+1)}(\cos z)(\sin z)^{s+1-j}.$$

Therefore, the following equation is satisfied by the critical points of $y_{j,n}^s(z)$:

$$y_{j,n}^s(z) = \frac{2s}{s-j}(\tan z) y_{j,n-1}^{s+1}(z).$$

When $j = 0$, it follows that $y_{j,n}^s(z)$ satisfies

$$y_{0,n}^s(z) = 2(\tan z) y_{j,n-1}^{s+1}(z). \tag{5.3}$$

Let z_n^s be the point at which $y_{0,n}^s(z)$ attains its maximum. By Lemma 3, the sequence of the relative extrema of $y_{0,n}^s(z)$ is decreasing and hence z_n^s is the smallest critical point of $y_{0,n}^s(z)$ in the interval $[0, \pi/2]$.

To estimate z_n^s we use Lemma 4. Recall that $y_{0,n}^s(z)$ satisfies the differential equation (8.5), with

$$\psi_0(z) = (n+s)^2 + \frac{s(s-1)}{(\sin z)^2} \geq (n+s)^2 + s(s-1).$$

By Lemma 4, $y_{0,n}^s(z)$ has a zero in each interval $[\varepsilon, \xi(s,n)+\varepsilon]$ for every $\varepsilon > 0$, where we have let $\xi(s,n) = \pi/\sqrt{(n+s)^2 + s(s-1)}$. Since $y_{0,n}^s(z)$ vanishes at $z = 0$, it follows that $\frac{d}{dz} y_{0,n}^s(z)$ vanishes at least once in $(0, \xi + \varepsilon]$. Therefore, $z_n^s \leq \xi$ and

$$\tan(z_n^s) = \frac{\sin(z_n^s)}{\cos(z_n^s)} \leq \frac{z_n^s}{\sqrt{1-(z_n^s)^2}} = \frac{\pi}{\sqrt{n^2 - \pi^2 - s + 2ns + 2s^2}}.$$

If $\|y_{0,n-1}^{s+1}\|_\infty \leq c_{n-1,s+1}$, then by (5.3) and the preceding estimate we have

$$\|y_{0,n}^s\|_\infty = |y_{0,n}^s(z_n^s)| \leq \frac{2\pi c_{n-1,s+1}}{\sqrt{n^2 - \pi^2 - s + 2ns + 2s^2}},$$

and the right-hand side of this inequality is $\leq c_{n,s}$ if

$$h(n,s) = \frac{2\pi c_{n-1,s+1}}{c_{n,s}\sqrt{n^2 - \pi^2 - s + 2ns + 2s^2}} \leq 1.$$

Recalling that $n-1$ is odd (since we have assumed that n is even), after easy simplifications we can write

$$h(n,s) = \frac{\pi n(n+2s)}{s\sqrt{n^2 - s + 2ns}\sqrt{n^2 - \pi^2 - s + 2ns + 2s^2}}, \tag{5.4}$$

which is easily seen to be at most 1.

6. Proof of Lemma 2

We show that Lemma 2 is a consequence of Proposition 3. The proof of Proposition 3 will be given in Section 7.

Let Y_m be as in Lemma 2. We shall prove (3.3), that is: for every $1 \leq p < q \leq 2$ and every $d \geq 2$,

$$\frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1},d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},d\sigma)}} \leq C m^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}. \tag{6.1}$$

First of all, observe that it suffices to prove Lemma 2 when $q = 2$. Indeed, say that $\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{2}$, where $\alpha = \left(\frac{1}{q} - \frac{1}{2}\right)/\left(\frac{1}{p} - \frac{1}{2}\right)$. By the Riesz–Thorin convexity theorem,

$$\|Y_m\|_{L^q(\mathbf{S}^{d-1},d\sigma)} \leq \|Y_m\|^{\alpha}_{L^p(\mathbf{S}^{d-1},d\sigma)} \|Y_m\|^{1-\alpha}_{L^2(\mathbf{S}^{d-1},d\sigma)},$$

and if (2.6) holds when $q = 2$ then

$$\|Y_m\|_{L^q(\mathbf{S}^{d-1},d\sigma)} \leq \left(C m^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\right)^{1-\alpha} \|Y_m\|_{L^p(\mathbf{S}^{d-1},d\sigma)} = C^{1-\alpha} m^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{q}\right)};$$

that is, it holds for all other $q \leq 2$. Then we observe that, in order to prove (6.1) for $q = 2$, it suffices to prove

$$\frac{\|Y_m\|_{L^{p'}(\mathbf{S}^{d-1},d\sigma)}}{\|Y_m\|_{L^2(\mathbf{S}^{d-1},d\sigma)}} \le Cm^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}, \tag{6.2}$$

where p' is the dual exponent of p. Indeed, we observe that

$$\|Y_m\|^2_{L^2(\mathbf{S}^{d-1},d\sigma)} \le \|Y_m\|_{L^{p'}(\mathbf{S}^{d-1},d\sigma)}\|Y_m\|_{L^p(\mathbf{S}^{d-1},d\sigma)}$$

by Hölder's inequality. Hence

$$\frac{\|Y_m\|_{L^2(\mathbf{S}^{d-1},d\sigma)}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1},d\sigma)}} \le \frac{\|Y_m\|_{L^{p'}(\mathbf{S}^{d-1},d\sigma)}}{\|Y_m\|_{L^2(\mathbf{S}^{d-1},d\sigma)}},$$

and if (6.2) holds then (6.1) also holds with $q = 2$. Finally, we can use Riesz–Thorin convexity theorem once more to reduce the proof of (6.2) to the case $p' = \infty$. We shall therefore prove that

$$\frac{\|Y_m\|_{L^\infty(\mathbf{S}^{d-1},d\sigma)}}{\|Y_m\|_{L^2(\mathbf{S}^{d-1},d\sigma)}} \le Cm^{\frac{d-2}{4}}. \tag{6.3}$$

We now recall that Y_m is as in Lemma 2, that is, as in (1.3). If we use spherical coordinates, (6.3) can be rewritten as

$$\frac{\displaystyle\sup_{\substack{z_{k+1}\in[0,\pi]\\ k=0,\dots,d-3}} \prod_{k=0}^{d-3}\left|P_{m_k-m_{k+1}}^{\left(m_{k+1}+\frac{d-k-1}{2}\right)}(\cos z_{k+1})\right|(\sin z_{k+1})^{m_{k+1}}}{\left(\displaystyle\int_0^\pi\cdots\int_0^\pi \prod_{k=0}^{d-3}\left|P_{m_k-m_{k+1}}^{\left(m_{k+1}+\frac{d-k-1}{2}\right)}(\cos z_{k+1})\right|^2(\sin z_{k+1})^{2m_{k+1}+d-2-k}\,dz_1\cdots dz_{d-2}\right)^{\frac{1}{2}}}$$

$$= \prod_{k=0}^{d-3}\frac{\displaystyle\sup_{z_{k+1}\in[0,\pi]}\left|P_{m_k-m_{k+1}}^{\left(m_{k+1}+\frac{d-k-1}{2}\right)}(\cos z_{k+1})\right|(\sin z_{k+1})^{m_{k+1}}}{\left(\displaystyle\int_0^\pi\left|P_{m_k-m_{k+1}}^{\left(m_{k+1}+\frac{d-k-1}{2}\right)}(\cos z_{k+1})\right|^2(\sin z_{k+1})^{2m_{k+1}+d-2-k}\,dz_{k+1}\right)^{\frac{1}{2}}}$$

$$\le Cm^{\frac{d-2}{4}}.$$

Thus, (6.3) follows if we can prove that, for every $0 \le k \le d-3$,

$$\frac{\displaystyle\sup_{z_{k+1}\in[0,\pi]}\left|P_{m_k-m_{k+1}}^{\left(m_{k+1}+\frac{d-k-1}{2}\right)}(\cos z_{k+1})\right|(\sin z_{k+1})^{m_{k+1}}}{\left(\displaystyle\int_0^\pi\left|P_{m_k-m_{k+1}}^{\left(m_{k+1}+\frac{d-k-1}{2}\right)}(\cos z_{k+1})\right|^2(\sin z_{k+1})^{2m_{k+1}+d-2-k}\,dz_{k+1}\right)^{\frac{1}{2}}} \le Cm^{\frac{1}{4}}. \tag{6.4}$$

To simplify notation, we will let $z = z_{k+1}$, $n = m_k - m_{k+1}$, $s = m_{k+1} + \frac{d-k-1}{2}$, and $j = \frac{d-2-k}{2}$.

We also observe that we can integrate over the interval $(0, \pi/2)$, since the ultraspherical polynomials are either even or odd. With the new formalism, the inequality that we shall prove is

$$\frac{\displaystyle\sup_{z\in[0,\pi/2]}|P_n^{(s)}(\cos z)|(\sin z)^{s-j}}{\left(\displaystyle\int_0^{\pi/2}|P_n^{(s)}(\cos z)|^2(\sin z)^{2s}\,dz\right)^{\frac{1}{2}}} \le Cm^{\frac{1}{4}}. \tag{6.5}$$

A change of variables shows that (6.5) is equivalent to (2.6), which will be proved in the next section.

7. Proof of Proposition 3

As observed at the end of the previous section, (2.6) is equivalent to (6.5). We therefore direct our attention to the proof of (6.5). We divide the proof of the inequality (6.5) into four steps.

Step 1. In what follows we will often denote by $I_{j,n}^s$ the ratio on the left-hand side of (6.5), and we will let $\|f\|_p = \|f\|_{L^p(0,\pi/2)}$.

The L^2 norm of $(\sin z)^s P_n^{(s)}(\cos z)$ is

$$q_{n,s} = \left(\frac{\pi 2^{1-2s}\Gamma(n+2s)}{(n+s)(\Gamma(s))^2\Gamma(n+1)}\right)^{\frac{1}{2}} \tag{7.1}$$

(see Section 8).

By Proposition 2 and (7.1) we obtain

$$I_{j,n}^s = \frac{\sup\limits_{z\in[0,\pi/2]} |(\sin z)^{(s-j)}P_n^{(s)}(\cos z)|^2}{\displaystyle\int_0^{\pi/2} |(\sin z)^s P_n^{(s)}(\cos z)|^2\,dz} \le e^{2j}\left(1+\frac{n}{s}\right)^{2j}\left(\frac{c_{n,s}}{q_{n,s}}\right)^2.$$

We will assume that n is even, since the proof in the other case is similar. Therefore,

$$\left(\frac{c_{n,s}}{q_{n,s}}\right)^2 = \frac{(n+s)\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{n}{2}+s\right)}{\pi\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(\frac{1+n}{2}+s\right)}.$$

By Lemma 5,

$$\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma\left(\frac{1+n}{2}+s\right)} \le \left(\frac{n}{2}+s\right)^{-\frac{1}{2}} \quad \text{and} \quad \frac{\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)} \le \left(\frac{1+n}{2}\right)^{-\frac{1}{2}}.$$

We thus obtain

$$\left(\frac{c_{n,s}}{q_{n,s}}\right)^2 \le \frac{2(n+s)}{\pi(n+1)^{\frac{1}{2}}(n+2s)^{\frac{1}{2}}},$$

and consequently

$$I_{j,n}^s \le e^{2j}\left(1+\frac{n}{s}\right)^{2j}\frac{2(n+s)}{\pi(n+1)^{\frac{1}{2}}(n+2s)^{\frac{1}{2}}}. \tag{7.2}$$

If $n \le \alpha s$ for some fixed $\alpha > 1$, then

$$I_{j,n}^s \le e^{2j}(1+\alpha)^{2j}\sup_{n\le\alpha s}\frac{2(n+s)}{\pi(n+1)^{\frac{1}{2}}(n+2s)^{\frac{1}{2}}}.$$

One can easily verify that $(n+1)(n+2s) \ge \left(n+\sqrt{s}\right)^2$; therefore,

$$I_{j,n}^s \leq e^{2j}(1+\alpha)^{2j}\frac{2(n+s)}{\pi\big(n+\sqrt{s}\big)} \leq \frac{4}{\pi}e^{2j}(1+\alpha)^{2j}s^{\frac{1}{2}},$$

which is what we shall prove.

In the next step we will show that we can always reduce matters to this case.

Step 2. In the proof of Lemma 3 we have observed that the following equation is satisfied by the critical points of $y_n^s(t)$:

$$y_{j,n}^s(z) = \frac{2s}{s-j}(\tan z)y_{j,n-1}^{s+1}(z).$$

By Lemma 3, the relative extrema of $y_{j,n}^s(z)$ in the interval $[0,\pi/2]$ are increasing whenever

$$z \leq z_{j,n}^s = \frac{1}{2}\arccos\left(\frac{j-3j^2+j^3+jn^2-s+2js+2jns+s^2-js^2}{j(-j^2+n^2+2ns+s^2)}\right)$$

and are decreasing otherwise.

Therefore, $y_{j,n}^s(z)$ attains its maximum at one of the two critical points that immediately follow or precede $z_{j,n}^s$. Let $\bar{z}$ be such a point. Then

$$\sup_{z\in[0,\pi/2]} y_{j,n}^s(z) \leq \frac{2s}{s-j}(\tan\bar{z})\sup_{z\in[0,\pi/2]} y_{j,n-1}^{s+1}(z).$$

In the next steps we will prove that there exists an $\alpha > \alpha_j$, where

$$\alpha_j = \frac{4\pi}{\sqrt{2j-1}},$$

such that the following inequalities hold whenever $j > \frac{1}{2}$, $s \geq j$, and $n \geq \alpha s$:

(a) $\bar{z} \leq 2z_{j,n}^s$;

(b) $\frac{2s}{s-j}\tan\bar{z} \leq 1$.

This will be enough to conclude the proof of the theorem. Indeed, from (a) and (b) it follows that

$$\sup_{0\leq z\leq\pi/2} y_{j,n}^s(z) \leq \sup_{0\leq z\leq\pi/2} y_{j,n-k}^{s+k}(z)$$

for every k such that $(n-k+1) \geq \alpha(s+k-1)$. If we let $k = \left[\frac{n-\alpha s}{\alpha+1}\right]$, we have $(n-k+1) \geq \alpha(s+k-1)$ and $n-k \leq \alpha(s+k)$.

By Step 1,

$$I_{j,n-k}^{s+k} \leq Ce^{2j}(1+j^2)^{2j}(s+k)^{\frac{1}{2}} \leq Ce^{2j}(1+j^2)^{2j}(s+n)^{\frac{1}{2}},$$

where C depends only on j, which is what we required.

Step 3. In proving (a) we suppose that $\bar{z} \geq z_{j,n}^s$, since the other case is trivial.

We recall that $\bar{z}$ is the first critical point of $y_{j,n}^s(z)$ in the interval $[z_{j,n}^s, \pi/2]$. By Lemma 4, the function $y_{j,n}^s(z)$ has at least a zero in the interval $[z_{j,n}^s, \sigma(s,n)+z_{j,n}^s]$

and at least two zeroes in $[z^s_{j,n}, 2\sigma(s,n)+z^s_{j,n}]$, where $\sigma(s,n) = \frac{\pi}{\sqrt{(n+s)^2+s(s-1)}}$. By Rolle's theorem, $y^s_{j,n}(z)$ has at least one critical point in $[z^s_{j,n}, 2\sigma(s,n)+z^s_{j,n}]$ and thus $\bar{z} \le 2\sigma(s,n)+z^s_{j,n}$.

We prove that $2\sigma(s,n) \le z^s_{j,n}$ whenever $n \ge \alpha_j s$ and $s \ge j$. Toward this end it is sufficient to prove that

$$\begin{aligned}(4\sigma(s,n))^2 &\le \sin(2z^s_{j,n})^2\\ &= 1 - \left(\frac{j-3j^2+j^3+jn^2+(2j+2jn-1)s+s^2(1-j)}{j(n^2-j^2+2ns+s^2)}\right)^2\\ &= 2u(s,n)-(u(s,n))^2,\end{aligned}$$

where we have let

$$u(s,n) = \frac{(2j-1)(s^2-s-j^2-j)}{j((n+s)^2-j^2)}.$$

Thus, we shall prove that

$$(u(s,n))^2 - 2u(s,n) + (4\sigma(s,n))^2 \le 0$$

or (equivalently) that

$$u(s,n) \le 1+\sqrt{1-(4\sigma(s,n))^2}.$$

We observe that $u(s,n) \le u(s,0) \le \lim_{s\to\infty} u(0,s) = 2-\frac{1}{j}$, since $u(s,n)$ is increasing with respect to s and decreasing with respect to n. Thus, we prove that

$$2-\frac{1}{j} \le 1+\sqrt{1+(4\sigma(s,n))^2}$$

or

$$(4\sigma(s,n))^2 \le \frac{2}{j}-\frac{1}{j^2}$$

whenever $n \ge \alpha_j s$ and $s \ge j$. But

$$(4\sigma(s,n))^2 \le \frac{16\pi^2}{(1+\alpha_j)^2 s^2} \le \frac{16\pi^2}{(1+\alpha_j)^2 j^2},$$

so the claim readily follows.

Step 4. We now prove (b). We shall prove that there exists an $\alpha \ge \alpha_j$ such that $\tan(2z^s_{j,n}) \le \frac{s-j}{2s}$ whenever $n \ge \alpha s$, $s > j$, and $j > \frac{1}{2}$. This is equivalent to $(2z^s_{j,n}) \le \arctan\left(\frac{s-j}{2s}\right)$ or to

$$\cos(2z^s_{j,n}) \ge \cos\left(\arctan\left(\frac{s-j}{2s}\right)\right) = \frac{1}{\left(1+\left(\frac{s-j}{2s}\right)^2\right)^{\frac{1}{2}}}, \tag{7.3}$$

since $t \to \cos t$ is a decreasing function in $[0, \pi/2]$.

The function on the right-hand side of (7.3) is an increasing function of s, and its supremum is $2/\sqrt{5}$. Therefore, it is sufficient to prove that

$$\cos(2z_{j,n}^s) = \frac{j - 3j^2 + j^3 + jn^2 - s + 2js + 2jns + s^2 - js^2}{j(-j^2 + n^2 + 2ns + s^2)} \geq \frac{2}{\sqrt{5}}.$$

Let $\cos(2z_{j,n}^s)) = A(s,n)$. It is easy to see that $n \to A(s,n)$ is increasing and hence that $A(s,n) \geq A(s,\alpha s)$. We now prove that, for some $\alpha > \alpha_j$,

$$A(s,\alpha s) - \frac{2}{\sqrt{5}} = \frac{-5j + 15j^2 - 5j^3 - 2\sqrt{5}j^3 + (5 - 10j)s + \psi(\alpha, j)s^2}{5j(j^2 - s^2 - 2\alpha s^2 - \alpha^2 s^2)} \geq 0,$$

where $\psi(\alpha, j) = \left(-5 + 5j + 2\sqrt{5}j - 10\alpha j + 4\sqrt{5}\alpha j - 5\alpha^2 j + 2\sqrt{5}\alpha^2 j\right)$, whenever $s \geq j > \frac{1}{2}$. But this is easily seen to be satisfied.

8. Appendix

We collect here the definitions and identities that we have used throughout this paper that are related to Jacobi polynomials and Bessel functions. Our main reference is the classical book of Szegö [12], but the formulas listed here can also be found in many other standard textbooks on special functions (see e.g. [1]).

Let $\alpha, \beta \in \mathbf{R}$. The Jacobi polynomials of degree n and order (α, β) are

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\left(\frac{d^n}{dx}\right)(1-x)^{\alpha+n}(1+x)^{\beta+n}. \tag{8.1}$$

They are a complete orthogonal system in $L^2([-1,1], (1-x)^\alpha(1+x)^\beta\,dx)$.

When $\alpha = \beta$, the Jacobi polynomials take the name of *ultraspherical* (or Gegenbauer) polynomials and are denoted by $P_n^{(s)}(x)$. Here is the customary notation and normalization:

$$P_n^{(s)}(t) = C_n^s P_n^{s-\frac{1}{2},s-\frac{1}{2}}(t), \quad s > -\frac{1}{2}, \tag{8.2}$$

where

$$C_n^s = \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(2s)}\frac{\Gamma(n+2s)}{\Gamma\left(n+s+\frac{1}{2}\right)}.$$

We can easily see that $P_n^{(s)}(x) \equiv 1$ when $n = 0$ and $P_n^{(s)}(x) = 2sx$ when $n = 1$. Furthermore,

$$P_n^{(s)}(-x) = (-1)^n P_n^{(s)}(x). \tag{8.3}$$

Observe that $P_n^{(s)}(t)$ satisfies the differential equation

$$(1-x^2)y'' - (2s+1)xy' + n(n+2s)y = 0 \tag{8.4}$$

and that $(\sin t)^s P_n^{(s)}(\cos t)$ satisfies the differential equation

$$u'' + \left((n+s)^2 + \frac{s(s-1)}{(\sin z)^2}\right)u = 0. \tag{8.5}$$

We also recall that

$$\frac{d}{dt}P_n^{(s)}(t) = 2sP_{n-1}^{(s+1)}(x) \tag{8.6}$$

and

$$\sup_{-1\le x\le 1} |P_n^{(s)}(x)| = P_n^{(s)}(1) = \frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}, \tag{8.7}$$

and also that

$$q_{n,s} = \left(\int_0^{\pi/2} (\sin t)^2 |P_n^{(s)}(\cos t)|^2\, dt\right)^{\frac{1}{2}} = \left(\frac{\pi 2^{1-2s}\Gamma(n+2s)}{(n+s)(\Gamma(s))^2\Gamma(n+1)}\right)^{\frac{1}{2}}. \tag{8.8}$$

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