THE DERIVED SETS OF A LINEAR SET

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For an arbitrary set E on the x-axis and an ordinal number b, let E^b denote the derived set of order b of E (the definition of the derived sets of a set, together with a discussion of many of their properties, was given by Cantor [2, pp. 98, 139-246]; for an excellent concise treatment of the material which is used here, see Baire [1, Chpt. III, especially §§ 32, 41, 42]). Since the set E^Ω, where Ω denotes the least nondenumerable ordinal number, is perfect, $E^b = E^\Omega$ for all $b > \Omega$. If a point x in E does not lie in E^Ω, the least ordinal b for which x does not lie in E^b is of the first kind; for if x lies in every set E^a $(a < b)$ and b is an ordinal of the second kind, then, since E^b is the intersection of the closed sets E^a $(a < b)$, the point x lies also in E^b. It follows that for every x in $E - E^\Omega$ there exists a greatest ordinal b for which x lies in E^b. This suggests the following definition of an ordinal-valued function p(x, E). For an arbitrary fixed set E and each point x, let

$$p(x, E) = \Omega \text{ if } x \text{ lies in } E^\Omega,$$

$$p(x, E) = 0 \text{ if } x \text{ does not lie in } E^1,$$

$$p(x, E) = b \text{ if } x \text{ lies in } E^b \text{ but not in } E^{b+1}.$$

It is the purpose of this note to characterize, among all the ordinal-valued functions p(x), those functions for which there exists a point set E such that $p(x, E) \equiv p(x)$.

THEOREM. *If* p(x) *is an ordinal-valued function defined on the* x-*axis, a necessary and sufficient condition for the existence of a point set* E *with* $p(x, E) = p(x)$ *is that* p(x) *have the following properties:*

(1) *the set where* $p(x) = \Omega$ *is perfect, and the set where* $p(x) > \Omega$ *is empty;*

(2) *for each* b $(0 < b < \Omega)$, *the set where* $p(x) = b$ *is isolated;*

(3) *for each point* x_0, *the inequality* $p(x) \le p(x_0)$ *holds throughout some neighborhood of* x_0;

(4) *if* $p(x_0) < \Omega$ *and* $a < p(x_0)$, *then* p(x) *takes the value* a *in every neighborhood of* x_0.

The necessity of the condition is not new. Indeed, if there exists a set E such that $p(x, E) = p(x)$, then (1) follows from the fact that E^Ω is perfect, and (2) and (3) follow from the fact that every limit point of E^b belongs to E^{b+1} $(b \ge 1)$. Finally, if x_0 lies in an open interval I in which $p(x) < \Omega$, and if p(x) does not take the value a in I, then $E^a = E^{a+1}$ in I; since $I \cap E^\Omega$ is empty, this implies that $I \cap E^a$ is also empty, and therefore $p(x) < a$ throughout I.

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To prove the sufficiency of the condition, suppose that $p(x)$ is an ordina function with the properties (1) to (4). It will be convenient to use the symb for the set where $p(x) = b$.

By (2), the set $M(1)$ is either empty or it consists of finitely or denume many isolated points x_k. Let $\{A_k\}$ be a family of disjoint open intervals (A_k); for each k, let $\{x_{ki}\}$ be a sequence of points in A_k ($x_{ki} \to x_k$); and let the set consisting of all the points x_{ki}. The set

$$E = X \cup M(\Omega)$$

has the desired property.

That $p(x, E) = p(x)$ in $M(\Omega)$ follows from (1). By (3), the set $M(0)$ is o since it contains no limit points of the isolated set X, the relation $p(x, E) =$ holds in $M(0)$. It remains to deal with the sets $M(b)$ $(0 < b < \Omega)$.

By (2), each point of $M(1)$ is an isolated limit point of the set X; theref $p(x, E) = 1$ in $M(1)$. Suppose now that $p(x, E) = p(x)$ in $M(a)$, for all a less b, and let x_0 be a point in $M(b)$. By (2) and (3), some neighborhood of x_0 c no points x (other than x_0) where $p(x) > b$; therefore some neighborhood of tains no points of E^{b+1}, and $p(x_0, E) \leq b$. On the other hand, (4) implies th neighborhood of x_0 contains points of each set M^a $(a < b)$, hence points of the corresponding sets E^a. Therefore x_0 lies in E^b, that is, $p(x_0, E) \geq b$, proof is complete.

In closing, it should be remarked that the properties (1) to (4) are indep If $p(x) = 0$ everywhere except at one point where $p(x) = \Omega$, then the functio has all the properties except (1). If $p(x) = 1$ everywhere on a Cantor set, a $p(x) = 0$ elsewhere, then $p(x)$ has all the properties except (2). If $p(x) = 1$ $x = 1/n$ $(n = 1, 2, \cdots)$ and $p(x) = 0$ elsewhere, then $p(x)$ has all the properti except (3). Finally, the function which takes the value 2 at one point and th 0 elsewhere has all the properties except (4).

REFERENCES

1. R. Baire, *Leçons sur les fonctions discontinues*, Paris, 1905.

2. G. Cantor, *Gesammelte Abhandlungen*, Berlin, 1932.

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