Michigan Math. J. 48 (2000)

On the WDVV Equation in Quantum K-Theory

Alexander Givental

To W. Fulton on his 60th birthday

0. Introduction

Quantum cohomology theory can be described in general terms as intersection theory in spaces of holomorphic curves in a given Kähler or almost Kähler manifold X. By quantum K-theory we may similarly understand the study of complex vector bundles over the spaces of holomorphic curves in X. In these notes, we will introduce a K-theoretic version of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation which expresses the associativity constraint of the "quantum multiplication" operation on $K^*(X)$.

Intersection indices of cohomology theory,

$$\int_{[\text{space of curves}]} \omega_1 \wedge \cdots \wedge \omega_k$$

obtained by evaluation on the fundamental cycle of cup products of cohomology classes are to be replaced in K-theory by Euler characteristics

$$\chi(\text{space of curves}; V_1 \otimes \cdots \otimes V_k)$$

of tensor products of vector bundles. The hypotheses needed in the definitions of the intersection indices and Euler characteristics—that the spaces of curves are compact and nonsingular, or that the bundles are holomorphic—are rarely satisfied. We handle this foundational problem by restricting ourselves throughout the notes to the setting where the problem disappears. Namely, we will deal with the so-called moduli spaces $X_{n,d}$ of degree-d genus-0 stable maps to X with n marked points *assuming that X is a homogeneous Kähler space.* Under this hypothesis, the moduli spaces $X_{n,d}$ (we will review their definition and properties when needed) are known to be compact complex orbifolds (see [1; 10]). We use their fundamental cycle $[X_{n,d}]$, well-defined over $\mathbb{Q}$, in the definition of intersection indices, and we use sheaf cohomology in the definition of the Euler characteristic of a holomorphic *orbi-bundle* V:

$$\chi(X_{n,d}; V) := \sum (-1)^k \dim H^k(X_{n,d}; \mathcal{O}(V)).$$

Received March 20, 2000. Revision received April 20, 2000.

1. Correlators

The WDVV equation is usually formulated in terms of the following generating function for *correlators*:

$$F(t, Q) = \sum_{d} \sum_{n=0}^{\infty} \frac{Q^d}{n!} (t, \ldots, t)_{n,d}.$$

Here $d \in H_2(X, \mathbb{Z})$ runs over the Mori cone of *degrees,* that is, homology classes represented by fundamental cycles of rational holomorphic curves in X, and the correlators $(\phi_1, \ldots, \phi_n)_{n,d}$ are defined using the *evaluation maps* at the marked points:

$$\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_n : X_{n,d} \to X \times \cdots \times X.$$

In cohomology theory, we pull back to the moduli space $X_{n,d}$ the n cohomology classes $\phi_1, \ldots, \phi_n \in H^*(X, \mathbb{Q})$ of X and define the correlator among them by

$$(\phi_1, \ldots, \phi_n)_{n,d} := \int_{[X_{n,d}]} \mathrm{ev}_1^*(\phi_1) \wedge \cdots \wedge \mathrm{ev}_n^*(\phi_n).$$

In K-theory, we pull back n elements $\phi_1, \ldots, \phi_n \in K^*(X)$ (representable under our restriction on X by holomorphic vector bundles or their formal differences) and put

$$(\phi_1, \ldots, \phi_n)_{n,d} := \chi(X_{n,d}; \mathrm{ev}_1^*(\phi_1) \otimes \cdots \otimes \mathrm{ev}_n^*(\phi_n)).$$

We will treat the series F as a formal function of $t \in H$ depending on formal parameters $Q = (Q_1, \ldots, Q_{\mathrm{Betti}_2(X)})$, where $H = H^*(X, \mathbb{Q})$ or $H = K^*(X)$.

Let $\{\phi_\alpha\}$ be a graded basis in $H^*(X, \mathbb{Q})$, and let

$$g_{\alpha\beta} := \langle \phi_\alpha, \phi_\beta \rangle = \int_{[X]} \phi_\alpha \wedge \phi_\beta$$

denote the intersection matrix. Let $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ be the inverse matrix (so that $\sum (\phi_\alpha \otimes 1) g^{\alpha\beta} (1 \otimes \phi_\beta)$ is Poincaré-dual to the diagonal in $X \times X$). In quantum cohomology theory, one defines the *quantum cup product* $\bullet$ on the tangent space $T_t H$ by

$$\langle \phi_\alpha \bullet \phi_\beta, \phi_\gamma \rangle := F_{\alpha\beta\gamma}(t)$$

(where the subscripts on the RHS mean partial derivatives in the basis $\{\phi_\alpha\}$). In this notation, the associativity of the quantum cup product is equivalent to the following WDVV identity:

$\sum_{\varepsilon, \varepsilon'} F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta}$ *is totally symmetric in* $\alpha, \beta, \gamma, \delta$.

2. Stable Maps, Gluing, and Contraction

In order to explain the proof of the WDVV identity, we must discuss some properties of the moduli spaces $X_{n,d}$ (see [1; 4; 10] for more details).

We consider prestable marked curves $(C, \mathbf{z})$, that is, compact connected complex curves C with at most double singular points and with n marked points

$\mathbf{z} = (z_1, \ldots, z_n)$ that are nonsingular and distinct. Two holomorphic maps, $f\colon (C, \mathbf{z}) \to X$ and $f'\colon (C', \mathbf{z}') \to X$, are called *equivalent* if they are identified by an isomorphism $(C, \mathbf{z}) \to (C', \mathbf{z}')$ of the curves. This definition introduces the concept of *automorphism* of a map $f\colon (C, \mathbf{z}) \to X$, and one calls f *stable* if it has no nontrivial infinitesimal automorphisms. The moduli spaces $X_{n,d}$ consist of equivalence classes of stable maps with fixed number n of marked points, degree d, and arithmetic genus 0 (it is defined as $g = \dim H^1(C, \mathcal{O}_C)$).

In plain terms, the space of degree-d holomorphic spheres in X with n marked points is compactified by prestable curves which are trees of $\mathbb{C}P^1$s and satisfy the stability condition: each irreducible component $\mathbb{C}P^1$ mapped to a point in X must carry at least three marked or singular points. Under the hypothesis that X is a homogeneous Kähler space, the moduli space $X_{n,d}$ has the structure of a compact complex orbifold of dimension $\dim_{\mathbb{C}} X + \int_d c_1(T_X) + n - 3$.

When X is a point, the moduli spaces coincide with the Deligne–Mumford compactifications $\bar{\mathcal{M}}_{0,n}$ of moduli spaces of configurations of marked points on $\mathbb{C}P^1$. For instance, $\mathcal{M}_{0,4}$ is the set $\mathbb{C}P^1 - \{0, 1, \infty\}$ of allowed values of the cross-ratio of four marked points on $\mathbb{C}P^1$. The compactification $\bar{\mathcal{M}}_{0,4} = \mathbb{C}P^1$ fills in the forbidden values of the cross-ratio by equivalence classes of reducible curves $\mathbb{C}P^1 \cup \mathbb{C}P^1$ with one double point and two marked points on each irreducible component.

For $n \geq 3$, there is a natural *contraction* map $X_{n,d} \to \bar{\mathcal{M}}_{0,n}$ defined by composing the map $f\colon (C, \mathbf{z}) \to X$ with $X \to pt$ (so that the components of C carrying < 3 special points become unstable) and contracting the unstable components. Similarly, one can define the *forgetting* maps $\mathrm{ft}_i\colon X_{n+1,d} \to X_{n,d}$ by disregarding the ith marked point and contracting the component if it has become unstable.

In particular, we will make use of the contraction map

$$\mathrm{ct}\colon X_{n+4,d} \to \bar{\mathcal{M}}_{0,4}$$

defined by forgetting the map $f\colon (C, \mathbf{z}) \to X$ and all the marked points except the first four. An allowed value $\lambda = \mathrm{ct}[f]$ of the cross-ratio means the following: the curve C has a component $C_0 = \mathbb{C}P^1$ carrying four special points with the cross-ratio λ, and the first four marked points are situated on the branches of the tree connected to C_0 at those four special points. A forbidden value $\mathrm{ct}[f] = 0$, 1, or ∞ means that C contains a *chain* $C_0, \ldots, C_k$ of $k > 0$ $\mathbb{C}P^1$s such that two of the four branches of the tree carrying the marked points are connected to the chain via C_0 and the other two via C_k. Such stable maps form a stratum of codimension k in the moduli space $X_{n,d}$. We will refer to them as strata (or stable maps) *of depth k*.

A stable map of depth 1 is glued from two stable maps obtained by disconnecting C_0 from C_1. This gives rise to the *gluing map*

$$X_{n_0+3,d_0} \times_\Delta X_{n_1+3,d_1} \to X_{n_0+n_1+4,d_0+d_1}$$

as follows. Consider the map from $X_{n_0+3,d_0} \times X_{n_1+3,d_1}$ to $X \times X$ defined by evaluation at the third marked points. Note that, for a homogeneous Kähler X, the map is conveniently transverse to the diagonal $\Delta \subset X \times X$. The source of the gluing map is the preimage of Δ. It consists of pairs of stable maps which have the same image of the third marked point and which therefore can be glued at this point into a single stable map of degree $d_0 + d_1$ with $n_0 + 2 + n_1 + 2$ marked points.

Similarly, gluing stable maps of depth k from $k+1$ stable maps subject to k diagonal constraints at the double points of the chain $C_0, \dots, C_k$ defines appropriate gluing maps parameterizing the strata of depth k.

3. Proof of the WDVV Identity

All points in $\bar{\mathcal{M}}_{0,4}$ represent the same (co)homology class. Thus, the analytic fundamental cycles of the fibers $\mathrm{ct}^{-1}(\lambda)$ are homologous in $X_{n+4,d}$. The cohomological WDVV identity follows from the fact that (for $\lambda = 0$, 1, or ∞) the fiber $\mathrm{ct}^{-1}(\lambda)$ consists of strata of depth > 0; moreover, the corresponding gluing maps (for all splittings $d = d_0 + d_1$ of the degree and all splittings of the $n = n_0 + n_1$ marked points) are isomorphisms at generic points and so identify the analytic fundamental cycle of the fiber with the sum of the fundamental cycles of $X_{n_0+3,d_1} \times_\Delta X_{n_1+3,d_2}$. This allows one to equate three quadratic expressions of the correlators that differ by the order of the indices $\alpha, \beta, \gamma, \delta$ associated with the first four marked points.

We leave the reader to work out some standard combinatorial details that are needed in order to translate this argument into the WDVV identity for the generating function F. Note that the contraction with the intersection tensor $(g^{\varepsilon\varepsilon'})$ in the WDVV equation takes care of the diagonal constraint $\Delta \subset X \times X$ for the evaluation maps.

In K-theory, similarly, the push-forward to $X \times X$ of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal is expressed as

$$\sum (\phi_\varepsilon \otimes 1) g^{\varepsilon\varepsilon'} (1 \otimes \phi_{\varepsilon'})$$

via $(g^{\varepsilon\varepsilon'})$ inverse to the "intersection matrix"

$$g_{\alpha\beta} := \langle \phi_\alpha, \phi_\beta \rangle = \chi(X; \phi_\alpha \otimes \phi_\beta).$$

The argument justifying the WDVV equation fails, however, since the above gluing map to $\mathrm{ct}^{-1}(\lambda)$ is one-to-one only at the points of depth 1 and does not identify the corresponding structure sheaves. Indeed, a stable map of depth k can be glued from two stable maps in k different ways and thus belongs to the k-fold self-intersection in the image of the gluing map.

Let us examine the variety $\mathrm{ct}^{-1}(\lambda)$ at a point of depth $k > 1$. One of the properties of Kontsevich's compactifications $X_{m,d}$ is that, after passing to the local nonsingular covers (defined by the orbifold structure of the moduli spaces), *the compactifying strata form a divisor with normal crossings* [1; 10]. Moreover, analyzing (inductively in k) the local structure of the contraction map $\mathrm{ct}\colon X_{n+4,d} \to \bar{\mathcal{M}}_{0,4}$ near a depth-k point, one easily finds the local model $\lambda(x_1, \dots, x_k, \dots) = x_1 \cdots x_k$ for the map ct in a suitable local coordinate system. In this model, the components $x_1 = 0, \dots, x_k = 0$ of the divisor with normal crossings represent the strata of depth 1, their intersections $x_{i_1} = x_{i_2} = 0$ represent the strata of depth 2, and so forth. Denote by $\mathcal{O}$ the algebra of functions on our local chart, so that $\mathcal{O}/(x_{i_1}, \dots, x_{i_l})$, $i_1 < \cdots < i_l$, are the algebras of functions on the depth-l strata. We have the following exact sequence of $\mathcal{O}$-modules:

$$0 \to \mathcal{O}/(x_1 \cdots x_k) \to \bigoplus \mathcal{O}/(x_i) \to \bigoplus \mathcal{O}/(x_{i_1}, x_{i_2}) \\ \to \bigoplus \mathcal{O}/(x_{i_1}, x_{i_2}, x_{i_3}) \to \cdots .$$

Notice that the $\bigoplus$-terms in the sequence are the algebras of functions on the normalized strata of depth 1, depth 2, ... Translating this local formula to a global K-theoretic statement about gluing maps, we conclude that, in the Grothendieck group of orbi-sheaves on $X_{n+4,d}$, the element represented by the structure sheaf of $\mathrm{ct}^{-1}(\lambda)$ (for $\lambda = 0, 1$, or ∞) is identified with the structure sheaf of the corresponding alternating disjoint sum over positive depth strata:

$$\sum X_{n_0+3,d_0} \times_\Delta X_{n_1+3,d_1} - \sum X_{n_0+3,d_0} \times_\Delta X_{n_1+2,d_1} \times_\Delta X_{n_2+3,d_2} + \cdots .$$

4. Formulation and Consequences

Now we can apply the preceding K-theoretic statement about moduli spaces to our generating functions. We introduce

$$G(t, Q) := \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} t_\alpha t_\beta + F(t, Q)$$

and let $(G^{\alpha\beta})$ be the matrix inverse to $(G_{\alpha\beta}) = (\partial_\alpha \partial_\beta G)$.

THEOREM.

$$\sum_{\varepsilon,\varepsilon'} G_{\alpha\beta\varepsilon} G^{\varepsilon\varepsilon'} G_{\varepsilon\gamma\delta} \textit{ is totally symmetric in } \alpha, \beta, \gamma, \delta.$$

Proof. We have rewritten

$$F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} - F_{\alpha\beta\varepsilon} g^{\varepsilon\mu} F_{\mu\mu'} g^{\mu'\varepsilon'} F_{\varepsilon'\gamma\delta} + \cdots$$

using the well-known matrix identity $1 - F + F^2 - \cdots = (1 + F)^{-1}$. □

Now introduce the *quantum tensor product* on $T_t H$ (with $H = K^*(X)$) by

$$(\phi_\alpha \bullet \phi_\beta, \phi_\gamma) := G_{\alpha\beta\gamma}(t),$$

where the metric $(\cdot, \cdot)$ on TH is defined by $(\phi_\mu, \phi_\nu) := G_{\mu\nu}(t)$.

COROLLARY 1. *The operations* $(\cdot, \cdot)$ *and* $\bullet$ *define on the tangent bundle the structure of a formal commutative associative Frobenius algebra with the unit* 1.

REMARK. At $Q = 0$, the algebra turns into the usual multiplicative structure on $K^*(X)$.

Proof. As in the cohomology theory, this is a formal corollary of the theorem—except that the statement about the unit 1 means that $G_{\alpha 1\beta} = G_{\alpha\beta}$ and follows from the simplest instance of the *string equation* in K-theory: $(1, t, \dots, t)_{n+1,d} = (t, \dots, t)_{n,d}$. The last equality is obvious. Indeed, the push-forward of the constant sheaf 1 along the map $\mathrm{ft}\colon X_{n+1,d} \to X_{n,d}$ (forgetting the first marked point) is the constant sheaf 1 on $X_{n,d}$ since the fibers are curves C of arithmetic genus $g = \dim H^1(C, \mathcal{O}_C) = 0$ while $H^0(C, \mathcal{O}_C) = \mathbb{C}$ by Liouville's theorem. □

We introduce on T^*H the 1-parameter family of connection operators

$$\nabla_q := (1-q)d - \sum_\alpha (\phi_\alpha \bullet) dt_\alpha \wedge .$$

COROLLARY 2. *The connections ∇_q are flat for any $q \neq 1$.*

Proof. This follows from $\phi_\alpha \bullet \phi_\beta = \phi_\beta \bullet \phi_\alpha$, $d^2 = 0$, and $\partial_\alpha(\phi_\beta \bullet) = \partial_\beta(\phi_\alpha \bullet)$:

$$\partial_\alpha(\phi_\beta \bullet)^\nu_\mu = G_{\mu\alpha\beta\varepsilon} G^{\varepsilon\nu} - G_{\mu\beta\varepsilon} G^{\varepsilon\varepsilon'} G_{\varepsilon'\alpha\varepsilon''} G^{\varepsilon''\nu}$$

is symmetric with respect to α and β because of the WDVV identity. □

PROPOSITION. *The operator ∇_{-1} is twice the Levi–Civita connection of the metric $(G^{\alpha\beta})$ on T^*H.*

Proof. For a metric of the form $G_{\alpha\beta} = \partial_\alpha \partial_\beta G$, the famous explicit formulas for the Christoffel symbols yield

$$2\Gamma^\gamma_{\alpha\beta} = [G_{\alpha\varepsilon\beta} + G_{\beta\varepsilon\alpha} - G_{\alpha\beta\varepsilon}] G^{\varepsilon\gamma} = G_{\alpha\beta\varepsilon} G^{\varepsilon\gamma} = (\phi_\beta \bullet)^\gamma_\alpha. \qquad \square$$

COROLLARY 3. *The metric $(\cdot,\cdot)$ on TH is flat.*

We complete this section with a description of flat sections of the connection operator ∇_q in terms of K-theoretic "gravitational descendents". Let us introduce the generating functions

$$S_{\alpha\beta}(t, Q) := g_{\alpha\beta} + \sum_{n,d} \frac{Q^d}{n!} \left(\phi_\alpha, t, \dots, t, \frac{\phi_\beta}{1-qL} \right)_{n+2,d},$$

where the correlators are defined by

$$(\psi_1, \dots, \psi_n L^k)_{m,d} := \chi(X_{m,d}; \mathrm{ev}_1^*(\psi_1) \otimes \cdots \otimes \mathrm{ev}_m^*(\psi_m) \otimes L^{\otimes k}).$$

Here L is the line *orbi*-bundle over the moduli space $X_{m,d}$ of stable maps $(C, \mathbf{z}) \to X$ formed by the cotangent lines to C at the *last* marked point (as specified by the position of the geometrical series $1 + qL + q^2L^2 + \cdots = (1-qL)^{-1}$ in the correlator).

THEOREM. *The matrix $S := (S_{\mu\nu})$ is a fundamental solution to the linear PDE system*

$$(1-q)\partial_\alpha S = (\phi_\alpha \bullet) S.$$

Proof. Taking ϕ_μ, ϕ_α, ϕ_β, and $\phi_\nu/(1-qL)$ for the content of the four distinguished marked points in the proof of the WDVV identity, we obtain its generalization in the form

$$G_{\mu\alpha\varepsilon}G^{\varepsilon\varepsilon'}\partial_\beta S_{\varepsilon'\nu} = G_{\mu\beta\varepsilon}G^{\varepsilon\varepsilon'}\partial_\alpha S_{\varepsilon'\nu},$$

or $(\phi_\alpha\bullet)\partial_\beta S = (\phi_\beta\bullet)\partial_\alpha S$. Now it remains to put $\phi_\beta = 1$ and use $(1-q)\partial_1 S = S$, which is another instance of the string equation:

$$(1, t, \dots, t, \phi L^k)_{n+2,d} = (t, \dots, t, \phi(1 + L + \dots + L^k))_{n+1,d}.$$

The last relation is obtained by computing the push-forward of $L^{\otimes k}$ along ft_1: $X_{n+2,d} \to X_{n+1,d}$. Some details can be found in [6; 11; 14; 15]. Briefly, one identifies the fibers of ft_1 with the curves underlying the stable maps $f\colon (C, \mathbf{z}) \to X$ with $n+1$ marked points. It is important to realize that the pull-back $L' := \mathrm{ft}_1^*(L)$ of the line bundle named L on $X_{n+1,d}$ differs from the line bundle named L on $X_{n+2,d}$. In fact, there is a holomorphic section of $\mathrm{Hom}(L', L)$ with the divisor D defined by the last marked point $z_{n+1} \in C$, and the bundle L restricted to D is trivial (while $L'\big|_D$ is therefore conormal to D). Since L' is trivial along the fibers C, we find that $H^1(C, L^k) = 0$ and $H^0(C, L^k) = (L')^k \otimes H^0(C, \mathcal{O}_C(kD)) \simeq (L')^k(1 + (L')^{-1} + \dots + (L')^{-k})$. $\square$

5. Some Open Questions

(a) *Definitions.* It is natural to expect that the foregoing results extend from the case of homogeneous Kähler spaces X to general compact Kähler and, even more generally, almost Kähler target manifolds.

In the Kähler case, the moduli of stable degree-d genus-g maps with n marked points form compact complex orbi-spaces $X_{g,n,d}$ equipped with the *intrinsic normal cone* [13]. The cone gives rise [3] to an element in the K-group of $X_{g,n,d}$ that should be used in the definition of K-theoretic correlators in the same manner as the virtual fundamental cycle $[X_{g,n,d}]$ is used in quantum cohomology theory.

The moduli space $X_{g,n,d}$ can also be described as the zero locus of a section of a bundle $E \to B$ over a nonsingular space. Owing to the famous “deformation to the normal cone” [3], the virtual fundamental cycle represents the Euler class of the bundle. This description survives in the almost Kähler case and yields a topological definition and symplectic invariance of the cohomological correlators. In K-theory, there exists a topological construction of the push-forward from B to the point based on the Whitney embedding theorem and the Thom isomorphism. However, we don’t know how to adjust the construction to our actual setting, where B is nonsingular only in the *orbi*-fold sense.

One (somewhat awkward) option is to define K-theoretic correlators topologically by the RHS of the Kawasaki–Riemann–Roch–Hirzebruch formula [8] for orbi-bundles over B. This proposal deserves further study even in the Kähler case, since it may lead to a “quantum Riemann–Roch formula”.

(b) *Frobenius-like Structures.* Our results in Section 4 show that K-theoretic Gromov–Witten invariants of genus 0 define on the space $H = K^*(X)$ a geometrical structure very similar (but not identical) to the Frobenius structure [2] of cohomology theory. One of the lessons is that the metric tensor on H (which can in both cases be described as $F_{\alpha 1\beta}$) is constant in cohomology theory and equal to $g_{\alpha\beta}$ only by an "accident", but it remains flat in K-theory even though it is no longer constant.

The translation $t \mapsto t + \tau 1$ in the direction of $1 \in H$ leaves the structure invariant in cohomology theory but causes multiplication by e^{τ} in K-theory—because of a new form of the string equation. Also, the $\mathbb{Z}$-grading missing in K-theory makes an important difference. It would be interesting to study the axiomatic structure that emerges here and to compare it with the structure implicitly encoded by K-theory on Deligne–Mumford spaces.

(c) *Deligne–Mumford Spaces.* When the target space X is the point, the moduli spaces $X_{g,n,0}$ are Deligne–Mumford compactifications of the moduli spaces of genus-g Riemann surfaces with n marked points. The parallel between cohomology and K-theory suggest several problems.

Holomorphic Euler characteristics of universal cotangent line bundles and their tensor products satisfy the string and dilation equations. (The same is true not only for $X = pt$ (see [12]). By the way, the push-forward $\mathrm{ft}_*(L)$ along $\mathrm{ft}\colon X_{g,n+1,d} \to X_{g,n,d}$, described by the dilation equation is equal to $\mathcal{H} + \mathcal{H}^* - 2 + n$. Here $\mathcal{H}$ is the g-dimensional *Hodge bundle* with the fiber $H^1(C, \mathcal{O}_C)$. This answer replaces a similar factor $2g-2+n$ in the cohomological dilation equation, but it also shows that tensor powers of $\mathcal{H}$ must be included to complete the list of "observables".)

The K-theoretic generalization of the rest of Witten–Kontsevich's intersection theory [9; 15] is unclear.

The case of genus 0 and 1 has been studied in [11; 12; 14]. The formula

$$\chi\left(\bar{\mathcal{M}}_{0,n};\ \frac{1}{(1-q_1L_1)\cdots(1-q_nL_n)}\right) = \frac{(1+q_1/(1-q_1)+\cdots+q_n/(1-q_n))^{n-3}}{(1-q_1)\cdots(1-q_n)}$$

found by Lee [11] is analogous to the well-known intersection theory result

$$\int_{[\bar{\mathcal{M}}_{0,n}]} \frac{1}{(1-x_1c_1(L_1))\cdots(1-x_nc_1(L_n))} = (x_1+\cdots+x_n)^{n-3}$$

[10; 15]. This second formula is the basis for fixed point computations [5; 10] in equivariant cohomology of the moduli spaces $X_{n,d}$ for toric X. As noted by Lee, the first formula is not sufficient for similar fixed point computation in K-theory: it requires Euler characteristics accountable for *invariants with respect to permutations of the marked points.* Finding an S_n-equivariant version of Lee's formula is an important open problem.

(d) *Computations.* The quantum K-ring is unknown even for $X = \mathbb{C}P^1$. It turns out that the WDVV equation is not powerful enough in the absence of grading constraints and the *divisor equation* (see e.g. [6]).

On the other hand, for $X = \mathbb{C}P^n$, it is not hard to compute the generating functions $G(t, Q)$ and even $S_{\alpha\beta}(t, Q, q)$ *at* $t = 0$ (see [12]). In cohomology theory, this would determine the *small* quantum cohomology ring due to the divisor equation which, roughly speaking, identifies the Q-deformation at $t = 0$ with the t-deformation at $Q = 1$ along the subspace $H^2(X, \mathbb{Q}) \subset H$. No replacement for the divisor equation seems to be possible in K-theory.

At the same time, the heuristic study [5] of S^1-equivariant geometry on the loop space LX suggests that the generating functions $S = S_{1\beta}(0, Q, q)$ should satisfy certain linear q-difference equations (instead of similar linear differential equations of quantum cohomology theory). This expectation is supported by the example of $X = \mathbb{C}P^n$, since Lee [12] has found that the generating functions are solutions to the q-difference equation $D^{n+1}S = QS$ (where $(DS)(Q) := S(Q) - S(qQ)$).

In the case of the flag manifold X, the generating functions S have been identified with the so-called *Whittaker functions*—common eigenfunctions of commuting operators of the q-difference Toda system. This result and its conjectural generalization [7] to the flag manifolds $X = G/B$ of complex simple Lie algebras links quantum K-theory to representation theory and quantum groups. Originally this conjecture served as a motivation for developing the basics of quantum K-theory.

Acknowledgments. The author is thankful to Yuan-Pin Lee and Rahul Pandharipande for useful discussions, and to the National Science Foundation for supporting this research.

References

[1] K. Behrend and Y. Manin, *Stacks of stable maps and Gromov–Witten invariants,* Duke Math. J. 85 (1996), 1–60.

[2] B. Dubrovin, *Geometry of 2D topological field theories,* Lecture Notes in Math., 1620, pp. 120–348, Springer-Verlag, Berlin, 1996.

[3] W. Fulton, *Intersection theory,* Ergeb. Math. Grenzgeb. (3), 2, Springer-Verlag, New York, 1984.

[4] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology,* Proc. Sympos. Pure Math., 62/2, pp. 45–96, Amer. Math. Soc., Providence, RI, 1997.

[5] A. Givental, *Homological geometry. I. Projective hypersurfaces,* Selecta Math. (N.S.) 1 (1995), 325–345.

[6] ———, *The mirror formula for quintic threefolds,* Amer. Math. Soc. Transl. Ser. 2, 196, pp. 49–62, Amer. Math. Soc., Providence, RI, 1995.

[7] A. Givental and Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups,* preprint, 1998.

[8] T. Kawasaki, *The signature theorem for V-manifolds,* Topology 17 (1978), 75–83.

[9] M. L. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function,* Comm. Math. Phys. 147 (1992), 1–23.

[10] ———, *Enumeration of rational curves via torus actions,* Progr. Math., 129, pp. 335–368, Birkhäuser, Boston, 1995.

[11] Y.-P. Lee, *A formula for Euler characteristics of tautological line bundles on the Deligne–Mumford moduli spaces,* Internat. Math. Res. Notices 8 (1997), 393–400.

[12] ———, *Quantum K-theory,* Ph.D. dissertation, Univ. of California, Berkeley, 1999.

[13] J. Li and G. Tian, *Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties,* J. Amer. Math. Soc. 11 (1998), 119–179.

[14] R. Pandharipande, *The symmetric function* $h^0(\bar{M}_{0,n}, L_1^{x_1} \otimes \cdots \otimes L_n^{x_n})$, J. Algebraic Geom. 6 (1997), 721–731.

[15] E. Witten, *Two-dimensional gravity and intersection theory on moduli space,* Surveys in Differential Geom. 1 (1991), 243–310.

Department of Mathematics
University of California – Berkeley
Berkeley, CA 94720

givental@math.berkeley.edu

Department of Mathematics
California Institute of Technology
Pasadena, CA 91125

givental@its.caltech.edu