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On an Example of Voisin

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To Bill Fulton on the occasion of his sixtieth birthday

1. Introduction

A general principle on smooth complete-intersection Calabi–Yau threefolds $X \subseteq P$ is that the obstruction theory for deforming families of rational curves on or with X is essentially determined by the Abel–Jacobi map (see [C2]). This principle can be generalized to families of nonrational curves as long as the Hilbert scheme Hilb^P of these curves in the underlying projective space P is strongly unobstructed. However, Voisin has pointed out that this principle fails badly if Hilb^P is obstructed. Indeed, she gave a beautiful counterexample for a family of nonrational curves on quintic hypersurfaces $X \subseteq P = \mathbb{P}^4$. The curve C in question is just the generic projection of a canonical curve of genus 5 into $\mathbb{P}^3 \subseteq \mathbb{P}^4$. (As we shall see, containing such a curve imposes two conditions on the moduli of X.) We propose to examine the deformation theory of Voisin's (C, X), our justification being the beauty of the geometry and the lack of examples where the deformation theory is understood in the obstructed case.

Concerning that deformation theory, a specific goal of this paper is the following. Given a complete family of curves on X, one wishes to associate a refined Gromov–Witten invariant, which should always be a nonnegative integer. (This is because the expected dimension of the Hilbert scheme at a curve $C \subseteq X$, the Euler characteristic of the normal bundle $N_{C\backslash X}$, is 0.) Again, this integer is often computable when Hilb^P is strongly unobstructed (see [CK]). However, we know of precious few examples where anything can be said in the obstructed case. Here we bound this integer for the Voisin example.

This paper is organized as follows. In Section 2 we consider the geometry of the Hilbert scheme of curves of degree 8 and genus 5 in $\mathbb{P}^4$. In Section 3 we consider those curves that lie in a hyperplane as well as the geometry of the quintic surfaces S that contain them. In Section 4 we consider S as a hyperplane section of a quintic threefold X and study the structure of Hilb^X near the curves in S. Finally, in Section 5 we bound the "refined Gromov–Witten invariant" for the relevant component of Hilb^X. One word about notation: If T' is a subscheme of the Hilbert scheme of curves, then $C_{T'}$ will denote the universal curve over T'.

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2. The Curve

2.1. Canonical Curves of Genus 5

Let $\hat{C}$ be a reduced, irreducible, nonhyperelliptic curve of (arithmetic) genus 5, canonically embedded in $\mathbb{P}^4$.

PROPOSITION 2.1. *Either $\hat{C}$ is a complete intersection of quadrics or $\hat{C}$ lies on a smooth cubic scroll. The latter occurs if and only if the curve $\hat{C}$ admits a g_3^1.*

Proof. The kernel of the map

$$\mathrm{Sym}^2 H^0(\omega_{\hat{C}}) \to H^0(\omega_{\hat{C}}^2)$$

has dimension at least 3, so that $\hat{C}$ in contained in a $\mathbb{P}^2$ of quadrics. If the dimension of the base locus of this linear system is greater than 1 (i.e., if $\hat{C}$ is not a complete intersection of quadrics) then, since $\hat{C}$ does not lie in a hyperplane, $\hat{C}$ must lie in a component of degree 3 that spans $\mathbb{P}^4$. The only such surface is a smooth cubic scroll. □

COROLLARY 2.2. *With $\hat{C}$ as before, $H^1(N_{\hat{C}\backslash\mathbb{P}^4}) = 0$.*

Proof. For $\hat{C}$ a complete intersection of quadrics, this is elementary. So suppose $\hat{C}$ lies on a cubic scroll $\hat{B}$. We have an exact sequence

$$0 \to N_{\hat{C}\backslash\hat{B}} \to N_{\hat{C}\backslash\mathbb{P}^4} \to N_{\hat{B}\backslash\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}} \to 0. \tag{1}$$

To prove the corollary, we will show that

$$h^1(N_{\hat{C}\backslash\hat{B}}) = 0 = h^1(N_{\hat{B}\backslash\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}}).$$

The scroll $\hat{B}$ is isomorphic to the Hirzebruch surface $\mathbb{F}_1$; let E be its exceptional curve and F a fiber. From $g(\hat{C}) = 5$ and $\deg \hat{C} = 8$, we conclude that

$$\hat{C} \equiv 3E + 5F. \tag{2}$$

Hence

$$\deg N_{\hat{C}\backslash\hat{B}} = \hat{C}^2 = 21 > 8 = 2g(\hat{C}) - 2$$

and thus $h^1(N_{\hat{C}\backslash\hat{B}}) = 0$.

Now, if we choose a quadric $Q \supset \hat{B}$ then we have the exact sequence

$$0 \to N_{\hat{B}\backslash Q} \otimes \mathcal{O}_{\hat{C}} \to N_{\hat{B}\backslash\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}} \to N_{Q\backslash\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}} \to 0.$$

But by (1), $\deg N_{\hat{B}\backslash\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}} = 27$. Hence, $\deg N_{\hat{B}\backslash Q} \otimes \mathcal{O}_{\hat{C}} = 11 > 8 = 2g(\hat{C}) - 2$, so that

$$h^1(N_{\hat{B}\backslash Q} \otimes \mathcal{O}_{\hat{C}}) = 0 = h^1(N_{Q\backslash\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}}).$$

Thus $h^1(N_{\hat{B}\backslash\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}}) = 0$. □

As a consequence of the proof we have the following.

COROLLARY 2.3. *Let $Q \supset \hat{C}$ be a quadric threefold that is smooth along $\hat{C}$. Then* $H^1(N_{\hat{C}\backslash Q}) = 0$.

2.2. Curves in $\mathbb{P}^3$

PROPOSITION 2.4. *If $C \subset \mathbb{P}^3$ is a subcanonically embedded, reduced, irreducible, l.c.i. curve of genus* 5, *then* $H^1(N_{C\backslash\mathbb{P}^3}) = 0$.

Proof. C must be the isomorphic image of its canonical model $\hat{C} \subset \mathbb{P}^4$ under the projection $\pi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$. Let $\mathfrak{I}$ and $\mathfrak{J}$ be the ideal sheaves of the curves in $\mathbb{P}^3$ and $\mathbb{P}^4$, respectively. Then the natural map $\pi^* \colon \mathfrak{I} \to \mathfrak{J}$ passes to a map of conormal sheaves and hence to a map of normal bundles

$$\pi_* \colon N_{\hat{C}\backslash\mathbb{P}^4} \to N_{C\backslash\mathbb{P}^3}. \tag{3}$$

At the smooth points of $\hat{C}$, this map is induced by the surjection $\mathfrak{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\hat{C}} \to \mathfrak{T}_{\mathbb{P}^3} \otimes \mathcal{O}_C$, so that the cokernel of (3) is torsion. The desired vanishing is now a corollary of Corollary 2.2. □

2.3. Submanifolds of $\mathrm{Hilb}^{\mathbb{P}^4}$

From now on, $\mathrm{Hilb}^{\mathbb{P}^4}$ will denote the open subscheme of the Hilbert scheme of 1-dimensional schemes of degree 8 and arithmetic genus 5 in $\mathbb{P}^4$ that parameterizes schemes that are reduced, irreducible, and have only singularities with embedding dimension 2. Let $\mathfrak{Q}'$ be the variety of complete irreducible quadric threefolds (i.e., the blow-up of the projective space of quadrics along the locus of double $\mathbb{P}^3$s). We let $\mathbf{Q}''$ be the locus of pairs

$$(\hat{C}, Q) \in \mathrm{Hilb}^{\mathbb{P}^4} \times \mathfrak{Q}'$$

such that $\hat{C}$ is canonically or subcanonically embedded in $\mathbb{P}^4$ and Q is a complete quadric containing $\hat{C}$ in its smooth locus. Let $\mathbf{Q}'$ denote the image of $\mathbf{Q}''$ under the projection

$$\mathrm{Hilb}^{\mathbb{P}^4} \times \mathfrak{Q}' \to \mathrm{Hilb}^{\mathbb{P}^4}.$$

If $C \in \mathbf{Q}'$ only spans a $\mathbb{P}^3$, then all the complete quadrics containing C are double covers of the $\mathbb{P}^3$ branched along a quadric surface tangent to C at each intersection point. Thus C is the projection of a canonical curve $\hat{C}$, and the set of branching quadric surfaces for fixed C is given by the limiting linear series

$$\lim_{\hat{C}\to C} \{\mathbb{P}^3 \cap Q : Q \supseteq \hat{C}\}.$$

Hence $\mathbf{Q}''$ is fibered over $\mathfrak{Q}'$ and, by Corollary 2.3 and Riemann–Roch, all fibers are smooth of dimension 24. Then the fibers of $\mathbf{Q}'' \to \mathbf{Q}'$ are always Zariski open dense subsets of $\mathbb{P}^2$. Thus we have the following proposition.

PROPOSITION 2.5. *$\mathbf{Q}''$ and $\mathbf{Q}'$ are smooth and of dimensions* 38 *and* 36, *respectively.*

On the other hand, let $\mathbf{P}'$ be the locus of $\{C\} \in \mathrm{Hilb}^{\mathbb{P}^4}$ lying in (and therefore spanning) a $\mathbb{P}^3 \subseteq \mathbb{P}^4$. Then $\mathbf{P}'$ is fibered over $(\mathbb{P}^4)^\vee$ and, by Proposition 2.4 and Riemann–Roch, the fibers are smooth of dimension 32. In this case we conclude as follows.

PROPOSITION 2.6. *$\mathbf{P}'$ is smooth of dimension* 36.

Let
$$\mathbf{U}' := \mathbf{P}' \cup \mathbf{Q}', \qquad \mathbf{R}' := \mathbf{P}' \cap \mathbf{Q}'.$$
Then we have the following.

PROPOSITION 2.7. *The scheme*
$$\mathbf{U}' - \mathbf{R}'$$
is smooth and locally closed in $\mathrm{Hilb}^{\mathbb{P}^4}$. *At* $\{C\} \in \mathbf{R}'$, *the embedding dimension of* $\mathrm{Hilb}^{\mathbb{P}^4}$ *is* 37. *The scheme* $\mathbf{U}'$ *is locally open in* $\mathrm{Hilb}^{\mathbb{P}^4}_{\mathrm{red}}$ *and, near* $\{C\} \in \mathbf{R}'$ *with* C *smooth*, $\mathbf{R}'$ *is a smooth reduced divisor along which the two components of* $\mathbf{U}'$ *meet transversely.*

Proof. Since $h^0(N_{C\backslash\mathbb{P}^4}) = 36$ when $\{C\} \in (\mathbf{U}' - \mathbf{R}')$, the first assertion is clear. Suppose $\{C\} \in \mathbf{R}'$. From the sequence
$$0 \to N_{C\backslash\mathbb{P}^3} \to N_{C\backslash\mathbb{P}^4} \to \mathcal{O}_C(1) \to 0, \tag{4}$$
we see that the imbedding dimension $h^0(N_{C\backslash\mathbb{P}^4})$ of $\mathrm{Hilb}^{\mathbb{P}^4}$ at $\{C\}$ is 37. Suppose now that C is smooth. Referring to (4), one component is given by $\mathbf{P}'$ with tangent space given by the preimage of
$$W := \mathrm{image}\big(H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_C(1))\big)$$
in $H^0(N_{C\backslash\mathbb{P}^4})$ and the other by $\mathbf{Q}'$ with tangent space given by the kernel of a rank-1 map
$$H^0(N_{C\backslash\mathbb{P}^4}) \to H^1(\mathcal{O}_C), \tag{5}$$
constructed as follows. If $\mathfrak{D}_1(\mathcal{O}_C(1))$ is the sheaf of first-order differential operators on sections of $\mathcal{O}_C(1)$, then $H^1(\mathfrak{D}_1(\mathcal{O}_C(1)))$ is the tangent space to the deformation space of the pair $(C, \mathcal{O}_C(1))$ (see [AC]). This identification has the property that a section $\alpha \in H^0(\mathcal{O}_C(1))$ extends with the first-order deformation $D \in H^1(\mathfrak{D}_1(\mathcal{O}_C(1)))$ if and only if
$$D(\alpha) = 0 \in H^1(\mathcal{O}_C(1)).$$

Since the tangent space to $\mathbf{Q}'$ is characterized by the property that *all* sections in $H^0(\mathcal{O}_C(1))$ extend to first-order deformations along it, the canonically defined subspace
$$K_1 = \ker\big(H^1(\mathfrak{D}_1(\mathcal{O}_C(1))) \to \mathrm{Hom}(H^0(\mathcal{O}_C(1)), H^1(\mathcal{O}_C(1)))\big)$$
defines the tangent space to $\mathbf{Q}'$. Because $\mathcal{O}_C(1) = \omega_C$, this subspace maps isomorphically onto $H^1(\mathfrak{T}_C)$ via the symbol map in the short exact sequence

$$0 \to H^1(\mathcal{O}_C) \to H^1(\mathfrak{D}_1(\mathcal{O}_C(1))) \to H^1(\mathfrak{T}_C) \to 0.$$

This defines a splitting

$$H^1(\mathfrak{D}_1(\mathcal{O}_C(1))) = K_1 \oplus H^1(\mathcal{O}_C).$$

Hence (5) is given as the composition

$$H^0(N_{C\backslash\mathbb{P}^4}) \to H^1(\mathfrak{D}_1(\mathcal{O}_C(1))) \xrightarrow{\text{projection}} H^1(\mathcal{O}_C),$$

where the first map is induced from the commutative diagram of exact sequences

$$\begin{tikzcd}
\mathcal{O}_C \arrow[r] \arrow[d, equal] & \mathfrak{D}_1(\mathcal{O}_C(1)) \arrow[r] \arrow[d] & \mathfrak{T}_C \arrow[d] \\
\mathcal{O}_C \arrow[r] & \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_{\mathbb{P}^4}(1)) \arrow[r] \arrow[d] & \mathcal{O}_C \otimes \mathfrak{T}_{\mathbb{P}^4}. \arrow[d] \\
 & N_{C\backslash\mathbb{P}^4} \arrow[r, equal] & N_{C\backslash\mathbb{P}^4}
\end{tikzcd} \tag{6}$$

To see that (5) is of rank 1, note that it maps into the subspace of $H^1(\mathfrak{D}_1(\mathcal{O}_C(1)))$ on which the sections in W extend—that is, into the nullspace for the pairing

$$H^1(\mathfrak{D}_1(\mathcal{O}_C(1))) \otimes W \to H^1(\mathcal{O}_C(1)).$$

This imposes $\dim W = 4$ conditions on $H^1(\mathfrak{D}_1(\mathcal{O}_C(1))) = H^1(\mathfrak{D}_1(\omega_C))$, which restrict to independent conditions on its summand $H^1(\mathcal{O}_C)$. □

As Voisin pointed out, something of interest occurs along the 35-dimensional singular locus

$$\mathbf{R}' = \mathbf{P}' \cap \mathbf{Q}'. \tag{7}$$

2.4. *Equations for C in $\mathbb{P}^3$*

Suppose now that, for $\{C\} \in \mathbf{R}'$, $\hat{C}$ lies in the base locus of the system of quadrics in $\mathbb{P}^4$ generated by

$$c_j x_0^2 + l_j(x_1, \ldots, x_4)x_0 + q_j(x_1, \ldots, x_4), \quad j = 0, 1, 2, \tag{8}$$

with center of projection (1, 0, 0, 0, 0) not lying on the base locus. Thus C lies on the cubic surface A given by the equation

$$\begin{vmatrix} c_0 & l_0 & q_0 \\ c_1 & l_1 & q_1 \\ c_2 & l_2 & q_2 \end{vmatrix} \equiv 0 \tag{9}$$

in the $\mathbb{P}^3$ given by $x_0 = 0$. We normalize our basis so that $c_1 = c_2 = 0$ and $c_0 = 1$; then the cubic (9) becomes

$$l_1 q_2 - l_2 q_1 = 0. \tag{10}$$

LEMMA 2.8. *For all* $\{C\} \in \mathbf{R}'$, (10) *is the unique cubic in* $\mathbb{P}^3$ *containing* C.

Proof. If the base locus of the system (8) is a smooth cubic scroll $\hat{B}$, then (10) is the equation of the projection of that scroll into $\mathbb{P}^3$ and

$$q_1 = a_{11}l_1 + a_{12}l_2, \qquad q_2 = a_{21}l_1 + a_{22}l_2$$

for some linear forms $a_{jj'}$. Now $\hat{C} \to C$ is an isomorphism onto a curve of degree 8 and genus 5, and we claim that (9) is the unique cubic containing C. To see this, notice that the intersection of (9) with a second cubic would be of the form $C + L$, where L is a line meeting C in six points. The trigonal curve C is the image of a plane quintic with a node p under a map taking $\mathbb{P}^2$ blown up at p to the surface (9), and so L must be the double line of (9) or come from a line through p or from the exceptional curve. But none of those lines intersect C six times.

Also, since the center of projection does not lie on $\hat{B}$, the projection $\hat{B} \to A$ is a finite birational morphism that maps a conic R in $\hat{B}$ two-to-one onto the line $l_1 = l_2 = 0$ and has inverse

$$A \to \mathbb{P}^4,$$

$$(x_1, \dots, x_4) \mapsto (q_j, l_j x_1, \dots, l_j x_4).$$

The surface

$$l_j x_0 + q_j, \quad j = 1, 2$$

is the union of $\hat{B}$ and the plane containing the conic R.

Otherwise, (8) defines the canonical embedding $\hat{C}$ of C. Using the Einstein summation convention, $H^0(\mathfrak{I}_{C,\mathbb{P}^3}(3))$ is given by homogeneous forms $f^j q_j$ such that $g^j q_j + f^j l_j \equiv 0$. This can be solved for g^j such that

$$g^j c_j \equiv 0 \quad \text{and} \quad g^j l_j + f^j c_j \equiv 0.$$

In normal form, this is given by f^j such that

$$f^j l_j \equiv -g^1 q_1 - g^2 q_2,$$

$$f^0 \equiv -g^1 l_1 - g^2 l_2$$

for some constants g^1 and g^2. Hence

$$f^1 l_1 + f^2 l_2 \equiv g^1(l_0 l_1 - q_1) + g^2(l_0 l_2 - q_2).$$

But, in the non–cubic-scroll case, the existence of a common solution of

$$l_1 = l_2 = q_1 = q_2 = 0$$

implies that $\hat{C} \to C$ is not an isomorphism. □

By a somewhat more elaborate calculation we have the following.

LEMMA 2.9. *For all* $\{C\} \in \mathbf{R}'$,

$$h^1(\mathfrak{I}_{C,\mathbb{P}^3}(4)) = 0.$$

Proof. First of all,

$$\begin{aligned} h^1(\mathfrak{I}_{C,\mathbb{P}^3}(4)) &= -\chi(\mathfrak{I}_C(4)) + h^0(\mathfrak{I}_C(4)) + h^2(\mathfrak{I}_C(4)) \\ &= \chi(\mathcal{O}_C(4)) - \chi(\mathcal{O}_{\mathbb{P}^3}(4)) + h^0(\mathfrak{I}_C(4)) \\ &= 28 - 35 + h^0(\mathfrak{I}_{C,\mathbb{P}^3}(4)). \end{aligned}$$

So, to complete the proof we must show that

$$h^0(\mathfrak{I}_{C,\mathbb{P}^3}(4)) \le 7.$$

If $\hat{C}$ lies on a smooth cubic scroll $\hat{B}$, then the projection

$$\hat{B} \to A \subseteq \mathbb{P}^3$$

maps a conic R on $\hat{B}$ two-to-one onto a (double) line on A. Any element of $H^0(\mathfrak{I}_{C,\mathbb{P}^3}(4))$ must vanish on this line, since C meets it in five points. Thus

$$h^0(\mathfrak{I}_{C,\mathbb{P}^3}(4)) \le 4 + h^0(\mathfrak{I}_{\hat{C}+R,\hat{B}}(4)).$$

But

$$\hat{C} \equiv 3E + 5F \quad \text{and} \quad R \equiv E + F,$$

so that

$$\mathfrak{I}_{\hat{C}+R,\hat{B}}(4) \equiv (4E + 8F) - (4E + 6F) = 2F$$

and

$$h^0(\mathfrak{I}_{\hat{C}+R,\hat{B}}(4)) = 3.$$

In the non–cubic-scroll case, the fact that there is no common solution of

$$l_1 = l_2 = q_1 = q_2 = 0$$

implies that A is smooth along the line

$$l_1 = l_2 = 0.$$

Let $\hat{A}$ denote the surface in $\mathbb{P}^4$ given by

$$l_j(x_1, \ldots, x_4)x_0 + q_j(x_1, \ldots, x_4), \quad j = 1, 2.$$

Then $\hat{A}$ has a distinguished point $p = (1, 0, \ldots, 0)$, which is a smooth point of $\hat{A}$ since C does not lie in a quadric. Then

$$h^0(\mathfrak{I}_{C,\mathbb{P}^3}(4)) \le 4 + h^0(\mathfrak{m}^4_{p,\hat{A}}(2)).$$

Since

$$H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^0\left(\frac{\mathcal{O}_{\mathbb{P}^4}(2)}{\mathfrak{m}^4_{p,\mathbb{P}^4}(2)}\right)$$

is surjective, we conclude that

$$h^0(\mathfrak{m}^4_{p,\hat{A}}(2)) = 15 - 2 - (1 + 2 + 3 + 4) = 3.$$

Thus, again we have

$$h^0(\mathfrak{I}_{C,\mathbb{P}^3}(4)) \le 7. \qquad \square$$

LEMMA 2.10. *There is a Zariski-open neighborhood* $\mathbf{U}'$ *of* $\mathbf{R}'$ *in* $\mathbf{P}'$ *such that, for any scheme* $T' \subseteq \mathbf{U}'$ *for which there is a cubic surface* $A_{T'}$ *containing* $C_{T'}$*, we have*

$$T' \subseteq \mathbf{R}'.$$

Proof. For $\{C\} \in \mathbf{R}'$ with C not trigonal (i.e., $\hat{C}$ does not lie on a cubic scroll), C lies in a cubic A given by an equation

$$l_1 q_2 - l_2 q_1 = 0$$

such that there is no common solution of

$$l_1 = l_2 = q_1 = q_2 = 0.$$

Thus the map

$$\begin{aligned} A &\to \mathbb{P}^4, \\ (x_1, \ldots, x_4) &\mapsto (q_j, l_j x_1, \ldots, l_j x_4) \end{aligned} \tag{11}$$

is everywhere well-defined. In fact, the morphism (11) has image $\hat{A}$ and is an isomorphism except that it blows down the line $l_1 = l_2 = 0$ to the distinguished point p. Since p does not lie on $\hat{C}$, (11) restricts to an isomorphism on C. By Lemma 2.8, any sufficiently small deformation of $\{C\}$ in $\mathbf{P}'$ that lies in a cubic surface gives a deformation of the l_j and q_j and hence a deformation of the map (11) and thus a deformation of $\hat{C} \subseteq \mathbb{P}^4$ that projects to the deformation of C. Since $\hat{C}$ is a canonical curve, we conclude that

$$\{C\} \in \mathbf{R}'.$$

If $\hat{C}$ lies on the cubic scroll $\hat{B}$, then the projection map

$$\hat{B} \to A$$

is finite and is an isomorphism except over $l_1 = l_2 = 0$, over which a conic R in $\hat{B}$ is mapped two-to-one. This map also restricts to an isomorphism on $\hat{C}$. The map (11) is given by

$$(x_1, \ldots, x_4) \mapsto \left(a_{jj} + a_{j(j\pm 1)} \frac{l_{j\pm 1}}{l_j}, x_1, \ldots, x_4 \right).$$

By Lemma 2.8, any deformation $C_{T'} \subseteq A_{T'}$ gives deformations $l_{j,T'}, q_{j,T'}$ for $j = 1, 2$ and so yields a rational map

$$A_{T'} \to \hat{A}_{T'}$$

restricting to an isomorphism on $C_{T'} \to \hat{C}_{T'}$ whose inverse is a family of projections. Thus $T' \subseteq \mathbf{R}'$. □

Lemma 2.10 immediately implies the following improvement on Proposition 2.7.

COROLLARY 2.11. $\mathbf{P}'$ *and* $\mathbf{Q}'$ *meet transversely at all points of* $\mathbf{R}'$.

Later we will also need the following result.

LEMMA 2.12. (i) *For* $\{C\} \in \mathbf{R}'$ *such that* C *is not trigonal,* C *is a Cartier divisor of the cubic* A *and*

$$N_{C\backslash A}(-1) = \omega_C.$$

(ii) *For* $\{C\} \in \mathbf{R}'$ *such that* C *is trigonal,*

$$h^1(N_{C\backslash \mathbb{P}^3}(-1)) = 0.$$

Proof. (i) Again use that

$$l_1 = l_2 = q_1 = q_2 = 0$$

has no common solutions. The morphism (11) has image $\hat{A}$ and is an isomorphism except that it blows down the line $l_1 = l_2 = 0$ to the distinguished point p. Since p does not lie on $\hat{C}$, we have

$$N_{C\backslash A} = N_{\hat{C}\backslash \hat{A}} = \mathcal{O}_C(2).$$

(ii) Consider the finite map $f\colon \hat{B} \to \mathbb{P}^3$ given by the restriction of the projection to the cubic scroll. We then have the exact sequence

$$0 \to N_{\hat{C}\backslash \hat{B}} \to N_{C\backslash \mathbb{P}^3} \to \mathcal{O}_{\hat{C}} \otimes N_f \to 0,$$

where

$$N_f = \frac{f^*\mathfrak{T}_{\mathbb{P}^3}}{\mathfrak{T}_{\hat{B}}}.$$

Now referring to (2), it follows that

$$\deg N_{\hat{C}\backslash \hat{B}} = (3E + 5F)^2 = 21$$

so that $h^1(N_{\hat{C}\backslash \hat{B}}(-1)) = 0$. Since $\deg N_{C\backslash \mathbb{P}^3} = 40$, we also have

$$\deg(\mathcal{O}_{\hat{C}} \otimes N_f) = 19.$$

Also, the natural injection of line bundles

$$\mathcal{O}_{\hat{C}} \otimes f^* N^\vee_{A\backslash \mathbb{P}^3} \to \mathcal{O}_{\hat{C}} \otimes N_f^\vee$$

has cokernel a skyscraper sheaf of length 5 corresponding to the five points of intersection of R and $\hat{C}$. Since f is of maximal rank except at two points, the torsion summand of $\mathcal{O}_{\hat{C}} \otimes N_f$ has length ≤ 2 and therefore its line bundle summand has degree ≥ 17, so that

$$h^1(\mathcal{O}_{\hat{C}} \otimes N_f(-1)) = 0. \qquad \square$$

3. The Surface

3.1. Quintics Containing C

Now fix a smooth curve $C \subseteq \mathbb{P}^3 \subseteq \mathbb{P}^4$ representing a general point in the set $\mathbf{R}'$ of (7). By a fundamental theorem of Gruson–Lazarsfeld–Peskine [GLP], for any nonrational reduced irreducible curve C of degree 8 spanning $\mathbb{P}^3$, the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathcal{O}_C(5)) \tag{12}$$

is surjective. This in turn implies that

$$\dim H^0(\mathfrak{I}_C(5)) = 20, \tag{13}$$

where $\mathfrak{I}_C$ is the ideal sheaf of C in $\mathbb{P}^3$.

PROPOSITION 3.1. *C is contained in a smooth quintic surface S whose Picard number is* 2.

Proof. The proof is based on the following construction.

As before, let $\hat{C}$ denote the canonical embedding of C in $\mathbb{P}^4$. Take a 1-parameter family of projections $\pi_t : \hat{C} \subseteq \mathbb{P}^4 \to \mathbb{P}^3$ with $C = \pi_1(\hat{C})$ and such that the center of the projection π_0 is the vertex of a quadric cone containing $\hat{C}$. Then $(C_0)_{\text{red}} = \pi_0(\hat{C})$ is a four-nodal curve, which is a (4, 4)-curve on a smooth quadric surface Q_0. We write

$$(C_0)_{\text{red}} = Q_0 \cap V_0$$

for some quartic surface V_0 in $\mathbb{P}^3$. Thus

$$h^0(\mathfrak{I}_{(C_0)_{\text{red}}}(5)) = h^0(\mathcal{O}_{\mathbb{P}^3}(3)) + h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 24. \tag{14}$$

The four nodes $\{p_1, \dots, p_4\}$ are coplanar because the canonical linear system on $\hat{C}$ is given by quadrics in $\mathbb{P}^3$ containing the p_i; hence, if $\hat{p}_{ij}$ denotes the two points of $\hat{C}$ above p_i for $j = 1, 2$,

$$\mathcal{O}_{\hat{C}}\Big(\sum \hat{p}_{ij}\Big) = \omega_{\hat{C}}.$$

However, no three points of $\{p_1, \dots, p_4\}$ are collinear, since $\hat{C}$ is not hyperelliptic. Thus the p_i impose independent conditions on cubics in $\mathbb{P}^3$. So, if we denote by C_0 the curve over 0 in the flat family associated to $\pi_t(\hat{C})$, then C_0 has embedded points at the p_i and all members of the linear system

$$\Lambda_0 = |\mathfrak{I}_{C_0}(5)| \subseteq |\mathcal{O}_{\mathbb{P}^3}(5)|$$

are singular at $\{p_1, \dots, p_4\}$; by (14), it follows that

$$h_0(\mathfrak{I}_{C_0}(5)) = 24 - 4 = 20.$$

Now a general member of Λ_0 is a general member of a general pencil spanned by the pair of reducible surfaces

$$Q_0 \cup K_0, \qquad V_0 \cup H_0,$$

where K_0 is a general cubic containing the p_i and H_0 is a hyperplane through those points p_i at which V_0 is smooth. Since $\dim \Lambda_0 = 19$, the only possibility is therefore that H_0 is a general hyperplane and V_0 is singular at all four points p_i. In fact, the tangent cone of a general element of Λ_0 at p_i is a general element of the pencil of conics spanned by two conics of the form

$$xl(x, y, z), \qquad yz + xm_0(x, y, z),$$

where l is a general and m_0 is a fixed (unknown) linear form. Thus, the general quintic surface containing C_0 has only ordinary nodes at each p_i. We also see immediately that $\mathfrak{I}_{C_0}(5)$ is generated by its global sections.

We are now ready to complete the proof of the Proposition. Let

$$\Lambda_t = |\mathfrak{I}_{C_t}(5)| \subseteq |\mathcal{O}_{\mathbb{P}^3}(5)|.$$

Take a generic section of the family $\{\Lambda_t\}$ near $t = 0$ and call the family of quintics $\{S_t\}$. By flatness, $\mathfrak{I}_{C_t}(5)$ is generated by global sections and so, by Bertini's theorem, S_t is smooth for $t \neq 0$. In fact, if $\tilde{\mathbb{P}}_t$ denotes the blow-up of $\mathbb{P}^3$ along C_t then the classical Lefschetz argument for hyperplane sections of the embedding

$$\tilde{\mathbb{P}}_t \to \mathbb{P}\big(H^0(\mathfrak{I}_{C_t}(5))^\vee\big)$$

shows that the Picard number of the generic S_t is equal to the Picard number of $\tilde{\mathbb{P}}_t$, which is 2. □

3.2. *C Moves in a Pencil on S*

We now have, for generic $\{C\} \in \mathbf{R}'$, that C lies in a smooth quintic surface S of Picard number 2. By adjunction,

$$N_{C\backslash S} = \mathcal{O}_C \tag{15}$$

so that, since S is regular,

$$h^0(\mathcal{O}_S(C)) = 2. \tag{16}$$

Let

$$p\colon S \to S' := \mathbb{P}\big(H^0(\mathcal{O}_S(C))^\vee\big)$$

be the corresponding pencil.

Using Riemann–Roch, Serre duality, standard exact sequences, and the fact that for no $\{C'\} \in |\mathcal{O}_S(C)|$ does C' lie in a hyperplane, one calculates

$$h^1(\mathcal{O}_S(C)) = 1, \quad h^2(\mathcal{O}_S(C)) = 0; \tag{17}$$

$$h^0(\mathcal{O}_S(2C)) = 3, \quad h^1(\mathcal{O}_S(2C)) = 6, \quad h^2(\mathcal{O}_S(2C)) = 0. \tag{18}$$

We want the surjectivity of (12) for all fibers of p, so we need the following lemma.

LEMMA 3.2. *All fibers of p are reduced and irreducible.*

Proof. Suppose not; that is, suppose that there exist

$$D_1 + D_2 \in |\mathcal{O}_S(C)|$$

with D_1 reduced and irreducible and that

$$1 \le k := D_1 \cdot K \le 4,$$

where K is a hyperplane section of S. Since C moves in a basepoint-free pencil, it must be that

$$C \cdot D_1 = 0$$

as well. One calculates that the intersection matrix of the divisors C_0, D_1, and K has determinant $-64D_1^2$ and so, since the Picard number of S is 2,

$$D_1^2 = 0.$$

Therefore, by the genus formula,

$$2g(D_1) - 2 = D_1^2 + D_1 \cdot K = k.$$

Now $k = 2$ is impossible because there are no curves of genus 2 and degree 2, so it must be that

$$k = 4 \quad \text{and} \quad g := g(D_1) = 3;$$

in other words, D_1 is a plane quartic. It follows that

$$K \equiv D_1 + L$$

for some line L in S. Plane quartics are canonically embedded, so $h^1(\mathcal{O}_{D_1}(K)) = 1$. Thus, using the exact sequence

$$0 \to \mathcal{O}_S(L) \to \mathcal{O}_S(K) \to \mathcal{O}_{D_1}(K) \to 0,$$

we conclude via Serre duality that

$$h^0(\mathcal{O}_S(D_1)) = h^2(\mathcal{O}_S(L)) = 2.$$

But, since $h^0(\mathcal{O}_S(C_0)) = 2$, this would imply that all curves $C' \in |\mathcal{O}_S(C)|$ are of the form

$$C' = D + D_2 \quad \text{with } \ D \equiv D_1,$$

which contradicts the irreducibility of C itself. □

By Proposition 2.4 we thus have the following corollary.

COROLLARY 3.3. *If $S \supseteq C$ is generic, then*

$$h^1(N_{D\backslash\mathbb{P}^3}) = 0$$

for every fiber $D \in |\mathcal{O}_S(C)|$.

Because the mapping of ideal sheaves

$$\mathfrak{I}_{D\backslash\mathbb{P}^3}/\mathfrak{I}^2_{D\backslash\mathbb{P}^3} \to \mathfrak{I}_{D\backslash S}/\mathfrak{I}^2_{D\backslash S}$$

is surjective, the mapping

$$T_{\{D\}}S' = \mathrm{Hom}(\mathfrak{I}_{D\backslash S}/\mathfrak{I}^2_{D\backslash S}, \mathcal{O}_D) \to \mathrm{Hom}(\mathfrak{I}_{D\backslash\mathbb{P}^3}/\mathfrak{I}^2_{D\backslash\mathbb{P}^3}, \mathcal{O}_D) \subseteq T_{\{D\}}\mathbf{R}'$$

is injective. We therefore conclude as follows.

PROPOSITION 3.4. *S' is a smooth subscheme of $\mathbf{R}'$.*

Finally, for fixed $\mathbb{P}^3$, we need to understand the local structure of

$$\mathbf{N}' := \left\{(C, S) \in \mathrm{Hilb}^{\mathbb{P}^3} \times \mathbb{P}\big(H^0(\mathcal{O}_S(5))\big) : C \subseteq S\right\},$$
$$\mathbf{N}'' := \{(C, S) \in \mathbf{N}' : \mathcal{O}_C(1) = \omega_C\}$$

near S'. Let

$$\mathbf{N}_2 := \left\{ S_1 \in \mathbb{P}\big(H^0(\mathcal{O}_S(5))\big) : \rho(S_1) = 2 \right\}$$

be the subscheme of quintic surfaces near S with Picard number 2, and let

$$\mathbf{N}_{2,\omega} := \mathrm{image}(\mathbf{N}'' \to \mathbf{N}_2).$$

PROPOSITION 3.5. *Suppose that the infinitesimal period mapping*

$$H^0(\mathcal{O}_S(5)) \to \mathrm{Hom}(H^0(\omega_S), H^1(\omega_C))$$

is surjective at (C, S). *Then* $\mathbf{N}_2$ *is a smooth submanifold of* $\mathbb{P}\big(H^0(\mathcal{O}_S(5))\big)$ *of codimension* 4, *and* $\mathbf{N}'$ *is the blow-up of* $\mathbf{N}_2$ *along the smooth codimension-*2 *submanifold* $\mathbf{N}_{2,\omega}$ *with exceptional locus* $\mathbf{N}''$.

Proof. The first assertion is an immediate consequence of the classical Lefshetz theorem. For the second, first notice that, for S_1 nearby with Picard number 2 and for the line bundle L that is the deformation of $c_1(\mathcal{O}_S(C))$, we have $\chi(L) = 1$ and, by (17), $h^2(L) = 0$; hence L has a section. Let C_1 denote its zero scheme. Then, by adjunction,

$$N_{C_1 \backslash S_1} = \omega_{S_1} \otimes \omega_{C_1}^{-1},$$

so that

$$h^0(L) > 1 \iff \mathcal{O}_{C_1} \otimes \omega_{S_1} = \omega_{C_1}.$$

Thus we locally have a birational morphism $\mathbf{N}' \to \mathbf{N}_2$ with fibers isomorphic to $\mathbb{P}^1$. A classical theorem of Moishezon characterizing smooth blow-ups then completes the proof. □

4. The Threefold

4.1. Quintics Containing S

For generic $C \in \mathbf{R}'$ and generic $S \supseteq C$ as in the previous section, we next let

$$X \supseteq S \supseteq C$$

be a generic quintic threefold $X \subseteq \mathbb{P}^4$ with hyperplane section S. Then X is given by an equation

$$F(x_0, \ldots, x_4) = G(x_1, \ldots, x_4) + x_0 \hat{V}(x_0, \ldots, x_4) = 0, \tag{19}$$

where S is given by $G = 0$. We wish to study Hilb^X in a neighborhood of S', the parameter space of the basepoint-free pencil S/S' with generic fiber C. We will break this study into two parts: the studies of $\mathrm{Hilb}^X \cap \mathbf{P}'$ and $\mathrm{Hilb}^X \cap \mathbf{Q}'$. In the next few subsections we will concentrate on $\mathrm{Hilb}^X \cap \mathbf{P}'$. Our main tool will be the identification of two symmetric bilinear forms on

$$H^0(\mathcal{O}_{\mathbb{P}^3}(1)) = H^0(\Omega_S^2) \subseteq H^0(\Omega_C^1).$$

For $l, l' \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$, the first form is

$$\langle l, l'\rangle := \int_C \tau'(\omega), \tag{20}$$

where $\omega \in H^0(\Omega_S^2)$ is the element given by $l \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ and where $\tau' \in H^1(\mathfrak{T}_S)$ is the Kodaira–Spencer class for the deformation of S in X given by

$$x_0 - \varepsilon l' = 0.$$

The second form is

$$\int_C \omega \cdot \theta\omega', \tag{21}$$

where $\omega, \omega' \in H^0(\Omega_C^1)$ are the elements given by l and l' (respectively) and

$$\theta \in \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C)$$

is the obstruction to splitting the sequence

$$0 \to N_{C\backslash S} \to N_{C\backslash X} \to N_{S\backslash X}\big|_C \to 0.$$

The form (21) of the pairing will later be shown to be nondegenerate. Hence this identification will allow us to apply Proposition 3.5 to show that our special surfaces S vary in a (generically) smooth set of codimension 6 in $\mathbb{P}\big(H^0(\mathcal{O}_{\mathbb{P}^3}(5))\big)$. So the set of $X \subseteq \mathbb{P}^4$ with a hyperplane section isomorphic to some S as before will turn out to be smooth of codimension $6-4=2$ in $\mathbb{P}\big(H^0(\mathcal{O}_{\mathbb{P}^4}(5))\big)$.

We begin by checking the coincidence of the two pairings (20) and (21). The pencil of hyperplane sections of X given by $x_0 - \varepsilon x_j$ has derivative

$$x_j V(x_1, \ldots, x_4) \tag{22}$$

at $\varepsilon = 0$, where

$$V(x_1, \ldots, x_4) := \hat{V}(0, x_1, \ldots, x_4).$$

If

$$\Omega := \left\langle \sum_{j=0}^{4} x_j \frac{\partial}{\partial x_j} \;\middle|\; dx_1 \ldots dx_4 \right\rangle$$

then the holomorphic 2-forms on S are given by

$$\mathrm{res}\, \frac{l(x_1, \ldots, x_4)\Omega}{G},$$

so that the obstruction to extending $c_1(C)$ to first order with a first-order deformation $l'V$ of S in X is given by the quadratic form

$$\langle l, l'\rangle = \int_C \mathrm{res}\, \frac{ll'V\Omega}{G^2} \tag{23}$$

on $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$.

To see that (20) and (21) coincide, we compute as in [G]. Consider the exact sequences

$$0 \to \hat{\Omega}^2_{\mathbb{P}^3}(S) \to \Omega^2_{\mathbb{P}^3}(S) \xrightarrow{d} \Omega^3_{\mathbb{P}^3}(2S) \to 0$$

and

$$0 \to N_{C\backslash S} \to N_{C\backslash \mathbb{P}^3} \to \mathcal{O}_C \otimes N_{S\backslash \mathbb{P}^3} \to 0$$

giving rise to the commutative diagram

$$\begin{array}{ccccc} H^0(\Omega^3_{\mathbb{P}^3}(2S)) & \xrightarrow{\text{res}} & H^0(\Omega^2_S \otimes N_{S\backslash\mathbb{P}^3}) & \longrightarrow & H^1(\Omega^2_S \otimes N_{C\backslash S}) \\ \downarrow & & \downarrow & & \|\wr \\ H^1(\hat{\Omega}^2_{\mathbb{P}^3}(S)) & \xrightarrow{\text{res}} & H^1(\Omega^1_S) & \longrightarrow & H^1(\Omega^1_C)\,. \end{array}$$

So the pairing (23) is given by

$$\begin{array}{c} H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \\ \Big\downarrow{\scriptstyle 1\otimes V} \\ H^0(\Omega^2_S) \otimes H^0(N_{S\backslash\mathbb{P}^3}) \\ \Big\downarrow \\ H^0(\Omega^2_S) \otimes H^1(N_{C\backslash S}) \\ \Big\downarrow \\ H^1(\omega_C) \end{array} \tag{24}$$

and so coincides with (21).

One final remark (that will not be used in what follows): One might envision a parallel study of the infinitesimal period map for the quadric sections of X at the complete quadric section $2\mathbb{P}^3_{Q_0} \cap X$, that is, the double cover of S branched along

$$S \cap \{Q_0 = 0\} \subseteq \mathbb{P}^3.$$

More precisely, recall the space $\mathbf{Q}''$ of pairs (curve, complete quadric) defined in Section 2.3. Let

$$\begin{aligned} F(x_0, \ldots, x_4) = G(x_1, \ldots, x_4) & \\ + x_0 V(x_1, \ldots, x_4) + x_0^2 K(x_1, \ldots, x_4) + x_0^3 \cdot \ldots, & \end{aligned} \tag{25}$$

and let

$$x_0^2 - \varepsilon(x_0 L_0(x_1, \ldots, x_4) + Q_0(x_1, \ldots, x_4))$$

be a path in $\mathbf{Q}''$ passing through $(C, 2\mathbb{P}^3_{Q_0})$. Taking resultants, we see that the derivative of the corresponding quadric section of X at $\varepsilon = 0$ projects into $\mathbb{P}^3$ with equation

$$Q_0(V^2 - 2GK) - L_0 GV = 0. \tag{26}$$

However, this infinitesimal period map seems harder to deal with, so we will handle the neighborhood of S' in $\mathbf{Q}'$ differently.

4.2. *Characterizing $N_{C\backslash X}$ as an Element of* $\mathrm{Ext}^1(\omega_C, \mathcal{O}_C)$

Again, for $S \subseteq X$ generic and any C in the distinguished pencil parameterized by S', we consider the short exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_{C\setminus S} & \longrightarrow & N_{C\setminus X} & \longrightarrow & \mathcal{O}_C \otimes N_{S/X} & \longrightarrow & 0 \\
 & & \| \wr & & & & \| \wr & & \\
 & & \mathcal{O}_C & & & & \omega_C & &
\end{array}
\tag{27}
$$

of normal bundles. This sequence is characterized by an element of

$$\mathrm{Ext}^1(\omega_C, \mathcal{O}_C) = H^1(\mathfrak{T}_C),$$

which we express in terms of the equation (25) defining X. From the Euler exact sequences

$$0 \to \mathcal{O}_C \to \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_{\mathbb{P}^n}(1)) \to \mathcal{O}_C \otimes \mathfrak{T}_{\mathbb{P}^n} \to 0$$

for $n = 3, 4$, we obtain two commutative diagrams of exact sequences of left $\mathcal{O}_C$-modules,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_S(1)) & \longrightarrow & \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_X(1)) & \longrightarrow & \mathcal{O}_C(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_{\mathbb{P}^3}(1)) & \longrightarrow & \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_{\mathbb{P}^4}(1)) & \longrightarrow & \mathcal{O}_C(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_4} & & & & \\
 & & \mathcal{O}_C \otimes N_{S\setminus\mathbb{P}^3} & = & \mathcal{O}_C \otimes N_{X\setminus\mathbb{P}^4} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{28}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_S(1)) & \longrightarrow & \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_X(1)) & \longrightarrow & \mathcal{O}_C(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N_{C\setminus S} & \longrightarrow & N_{C\setminus X} & \longrightarrow & \mathcal{O}_C(1) & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{29}
$$

Referring to (28), the vertical maps ψ_n are given by

$$\psi_3\left(\alpha_1\frac{\partial}{\partial x_1} + \cdots + \alpha_4\frac{\partial}{\partial x_4}\right) = \alpha_1\frac{\partial G}{\partial x_1} + \cdots + \alpha_4\frac{\partial G}{\partial x_4},$$

$$\psi_4\left(\alpha_0\frac{\partial}{\partial x_0} + \alpha_1\frac{\partial}{\partial x_1} + \cdots + \alpha_4\frac{\partial}{\partial x_4}\right) = \alpha_0 V + \alpha_1\frac{\partial G}{\partial x_1} + \cdots + \alpha_4\frac{\partial G}{\partial x_4}.$$

A simple diagram chase, together with the fact that the exact sequence given by the middle row of (28) is split, yields the following.

PROPOSITION 4.1. *The extension class giving* (27) *is the image of* V *under the composition*

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(5)) &\to \mathrm{Ext}^1(\mathcal{O}_C(1), \mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_S(1))) \\ &\to \mathrm{Ext}^1(\mathcal{O}_C(1), N_{C\backslash S}) \end{aligned} \tag{30}$$

induced by the diagrams (28) *and* (29). (*Of course,* Ext^1 *here refers to extensions with respect to the left* $\mathcal{O}_C$*-module structure on* $\mathcal{O}_C \otimes \mathfrak{D}_1(\mathcal{O}_S(1))$.)

We next wish to explore which elements of $\mathrm{Ext}^1(\mathcal{O}_C(1), N_{C\backslash S}) = \mathrm{Ext}^1(\omega_C, \mathcal{O}_C) = H^1(\mathfrak{T}_C)$ occur as we vary $V \in H^0(\mathcal{O}_{\mathbb{P}^3}(4))$.

LEMMA 4.2. *The dimension of the cokernel of* (30) *is* ≤ 1 *and, for generic* $\{C\} \in S'$, *the mapping* (30) *is surjective.*

Proof. We factor the mapping (30) as follows:

$$H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_C(4)) \tag{31}$$

gives

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(5)) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_C(5))$$

followed by

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_C(5)) \to \mathrm{Ext}^1(\mathcal{O}_C(1), N_{C\backslash S}) \tag{32}$$

coming from the short exact sequence

$$\begin{aligned}0 \to \mathcal{H}om(\mathcal{O}_C(1), N_{C\backslash S}) &\to \mathcal{H}om(\mathcal{O}_C(1), N_{C\backslash \mathbb{P}^3}) \\ &\to \mathcal{H}om(\mathcal{O}_C(1), \mathcal{O}_C \otimes N_{S\backslash \mathbb{P}^3}) \to 0.\end{aligned}$$

By Lemma 2.9, the map (31) is surjective.

The cokernel of (32) is $H^1(N_{C\backslash \mathbb{P}^3}(-1))$. Recall that C lies on an irreducible cubic surface $A \subseteq \mathbb{P}^3$ given by (10). If C is nontrigonal then the exact sequence

$$0 \to N_{C\backslash A}(-1) \to N_{C\backslash \mathbb{P}^3}(-1) \to \mathcal{O}_C(2) \to 0$$

in Lemma 2.12 gives

$$h^1(N_{C\backslash \mathbb{P}^3}(-1)) \leq h^1(N_{C\backslash A}(-1)).$$

Yet by the same lemma we have

$$N_{C\backslash A}(-1) = \omega_C.$$

If C is trigonal, then Lemma 2.12 yields

$$h^1(N_{C\backslash \mathbb{P}^3}(-1)) = 0.$$

Since the generic S' must contain some trigonal curves, the proof is complete. □

COROLLARY 4.3. (i) *For generic* $V \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(5)) = H^0(\mathcal{O}_{\mathbb{P}^3}(4))$, *the pairing* (23) *is of maximal rank*.

(ii) *There is a Zariski open set* $\mathbf{Z}' \subseteq \mathbf{P}'$ *such that* $S' \subseteq \mathbf{Z}'$, *and we have a scheme-theoretic equality*

$$S' = \mathrm{Hilb}^X \cap \mathbf{Z}'.$$

Proof. (i) The image

$$\mathrm{image}\big(H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^1(\mathfrak{T}_C)\big)$$

of the map (30) in Proposition 4.1 has codimension ≤ 1. However, by (24) the pairing (23) is given by the map

$$\begin{array}{c} H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \\ \big\downarrow{\scriptstyle \text{inclusion}\otimes(30)} \\ H^0(\omega_C) \otimes H^0(\omega_C) \otimes H^1(\mathfrak{T}_C) \\ \big\downarrow \\ H^0(\omega_C^2) \otimes H^1(\mathfrak{T}_C) \\ \big\downarrow \\ H^1(\omega_C)\,. \end{array}$$

Since no quadric in $\mathbb{P}^3$ contains C, the mapping

$$\mathrm{Sym}^2 H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\omega_C^2)$$

is injective and so

$$H^1(\mathfrak{T}_C) \to \mathrm{Sym}^2 H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$$

takes the image of $H^0(\mathcal{O}_{\mathbb{P}^3}(4))$ onto a subspace of codimension ≤ 1. However, the degenerate quadratic forms in $\mathrm{Sym}^2 H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$ are an irreducible quartic hypersurface which therefore cannot contain a hyperplane.

(ii) Now the cohomological obstruction to deforming $c_1(C) \in H^{1,1}(S)$ to first order with a deformation of the hyperplane section S in X is exactly the pairing (23) considered as a map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to \mathrm{Hom}(H^0(\omega_S), H^1(\omega_C)).$$

Hence, for generic V, this map is injective. So $c_1(C)$ is obstructed to first order for every first-order deformation of S in X. So the only first-orders deformations of curves on S' staying inside $\mathrm{Hilb}^X \cap \mathbf{P}'$ stay inside S. □

Our next step is to recall the analysis in [B] of the structure of the space

$$\mathbb{P}(H^1(\mathfrak{T}_C)) = \mathbb{P}(H^0(\omega_C^2)^\vee)$$

of vector bundle extensions of ω_C by $\mathcal{O}_C$. For any effective divisor D on C, we let

$$k(D) = \ker\big(H^1(\mathfrak{T}_C) \to H^1(\mathfrak{T}_C(D))\big).$$

The subspace

$$\mathbb{P}(k(D)) \subseteq \mathbb{P}(H^1(\mathfrak{T}_C))$$

is defined by equations spanning the subspace $H^0(\omega_C^2(-D))$ of $H^0(\omega_C^2)$.

PROPOSITION 4.4. *An extension*

$$0 \to \mathcal{O}_C \to N \to \omega_C \to 0$$

lies in $\mathbb{P}(k(D))$ *if and only if* N *has a quotient of the form* $\omega_C(-D)$.

Proof. Consider the bundle N' containing N that is generated by N and rational sections s of $\mathcal{O}_C$ with $\operatorname{div}(s) + D \geq 0$. The extension

$$0 \to \mathcal{O}_C(D) \to N' \to \omega_C \to 0$$

is the image of the extension in the proposition under the map

$$H^1(\mathfrak{T}_C) \to H^1(\mathfrak{T}_C(D)).$$

The image extension is split if and only if there is a map

$$\omega_C(-D) \to N \subseteq N'$$

for which the composition

$$\omega_C(-D) \to N \to \omega_C$$

has cokernel $\mathbb{C}D$. □

PROPOSITION 4.5.

$$h^0(N) > 1$$

if and only if there exists a $D > 0$ *such that* N *has a subbundle* $\omega_C(-D)$ *with* $h^0(\omega_C(-D)) > 0$.

Proof. Let t denote a nontrivial section of $\mathcal{O}_C \subseteq N$. If $h^0(N) > 1$ there is a section s of N whose saturation $L = \mathcal{O}_C(D)$ is a sub–line bundle L of N such that the natural map

$$\mathcal{O}_C + L \to N$$

is generically surjective. (Here D is given by the zeros of s.) Since N is nonsplit, the cokernel of this map is a skyscraper sheaf of positive degree and so, replacing s by a linear combination of s and t if necessary, we can assume that s has a zero and so $\deg L > 0$. The nontrivial section of $\mathcal{O}_C$ gives a nontrivial section of the quotient line bundle N/L whose divisor D' is such that $D + D'$ is a canonical divisor. Conversely, if

$$N \to \omega_C(-D)$$

then the kernel is $\mathcal{O}_C(D)$ and so has a section. □

The last two propositions tell us that the subvariety

$$B = \{N \in \mathbb{P}(H^1(\mathfrak{T}_C)) : h^0(N) > 1\}$$

is given as follows. Let

$$b\colon C \to \mathbb{P}(H^0(\omega_C^2)^\vee) = \mathbb{P}(H^1(\mathfrak{T}_C)) = \mathbb{P}^{11}$$

denote the bi-canonical embedding. For each proper subset D of a canonical divisor of C, $k(D)$ defined as before is the linear span of the set $b(D)$, and B is simply the union of the $k(D)$ for all D such that there exists a $D' > 0$ with $D + D' \in |\omega_C|$. Thus

$$\dim B \leq 4 + 6 = 10.$$

Hence the generic $N \in \mathbb{P}(H^1(T_C))$ has $h^0(N) = 1$ and so, by Lemma 4.2, we conclude as follows.

PROPOSITION 4.6. *For $C \subseteq S \subseteq X$ generic, the standard inclusion*

$$H^0(N_{C\backslash S}) \to H^0(N_{C\backslash X})$$

is an isomorphism of 1*-dimensional vector spaces.*

4.3. Hilb^X *at Generic C*

We now consider the diagram

$$\begin{array}{ccc} S' \times X & \xrightarrow{q} & X \\ \big\downarrow{\scriptstyle p} & & \\ S' & & \end{array}$$

and let $\Delta \subseteq S' \times X$ denote the incidence variety of the family of curves S' on X. Notice that q maps Δ isomorphically onto $S \subseteq X$ and, under that identification, $p\colon \Delta \to S'$ is simply the fibration S/S'. From the short exact sequence

$$0 \to N_{\Delta\backslash S'\times S} \to N_{\Delta\backslash S'\times X} \to q^*N_{S\backslash X} \to 0 \tag{33}$$

we have the map

$$\mu_V\colon p_*q^*N_{S\backslash X} \to R^1p_*N_{\Delta\backslash S'\times S} \tag{34}$$

of bundles, which is given at a point $\{C\} \in S'$ as the map

$$H^0(\mathcal{O}_C \otimes N_{S\backslash X}) \to H^1(N_{C\backslash S}),$$

that is, the map

$$H^0(\omega_C) \to H^1(\mathcal{O}_C)$$

given by the extension data of the exact sequence

$$0 \to \mathcal{O}_C \to N_{C\backslash X} \to \omega_C \to 0. \tag{35}$$

LEMMA 4.7. *For generic $S \subseteq X$, the following statements hold.*

(i) Hilb^X *is given at generic $\{C\}$ by the reduced scheme S'.*

(ii) *The Fitting scheme given by the locus at which μ is not of maximal rank has length* 2*; in fact, it has length* 1 *at two distinct points of S' that can be taken to be outside any pre-given finite subset of S'.*

Proof. (i) We have by Proposition 4.6 that μ can be made to be fiberwise injective at any $\{C\} \in S'$. Thus $H^0(N_{C\setminus S}) \to H^0(N_{C\setminus X})$ is an isomorphism.

(ii) By Porteous's formula, the desired length is given by

$$c_1(R^1 p_* N_{\Delta\setminus S'\times S}) - c_1(p_* q^* N_{S\setminus X}).$$

Now

$$N_{\Delta\setminus S'\times S} = p^* N_{\Delta'\setminus S'\times S'} = p^* \mathcal{O}_{S'}(2),$$

so that

$$p_* N_{\Delta\setminus S'\times S} = \mathcal{O}_{S'}(2) \otimes p_* \mathcal{O}_S,$$

$$R^1 p_* N_{\Delta\setminus S'\times S} = \mathcal{O}_{S'}(2) \otimes R^1 p_* \mathcal{O}_S.$$

Also,

$$q^* N_{S\setminus X} = \omega_S = \omega_{S/S'} \otimes p^* \mathcal{O}_{S'}(-2).$$

Since $p_* \omega_{S/S'} = (R^1 p_* \mathcal{O}_S)^\vee$, we have

$$p_* q^* N_{S\setminus X} = (R^1 p_* N_{\Delta\setminus S'\times S})^\vee.$$

Now

$$h^0(R^1 p_* \mathcal{O}_S) + h^1(p_* \mathcal{O}_S) = h^1(\mathcal{O}_S) = 0$$

and

$$h^1(R^1 p_* \mathcal{O}_S) = h^2(\mathcal{O}_S) = 4,$$

so

$$R^1 p_* \mathcal{O}_S = \bigoplus_{j=1}^{5} \mathcal{O}_{S'}(-a_j)$$

with each $a_j > 0$ and $\sum_j a_j = 9$. Hence $c_1(R^1 p_* N_{\Delta\setminus S'\times S}) = 5 \cdot 2 - 9 = 1$.

Since no fiber of S/S' lies in a hyperplane, we know that

$$p_* \omega_S = \bigoplus_{j=1}^{5} \mathcal{O}_{S'}(a_j - 2)$$

can have no global sections with zeros. Thus

$$p_* \omega_S = \mathcal{O}_{S'}(-1) \oplus \bigoplus_{j=1}^{4} \mathcal{O}_{S'} \tag{36}$$

and so the map μ given in (34) becomes

$$\mu_V : \mathcal{O}_{S'}(-1) \oplus \bigoplus_{j=1}^{4} \mathcal{O}_{S'} \xrightarrow{A} \mathcal{O}_{S'}(1) \oplus \bigoplus_{j=1}^{4} \mathcal{O}_{S'}, \tag{37}$$

where

$$A = \begin{pmatrix} a & B \\ {}^tB & C \end{pmatrix} = \begin{pmatrix} a & b_1 & b_2 & b_3 & b_4 \\ b_1 & c_{11} & c_{12} & c_{13} & c_{14} \\ b_2 & c_{21} & c_{22} & c_{23} & c_{24} \\ b_3 & c_{31} & c_{32} & c_{33} & c_{34} \\ b_4 & c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}$$

for some constant symmetric matrix C, vector B of linear forms, and quadratic form a. Now $\det C \neq 0$, since it is the determinant of the quadratic form (24). So, after change of basis of $H^0(p_*\omega_S)$, the map A can be rewritten as

$$A = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, \tag{38}$$

where I is the identity matrix. Let

$$\mathfrak{L} \cong \mathcal{O}_{S'}(-1) \subseteq p_*\omega_S$$

denote the line bundle summand corresponding to the factor a. We then have the commutative diagram

$$\begin{tikzcd}
p_*q^*N_{S\backslash X} \otimes p_*q^*N_{S\backslash X} \arrow[r] \arrow[d, "1\otimes A"] & R^1p_*\omega_{S/S'} \arrow[d, equal] \\
p_*q^*N_{S\backslash X} \otimes R^1p_*N_{\Delta\backslash S'\times S} \arrow[r] \arrow[d, equal, "a\otimes a^{-1}"] & R^1p_*\omega_{S/S'} \arrow[d, equal] \\
p_*\omega_{S/S'} \otimes R^1p_*\mathcal{O}_S \arrow[r] & R^1p_*\omega_{S/S'} \\
p_*\omega_S \otimes R^1p_*\mathcal{O}_S \arrow[r] \arrow[u, "a"] & R^1p_*\omega_S \arrow[u, "a"]
\end{tikzcd}$$

restricts to

$$\mathfrak{L}^2 \xrightarrow{a} R^1p_*\omega_{S/S'} \cong \mathcal{O}_{S'}.$$

Finally, by (38) the Fitting divisor on S' depends linearly on the choice of V and corresponds to a linear series of degree 2 without basepoints. □

So, by Lemma 4.7, S' is not an isolated subscheme of Hilb^X but must have at least two embedded points along S' and seemingly could have other components of positive dimension meeting S'. Since all nearby (reduced, irreducible) curves are either canonically embedded or lie in a $\mathbb{P}^3$, any other positive-dimensional component (as a reduced variety) would have to lie in $\mathbf{P}'$ or $\mathbf{Q}'$. As we will see at the beginning of the next section, such positive-dimensional components do not exist.

5. Main Theorem and Numerical Invariants

THEOREM 5.1. *For generic $S \subseteq X$ as before, there is a Zariski open neighborhood $\mathbf{U}'$ of S' in $\mathrm{Hilb}^{\mathbb{P}^4}$ such that*

$$\mathrm{Hilb}^X \cap \mathbf{U}' = S' \cup C_{1,\mathbf{2}} \cup C_{2,\mathbf{2}},$$

where $\{C_1\}$ and $\{C_2\}$ are (general) points of S' and, at each C_i, there is an analytic isomorphism

$$(\mathrm{Hilb}^X \cap \mathbf{U}')_{\{C_i\}} \cong \mathrm{Spec}\, \frac{\mathbb{C}[x, y]}{\{xy, y^2\}}.$$

Furthermore,

$$\mathrm{Hilb}^X \cap \mathbf{U}' \cap \mathbf{P}' = S'.$$

Proof. We have already seen in Corollary 4.3 that

$$\mathrm{Hilb}^X \cap \mathbf{P}' = S'.$$

By Lemma 4.7, we have locally

$$(\mathrm{Hilb}^X \cap \mathbf{U}')_{\{C\}} = (S')_{\{C\}}$$

at all $\{C\} \in S'$ except possibly at two points $\{C_1\}$, $\{C_2\}$ on S'. At each C_i,

$$\dim \frac{H^0(N_{C\backslash X})}{H^0(N_{C\backslash S})} = 1. \tag{39}$$

Again we will argue by specialization. We consider the family of quintics $X_{\delta,\varepsilon}$ in $\mathbb{P}^4$ given by

$$F_{\delta,\varepsilon} = G + \delta x_0 V + \varepsilon x_0^2 K. \tag{40}$$

Notice that $S \subseteq X_0$ is stationary under the deformation $X_{\delta,\varepsilon}$ of X_0.

The Hilbert scheme of $X_{0,0}$ is easy. It is simply given by the sections ω of the normal bundle $H^0(N_{S\backslash X_{0,0}}) = H^0(\omega_C)$ for each $\{C\} \in S'$. The curve $\hat{C} = (C, \omega)$ in $X_{0,0}$ deforms to first order with the first-order deformation $X_{\delta,0}$ if and only if

$$\omega V \in \sum_{j=1}^{4} H^0(\omega_C) \cdot \frac{\partial G}{\partial x_j},$$

that is, if and only if (referring now to (34) and (37)) we have

$$\mu_V(\omega) = 0 \in H^1(N_{C\backslash S}).$$

Thus $C = C_i$. Since the Fitting ideal of μ_V is reduced at each $\{C_i\}$, the subscheme of the Hilbert scheme of $X_{0,0}$ which deforms to first order in the direction $X_{\delta,0}$ is precisely the (reduced) union of S' and a copy of the 1-dimensional kernel of

$$\mu_V : H^0(\omega_{C_i}) \to H^1(N_{C_i\backslash S})$$

for $i = 1, 2$. Denote the families of curves corresponding to these last two components as $C_{i,\infty}/\mathrm{Spec}\,\mathbb{C}[t]$. (Notice that their first-order deformations in the direction $X_{\delta,0}$ lie in $\mathbf{Q}'$.) Since $X_{\delta,0} \cong X_{\delta,\varepsilon}$ modulo x_0^2, the subscheme $C_{i,\mathbf{2}}$ of $C_{i,\infty}$ over $\mathrm{Spec}(\mathbb{C}[t]/\{t^2\})$ deforms (and continues to lie in $\mathbf{Q}'$) over all of

$$\frac{X_{\delta,\varepsilon}}{\{\delta^2, \varepsilon^2\}}. \tag{41}$$

The obstruction to deforming the subscheme $C_{i,\mathbf{3}}$ of $C_{i,\infty}$ over (41) is the image of the element $x_0^2 K$ under the natural map

$$H^0(N_{X_{0,0}\backslash\mathbb{P}^4}) \to R^1 p_*(N_{C_{i,\mathbf{3}}\backslash I' \times X_0}),$$

where $I' = \mathrm{Spec}(\mathbb{C}[t]/\{t^3\})$. By construction, this image vanishes modulo t^2 and so produces an element of the vector space

$$\begin{aligned}
\frac{t^2R^1p_*(N_{C_{3,i}\backslash I'\times X_0})}{\{t^3\}} &= H^1(N_{C_i\backslash X_{0,0}})\\
&= \frac{H^0(\mathcal{O}_{C_i}\otimes N_{X_{0,0}\backslash\mathbb{P}^4})}{\text{image } H^0(N_{C_i\backslash\mathbb{P}^4})} + H^1(\mathcal{O}_C\otimes N_{S\backslash X_{0,0}})\\
&= \frac{H^0(\mathcal{O}_{C_i}(5))}{\text{image } H^0(N_{C_i\backslash\mathbb{P}^4})} + H^1(\mathcal{O}_C\otimes N_{S\backslash X_{0,0}}).
\end{aligned}$$

Now, by varying the center of projection (and hence the element ω) and varying the choice of K, it is easy to see that the expressions ω^2K generate the entire vector space $H^0(\mathcal{O}_{C_i}(5))$ so that, for generic (41), $C_{C_{i,\mathbf{3}}}$ is obstructed. Setting $\delta=\varepsilon$ we conclude that, for the first-order deformation of $X_{0,0}$ in the family

$$F_\varepsilon = G+\varepsilon(x_0V+x_0^2K),$$

only the scheme

$$S'\cup C_{1,\mathbf{2}}\cup C_{2,\mathbf{2}} \tag{42}$$

deforms (and that this first-order deformation lies in $\mathbf{Q}'$). Since we know that

$$h^0(N_{C_i\backslash X_{\delta,\varepsilon}})=2$$

for $\delta\neq 0$ and $\varepsilon\neq 0$, we conclude that (42) must give $\mathrm{Hilb}^{X_{\delta,\varepsilon}}\cap\mathbf{U}'$ for generic δ and ε. (Note, however, that we can not conclude that the first-order deformations $C_{i,\mathbf{2}}$ of C_i continue to lie in Q' for generic $X=X_{\delta,\varepsilon}$.) $\square$

So, for F as in (19), the zero scheme of $p_*q^*F\big|_{\mathbf{U}'\cap\mathbf{Q}'}$ is the smooth (reduced) curve S' possibly together with simple embedded points at the $\{C_i\}$ for $i=1,2$. By intersection theory as in [F], the normal cone $\mathfrak{C}_{\mathbf{Q}'}$ associated to the section $p_*q^*F\big|_{\mathbf{U}'\cap\mathbf{Q}'}$ of the vector bundle

$$E_{\mathbf{U}'\cap\mathbf{Q}'}=p_*q^*\mathcal{O}_X(5)$$

is given by a subbundle $\mathfrak{C}\subseteq E_{S'}$ of co-rank 1 possibly together with the entire fiber E_i of $E_{S'}$ at the two points $\{C_i\}$.

Referring to the components $\mathbf{P}'$ and $\mathbf{Q}'$ of $\mathbf{U}'$ defined in Section 2, let

$$\mathbf{P}\subseteq\mathbf{P}'\times\mathbb{P}^4,\quad \mathbf{Q}\subseteq\mathbf{Q}'\times\mathbb{P}^4,\quad \mathbf{U}\subseteq\mathbf{U}'\times\mathbb{P}^4$$

be the universal curves. For the standard map

$$\begin{array}{ccc}\mathbf{U} & \xrightarrow{q} & \mathbb{P}^4\\ \big\downarrow{\scriptstyle p} & & \\ \mathbf{U}', & & \end{array}$$

the surjectivity of (12) implies that

$$E=p_*q^*\mathcal{O}_{\mathbb{P}^4}(5)$$

is a vector bundle that is generated by the linear space of sections

$$\{p_*q^*F' : F'\in H^0(\mathcal{O}_{\mathbb{P}^4}(5))\}.$$

Because there is an $F' \in H^0(\mathcal{O}_{\mathbb{P}^4}(5))$ near F such that zeros$(p_*q^*F')\big|_{S'}$ are given locally by a deformation of S' that meets S' transversely at $\{C\}$, it follows by Theorem 5.1 and Sard's theorem that, given any analytic neighborhood of $\{F\}$ in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5)))$, the general section F' in that neighborhood has the property that p_*q^*F' has reduced simple zeros on $\mathbf{P}'$ and reduced simple zeros on $\mathbf{Q}'$. Since a simple constant count shows that generically none of these lie on $\mathbf{R}'$, we have established that, for a general deformation X' of X that is sufficiently near to X, the components of Hilb^X supported in S' contribute a finite nonnegative number $n_{\mathbf{P}'}$ of infinitesimally rigid curves from $\mathbf{P}'$ as well as a finite nonnegative number $n_{\mathbf{Q}'}$ of infinitesimally rigid curves from $\mathbf{Q}'$ to the Hilbert scheme $\mathrm{Hilb}^{X'}$ of X'.

THEOREM 5.2.

$$n_{\mathbf{P}'} = 1.$$

Proof. This is an immediate concequence of Proposition 3.5 and Corollary 4.3. More precisely, for generic $S \subseteq X$ we have seen in Corollary 4.3 that the 4-dimensional set of hyperplane sections of X meet $\mathbf{N}_2$ transversely at $\{S\} \in \mathbf{N}_{2,\omega}$. If G' is chosen as in the previous proof, the deformation

$$F_\delta = G + \delta G' + x_0 \hat{V} = 0$$

still has the property that the period map (23)

$$H^{2,0}(S_\delta) \to H^{1,1}(S_\delta)$$

is of maximal rank for small δ. Therefore, all of Hilb^{X_δ} near S' lies in Hilb^{S_δ}. But as we have seen,

$$c_1(C_\delta) \neq \omega_{S_\delta}$$

so that $h^0(\mathcal{O}_{S_\delta}(C_\delta)) = 1$. □

The final goal of this paper is to estimate $n_{\mathbf{Q}'}$. Again by intersection theory, $n_{\mathbf{Q}'}$ is the intersection of the normal cone $\mathfrak{C}_{\mathbf{Q}'}$ associated to the section p_*q^*F of the vector bundle

$$E_{\mathbf{Q}'} = p_*q^*\mathcal{O}_X(5)$$

with the zero section. Since the zero scheme of $p_*q^*F\big|_{\mathbf{Q}'}$ is the smooth (reduced) curve S' possibly together with simple embedded points at the $\{C_i\}$ for $i = 1, 2$, it follows that the normal cone $\mathfrak{C}_{\mathbf{Q}'}$ is simply a subbundle

$$\mathfrak{C} \subseteq E_{S'}$$

of co-rank 1 possibly together with the entire fiber E_i of $E_{S'}$ at each of the two points $\{C_i\}$. Hence, by intersection theory (see [F]) we have either

$$n_{\mathbf{Q}'} = c_1(E_{S'}) - c_1(\mathfrak{C})$$

if the embedded points at the $\{C_i\}$ do not lie in $\mathbf{Q}'$ or

$$n_{\mathbf{Q}'} = c_1(E_{S'}) - c_1(\mathfrak{C}) + 2$$

if they do.

To compute $c_1(E_{S'})$ we use that, since $S' = \mathbb{P}^1$, $E_{S'}$ is a sum of line bundles. But

$$h^0(E_{S'}) = h^0(\mathcal{O}_S(5)) = \frac{4 \cdot 5 \cdot 5}{2} + 5 = 55,$$
$$h^1(E_{S'}) = h^1(\mathcal{O}_S(5)) = 0,$$

so that

$$c_1(E_{S'}) = 55 - \operatorname{rank}(E_{S'}) = 19.$$

To compute $c_1(\mathfrak{C})$, consider the deformation

$$F_\delta = G + \delta x_0 \hat{V}$$

under which X deforms to the cone over S and $\mathfrak{C}$ deforms to the normal cone $\mathfrak{C}_0$ of

$$J' := \mathbf{Q}' \cap \mathrm{Hilb}^{X_0}$$

in $E_{\mathbf{Q}'}$. Now

$$S' \subseteq J'$$

and, near S', J' is smooth (reduced) of dimension 6 and so, near S', $\mathfrak{C}_0$ is a sub-bundle of $E_{J'}$ of co-rank 6. Thus

$$c_1(\mathfrak{C}) = c_1(N_{S' \backslash \mathfrak{C}_0}) = c_1(\mathfrak{C}_0)\big|_{S'} + c_1(N_{S' \backslash J'}).$$

Now from the split exact sequence

$$0 \to N_{\Delta \backslash S' \times S} \to N_{\Delta \backslash S' \times X_0} \to q^* N_{S \backslash X_0} \to 0$$

and the proof of Lemma 4.7, we have

$$\begin{aligned} c_1(N_{S' \backslash J'}) &= c_1(p_* N_{\Delta \backslash S' \times X_0}) - c_1(\mathfrak{T}_{S'}) = c_1(p_* N_{\Delta \backslash S' \times S}) + c_1(p_* \mathcal{O}_S(1)) - 2 \\ &= 2 - 1 - 2 = -1. \end{aligned}$$

On the other hand,

$$\mathfrak{C}_0\big|_{S'} = N_{J' \backslash \mathbf{Q}'}\big|_{S'}.$$

Hence, if we let $\mathbf{P}_0' \subseteq \mathbf{P}'$ denote the curves lying in the *fixed* $\mathbb{P}^3$ given by the equation $x_0 = 0$, then the natural projection map from $(1, 0, \ldots, 0)$ gives a morphism

$$\mathbf{Q}' \to \mathbf{R}_0' := \mathbf{P}_0' \cap \mathbf{R}'$$

such that J' is the inverse image of S'. Since $N_{S' \backslash \mathbf{R}_0'}$ is generated by global sections, $h^1(N_{S' \backslash \mathbf{R}_0'}) = 0$ and so we have

$$\begin{aligned} c_1(N_{J' \backslash \mathbf{Q}'}\big|_{S'}) &= c_1(N_{S' \backslash \mathbf{R}_0'}) \\ &= h^0(N_{S' \backslash \mathbf{R}_0'}) - \operatorname{rank}(N_{S' \backslash \mathbf{R}_0'}) \\ &= (55 - 6) - (31 - 1) = 19. \end{aligned}$$

Thus we conclude as follows.

THEOREM 5.3.

$$1 \leq n_{\mathbf{Q}'} \leq 3.$$

We wish, of course, to compute $n_{\mathbf{Q}'}$ exactly for generic F. Yet despite repeated attempts we have not been able to determine whether the entire scheme $\mathrm{Hilb}^X \cap \mathbf{U}'$ remains inside $\mathbf{Q}'$.

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