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McMillan's Area Problem

Michael D. O'Neill & Robert E. Thurman

1. Introduction

Let A denote the set of ideal accessible boundary points of a simply connected domain Ω. Recall that these are the finite radial limit points of the Riemann map from the unit disk onto Ω and that each radius along which the limit exists gives a distinct ideal boundary point. In particular, distinct ideal accessible boundary points may have the same complex coordinate. Fix $w_0 \in \Omega$ and for each $a \in A$ and $r < |w_0 - a|$ let $\gamma(a,r) \subset \{z : |z-a| = r\}$ be the circular crosscut of Ω separating a from w_0 that can be joined to a by a Jordan arc contained in $\Omega \cap \{z : |z-a| < r\}$. Throughout this paper we will refer to $\gamma(a,r)$ as the *principal separating arc* for a of radius r.

Let $L(a,r)$ denote the Euclidean length of $\gamma(a,r)$ and let

$$A(a,r) = \int_0^r L(a,\rho)\,d\rho.$$

In [5], McMillan showed that

$$\limsup_{r\to 0} \frac{A(a,r)}{\pi r^2} \geq \frac{1}{2}$$

almost everywhere on $\partial\Omega$ with respect to harmonic measure (denoted hereafter by a.e.-ω).

The purpose of this paper is to prove Theorem A.

Theorem A.

$$\liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} \leq \frac{1}{2} \quad \text{a.e.-}\omega.$$

This answers a question raised at the end of [5]. In an earlier paper [7], we proved the following theorem.

Theorem B.

$$\liminf_{r\to 0} \frac{L(a,r)}{2\pi r} \leq \frac{1}{2} \quad \text{a.e.-}\omega.$$

This is also in answer to the last paragraph of [5]. Theorem A implies Theorem B but the basic idea of the proof is the same as in [7]. Let

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$$E_{m,k} = \{a \in A \mid A(a, r) > (1/2 + 1/m)\pi r^2 \ \forall r < 1/k\}$$

and consider a Riemann map $f : \mathbb{D} \to \Omega$ from the unit disk to Ω such that $f(0) = w_0$. We will show that $f^{-1}(E_{m,k})$ has zero Lebesgue measure in the unit circle $\mathbb{T}$ for each m and k. We do this by showing that, if $f^{-1}(E_{m,k})$ has a point of density for some m and k, then the image of that point would be surrounded by a closed curve contained in Ω. Since the union of all such sets then has measure zero, this completes the proof.

The details of the present argument are more complicated than in [7], so it may be helpful to read [7] first to get the main idea with fewer technicalities. It may also be helpful to take an early glance at Figures 1 and 2 near the end of this paper. For more detailed background on the problem, one can also refer to [4], [5], and [6]. For the ideas from geometric function theory used here, we refer to [1], [3], and [8].

2. Proof of Theorem A

In order to construct a curve in Ω that will surround a boundary point and thus give the contradiction proving Theorem A, we will need to know that centered at almost every point of $E_{m,k}$ is a wide-angled annular corridor whose thickness is bounded from below. That such corridors exist will be a consequence of the accumulation of $E_{m,k}$ near the image of a point of density of $f^{-1}(E_{m,k})$. In fact, the abundance of points of $E_{m,k}$ will allow us to construct a chain of such corridors in Ω that will wrap around a boundary point.

We will require the following lemma. Let $\omega(z, E, \Omega)$ denote the harmonic measure of the set $E \subset \partial\Omega$ from the point $z \in \Omega$.

LEMMA 2.1. *Let Ω be a simply connected domain in $\mathbb{C}$ and let f be a Riemann map $f : \mathbb{D} \to \Omega$. Let $E \subset \partial\Omega$ be a Borel set such that $f^{-1}(E)$ has a point of density. Then, given $\delta > 0$, there is a point $w \in \Omega$ such that*

$$\omega(w, E, \partial\Omega) > 1 - \delta.$$

Proof. Let η be a point of density of $f^{-1}(E) \subset \mathbb{T}$. For any interval $I \subset \mathbb{T}$ centered at η, there is a unique $r(I, \delta)$ with $0 < r(I, \delta) < 1$ such that

$$\omega(r(I, \delta)\eta, I, \mathbb{D}) = 1 - \delta/2.$$

Let $z_I = r(I, \delta)\eta$. Given any $\varepsilon > 0$, there is an interval I centered at η such that

$$|I \setminus f^{-1}(E)| < \varepsilon |I|,$$

where $|\cdot|$ denotes linear measure. Integrating the Poisson kernel at z_I over $I \setminus f^{-1}(E)$ then gives

$$\omega(z_I, I \setminus f^{-1}(E), \mathbb{D}) < \delta/2$$

if $|I|$ is sufficiently small. Therefore

$$\omega(z_I, I \cap f^{-1}(E), \mathbb{D}) > 1 - \delta,$$

and taking $w = f(z_I)$ finishes the proof of the lemma. □

Let $d_f(z_I)$ denote the Euclidean distance from $f(z_I)$ to $\partial\Omega$. Actually, results of Beurling [2] imply the existence of a constant K such that a disk of radius $Kd_f(z_I)$ contains all the harmonic measure of the set E found in the lemma (see [8, p. 142]).

Let $w_0 = f(0)$ and assume that $\eta \in \mathbb{T}$ is a point of density of $f^{-1}(E_{m,k}) \subset \mathbb{T}$. The finite number of steps required to obtain a contradiction in the construction to follow will depend only on the number m in the definition of $E_{m,k}$. It will be clear from the construction that if $\delta > 0$ is sufficiently small and if $\omega(w_1, E_{m,k}, \Omega) > 1 - \delta$ for some point w_1, then the required number of steps can be completed. Moreover, the choice of δ depends only on m. We choose δ to be this small and apply Lemma 2.1 with $E = E_{m,k}$, thus obtaining the desired point w_1.

Let d_0 be the Euclidean distance from w_1 to $\partial\Omega$ and let $x_0 \in \partial\Omega$ be a point such that $|x_0 - w_1| = d_0$. Since $f(\eta) \in A$ we can assume $d_0 \ll 1/k$, where k is the integer in the definition of $E_{m,k}$.

We will introduce positive constants $c_0, c_1, c_2, \ldots$ and $C_1, C_2, \ldots$. Their values will be determined in the discussion to follow and will either be purely numerical or depend only on m (in the definition of $E_{m,k}$). For any $w \in \mathbb{C}$ and $r > 0$, let $D(w, r)$ denote the set

$$\{z \in \mathbb{C} : |z - w| < r\}.$$

Let N be a large integer to be determined later. We will see that it can be chosen so that $N \leq (\text{const} \cdot m^{3/2})$. Since x_0 is a boundary point nearest to w_1, we may choose R_0 so that $D(w_1, d_0) \cap D(x_0, 2^N R_0)$ has area greater than $\left(\frac{1}{2} - \frac{1}{8m}\right)\pi(2^N R_0)^2$. Choose c_0 so that if x_0^* is any point in $D(x_0, c_0 R_0)$ then the area of $D(w_1, d_0) \cap D(x_0^*, R_0)$ is greater than $\left(\frac{1}{2} - \frac{1}{4m}\right)\pi R_0^2$. Later, we will also need $c_0 \ll 1/\sqrt{2m}$. It is clear that R_0 is proportional to d_0 in a ratio that depends only on m.

If $\delta > 0$ is sufficiently small, then there exists a set of points of $E_{m,k}$ of positive harmonic measure contained in $D(x_0, c_0 R_0)$. In fact, the circular arc $\partial D(x_0, c_0 R_0) \cap D(w_1, d_0)$ extends to a circular crosscut of Ω that determines a unique subdomain, U_0, of Ω not containing w_1. The midpoint, w^*, of the circular arc $\partial D(x_0, c_0 R_0/2) \cap D(w_1, d_0)$ is contained in U_0. By the comparison principle for harmonic measure and the Beurling projection theorem, there exists a constant $C_1 > 0$ such that

$$\omega(w^*, \partial U_0 \cap \partial\Omega \cap D(x_0, c_0 R_0), \Omega) \geq C_1 > 0;$$

by repeated application of Harnack's inequality in $D(w_1, d_0) \cup U_0$, there is then a constant C_2 such that

$$\omega(w_1, \partial U_0 \cap \partial\Omega \cap D(x_0, c_0 R_0), \Omega) \geq C_2 > 0.$$

By Lemma 2.1, if δ is sufficiently small then

$$\omega(w_1, \partial U_0 \cap \partial\Omega \cap D(x_0, c_0 R_0) \cap E_{m,k}, \Omega) \geq C_2/2 > 0, \tag{1}$$

as claimed.

Let x_0^* be an element of $\partial U_0 \cap D(x_0, c_0 R_0) \cap E_{m,k}$. Note that, because $x_0^* \in E_{m,k}$, we have

$$\int_{R_0/\sqrt{2m}}^{R_0} L(x_0^*, \rho)\, d\rho \geq \left(\frac{1}{2} + \frac{1}{2m}\right)\pi r^2$$

and, by the choice of c_0, the area of

$$\left\{z \in \mathbb{C} : R_0/\sqrt{2m} \le |z - x_0^*| \le R_0\right\} \cap D(w_1, d_0)$$

is greater than $\left(\frac{1}{2} - \frac{1}{2m}\right)\pi R_0^2$. If

$$\gamma(x_0^*, r) \cap D(w_1, d_0) = \emptyset$$

for each $r \in \left[R_0/\sqrt{2m}, R_0\right]$, then the area of the annulus

$$\left\{z \in \mathbb{C} : R_0/\sqrt{2m} \le |z - x_0^*| \le R_0\right\}$$

is greater than

$$\left(\frac{1}{2} + \frac{1}{2m}\right)\pi R_0^2 + \left(\frac{1}{2} - \frac{1}{2m}\right)\pi R_0^2 = \pi R_0^2.$$

This contradiction shows that there exists an $r \in \left[R_0/\sqrt{2m}, R_0\right]$ such that $\gamma(x_0^*, r) \cap D(w_1, d_0) \ne \emptyset$. Simple topological considerations show that circular crosscuts of smaller radius centered at x_0^* that intersect $D(w_1, d_0)$ must be principal separating arcs for x_0^*. Let $c_1 = 1/\sqrt{2m}$. Thus, shrinking R_0 by a factor no smaller than $c_1/3$, we may assume that for each $r \le 3R_0$ we have $\gamma(x_0^*, r) \cap D(w_1, d_0) \ne \emptyset$. It follows that for each $r \le 2R_0$ we have $\gamma(x_0, r) \cap D(w_1, d_0) \ne \emptyset$.

By a slight strengthening of the preceding argument it is clear that there are constants $c_2, c_3 > 0$ such that, if $0 < R < 1/k$ and $a \in E_{m,k}$, then

$$|\{r \in [c_1 R, R] : L(a, r) > (1 + c_2/m)\pi r\}| \ge c_3 R. \tag{2}$$

We will now assume without loss of generality that x_0 is the origin and that w_1 is on the positive imaginary axis. Let

$$A_0 = \{z : R_0 < |z| < 2R_0\}.$$

Let

$$\theta_0 = \inf\{\theta \in (-\pi/2, \pi) : J_\theta \cap \partial\Omega \ne \emptyset\},$$

where

$$J_\theta = \{z : \arg(z) = -\theta,\ R_0 < |z| < 2R_0\}.$$

Let

$$S_0 = \{z : R_0 < |z| < 2R_0,\ -\theta_0 \le \arg(z) < \pi/2\}.$$

(See Figure 2.)

Choose $x_1 \in J_{\theta_0} \cap \partial\Omega$. Let $R_1 = |x_1|/2$ and consider the annulus $A_1 = \{z : c_1 R_1 \le |z - x_1| \le R_1\}$. Any circular arc K centered at x_1 in A_1 with an angle of at least $(1 + c_2/m)\pi$ is divided into two or three subarcs by the ray $\{z : \arg z = -\theta_0\}$. At least two of the arcs have an angle larger than $c_2\pi/2m$. If $\alpha > 0$ is sufficiently small then the ray $L_1 = \{z : \arg z = -(\theta_0 + \alpha)\}$ also divides K into two or three subarcs, at least two of which have an angle larger than $c_2\pi/4m$. The same angle α will be used in each step of the construction. It is determined that, in each new step, newly constructed annular corridors centered at points x_{j+1} with $\arg x_{j+1} = -\theta_j$ will cross the ray $\{z : \arg z = -(\theta_j + \alpha)\}$. The angle α depends not on the size of R_1 (or R_j for later j) but only on c_1 and c_2. Specifically, choose $\alpha < \alpha^*$ where α^* is found by solving the triangle with sides $A = 1,\ B = c_1/2,$

and C and with angles $\angle AB = \pi - c_2\pi/2m$, $\angle CA = \alpha^*$, and $\angle BC$. The choice $\alpha = c_1c_2\pi/32m$ is sufficient for our purposes.

We can further choose a sufficiently small constant $c_4 > 0$ such that any circular arc centered at $a \in D(x_1, c_4R_1)$ with an angle of at least $(1 + c_2/m)\pi$ and with radius between c_1R_1 and R_1 will also be divided by the ray L_1 into at least two subarcs with angle larger than $c_2\pi/8m$. Notice that c_4 depends only on c_1 and c_2 and not on R_1. We will use the same constant c_4 in subsequent similar steps of the construction with different radii R_j.

The circular arc $\partial D(x_1, c_4R_0) \cap S_0$ extends to a crosscut of Ω that determines a subdomain U_1 not containing w_1. Because the width of S_0 is greater than const$\cdot d_0$, we may argue as before using Harnack's inequality and the Beurling projection theorem in $D(w_1, d_0) \cup S_0 \cup U_1$ to find a constant $C_3 > 0$ depending only on m such that

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_4R_0), \Omega) > C_3 > 0. \tag{3}$$

Therefore,

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_4R_0) \cap E_{m,k}, \Omega) > C_3/2 > 0 \tag{4}$$

by Lemma 2.1 with a sufficiently small initial choice of $\delta > 0$.

For each point $a \in E_{m,k} \cap \partial U_1 \cap D(x_1, c_4R_0)$, let $F_a \subset [c_1R_1, R_1]$ denote the set of r such that $L(a, r) > (1 + c_2/m)\pi r$. By (2), the set F_a has $|F_a| > c_3R_1$ and, for each $r \in F_a$, $\gamma(a, r)$ intersects the ray L_1. Let x denote the orthogonal projection of x_1 on the line L_1. For points z, w in the plane, let $\overline{zw}$ denote the line segment with endpoints z and w. Then $L_1 = \overline{x_0x} \cup \overline{x\{\infty\}}$ and we write $F_a = F_a^+ \cup F_a^-$, where F_a^+ (resp., F_a^-) is the set of $r \in F_a$ such that $\overline{x\{\infty\}}$ (resp., $\overline{x_0x}$) divides $\gamma(a, r)$ into two subarcs, the smaller of which has an angle at least $c_2\pi/8m$. Then either $|F_a^+| \geq (c_3/2)R_1$ or $|F_a^-| \geq (c_3/2)R_1$. Making a choice of $+$ or $-$ so that the previous inequality holds, we rename the chosen set F_a^*. Let L_1^* denote the corresponding side of L_1 with respect to the point x and let

$$G_a = \{L_1^* \cap \gamma(a, r) : r \in F_a^*\}.$$

By (4) and the pigeonhole principle we find a_1, a_1^* in $E_{m,k} \cap \partial U_1 \cap D(x_1, c_4R_0)$ and constants $c_5 > 0$ and $c_6 > 0$ such that $c_5R_0/2 < |a_1 - a_1^*| < c_5R_0$ and $|G_{a_1} \cap G_{a_1^*}| > c_6R_1$. Note that here, $c_5 \ll c_4$. In fact, it will be seen in the following paragraph that c_5 should be chosen to be small compared to the angle $c_2\pi/8m$.

There are now two cases to consider.

Case I. For each ρ such that $c_1R_1 \leq \rho \leq R_1$, we have $\gamma(a_1, \rho) \cap S_0 \neq \emptyset$.

Case II. There is a radius ρ with $c_1R_1 \leq \rho \leq R_1$ such that $\gamma(a_1, \rho) \cap S_0 = \emptyset$.

Assume that we are in Case I. Given a and b in $G_{a_1} \cap G_{a_1^*}$, let $S(a, b) \subset \Omega$ be the subdomain of Ω between the crosscuts $\gamma(a_1, |a_1 - a|)$ and $\gamma(a_1, |a_1 - b|)$. Let $S^*(a, b)$ denote the annular corridor bounded by $\gamma(a_1, |a_1 - a|)$, $\gamma(a_1, |a_1 - b|)$, $\overline{ab}$, and ∂S_0. We claim that there is a constant $c_7 > 0$ and points a and b in $G_{a_1} \cap G_{a_1^*}$ such that $|a - b| > c_7R_1$ and $S^*(a, b)$ contains no point of $\partial\Omega$. In fact, if $|a - b| < c_7^*R_1$ and if there is a point $\tau \in \partial\Omega$ contained in $S^*(a, b)$, then some piece of $\partial\Omega$ must connect τ to $\overline{ab}$ and then must extend past L_1 through an angle of

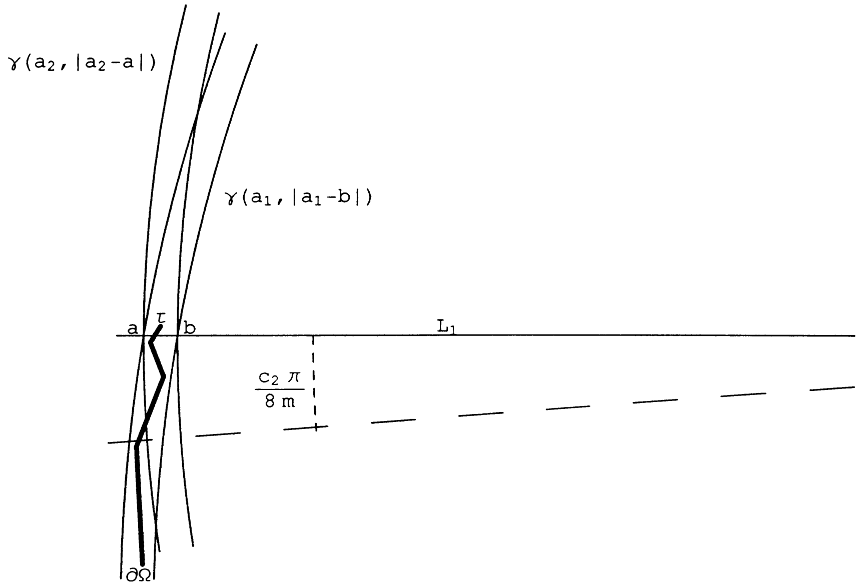

Figure 1 $S^*(a, b)$ can contain no point of $\partial\Omega$

at least $c_2\pi/8m$ in $S(a, b)$. Since c_5 is very small compared to $c_2\pi/8m$ and since $|a_1^* - a_1| \geq c_5R_0/2$, simple geometric considerations show that if c_7^* is sufficiently small then one of the arcs $\gamma(a_1^*, |a_1^* - b|)$ or $\gamma(a_1^*, |a_1^* - a|)$ would intersect $\partial\Omega$ at a point too close to L_1 for the points a and b to be contained in $G_{a_1^*}$ (see Figure 1). Because $|G_{a_1} \cap G_{a_1^*}| > c_6R_1$ and $\operatorname{diam}(G_{a_1} \cap G_{a_1^*}) < (1 - c_1)R_1$, we find the desired constant c_7 with $c_7^* > c_7 > 0$ and the points a and b with $c_7R_1 < |a - b| \leq c_7^*R_1$. Note that the constant c_7 depends only on previously introduced constants and thus only on m. We rename this annular corridor $S^*(a, b) \subset \Omega$ as S_0^*.

Now, still assuming Case I, let

$$J_\theta = \{z : \arg(z) = -\theta,\ |a| < |z| < |b|\}$$

and let

$$S_1 = \{z : |a| < |z| < |b|,\ -\theta_1 \leq \arg z \leq -(\theta_0 + \alpha)\},$$

where

$$\theta_1 = \inf\{\theta \in ((\theta_0 + \alpha), \pi) : J_\theta \cap \partial\Omega \neq \emptyset\};$$

see Figure 2.

Choose $x_2 \in J_{\theta_1} \cap \partial\Omega$. Let $R_2 = |x_2|/2$ and let L_2 be the ray $\{z : \arg z = -(\theta_1 + \alpha)\}$. The arc $\partial D(x_2, c_4R_2) \cap S_0 \cup S_0^* \cup S_1$ defines a subdomain U_2 not containing w_1. Arguing as before (with Harnack's inequality, the comparison principle, and the Beurling projection theorem) but now in $D(w_1, d_0) \cup S_0 \cup S_0^* \cup S_1 \cup U_2$ we find, using Lemma 2.1 with a sufficiently small choice of $\delta > 0$, a constant $C_4 > 0$ such that

$$\omega(w_1, \partial U_2 \cap \partial\Omega \cap D(x_2, c_4R_2) \cap E_{m,k}, \Omega) \geq C_4 > 0.$$

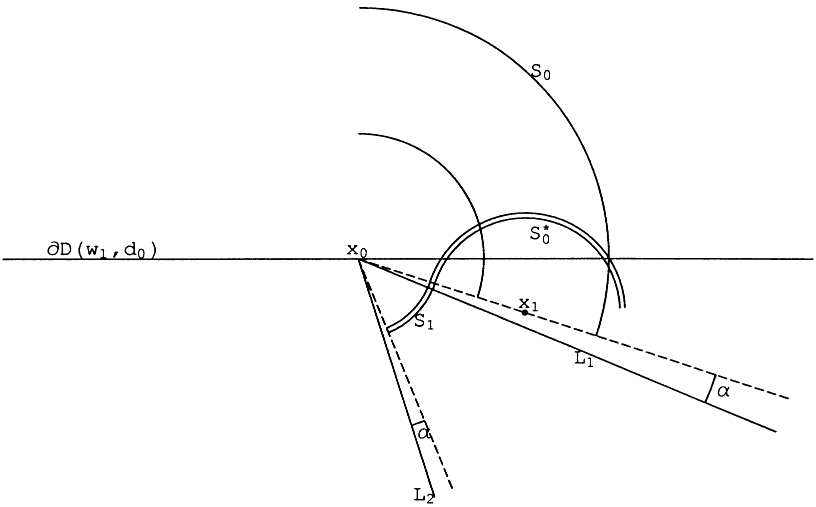

Figure 2 Step 1 of the construction

As in the preceding step, we find points a_2 and a_2^* in $D(x_2, c_4 R_2) \cap \partial U_2 \cap E_{m,k}$ and sets $G_{a_2}, G_{a_2^*} \subset L_2$ with the same properties as before. Then we again have Cases I and II as described previously.

Assume we are again in Case I. We repeat the argument made for the point x_1 at the new point x_2 and find two annular sectors. First S_1^* is found by the pigeonhole argument in the same way that S_0^* was found in the previous step. The new annular corridor S_1^* is centered at the point a_2 near x_2 and ends on the ray L_2 after having passed through the additional angle of α clockwise around x_0. Now S_2 is obtained in the same manner that S_1 was previously. That is, S_2 is centered at x_0, begins where S_1^* ends on L_2, and is stopped in its clockwise course around x_0 by a point $x_3 \in J_{\theta_2} \cap \partial\Omega$. In the jth subsequent step, a point x_j is found at the end of S_{j-1} and nearby points $a_j, a_j^* \in E_{m,k}$ are found as before. Case I at the jth step means that every principal separating arc for a_j with radius ρ between $c_1 R_j$ and R_j intersects the union of the previously constructed annular corridors $S_0, S_0^*, S_1, S_1^*, \ldots, S_{j-1}$. The new annular corridors S_{j-1}^* and S_j are now found as in previous steps. Note that, after the jth step, the union of annular corridors so far constructed has turned through an angle of at least $j\alpha$ clockwise from the horizontal through x_0. A sufficiently small initial choice of $\delta > 0$ ensures that there is an abundance of points of $E_{m,k}$ near the point x_j at the end of S_{j-1} so that the construction may continue to the $(j+1)$th step.

Assuming that we only encounter Case I in each step, a sufficiently small choice of δ at the beginning of the proof allows us to repeat the argument $N = [2\pi/\alpha]$ times, and this determines the choice of N at the beginning of the construction. Since the union of constructed corridors turns by an additional angle of at least α with each step, we will have constructed a connected union of annular corridors $\mathcal{C}$ in Ω contained in the annulus

$$\{z : 2^{-N}R_0 < |z - x_0| < 2^N R_0\}.$$

The union of $\mathcal{C}$ with $D(w_1, d_0)$ contains a closed curve in Ω surrounding the boundary point x_0.

If Case II occurs at any step n before the Nth then there is a principal separating arc for a_n of radius ρ $(c_1 R_n \le \rho \le R_n)$ that does not intersect $S_0 \cup S_0^* \cup S_1 \cup S_1^* \cup \cdots \cup S_{n-1}$. It follows that the circular crosscut centered at a_n of radius ρ that does intersect $S_0 \cup S_0^* \cup S_1 \cup S_1^* \cup \cdots \cup S_{n-1}$ cannot be a separating arc for a_n at all. This means that w_0 is located in Ω on the concave side of this arc but on the convex side of the arcs that make up S_{n-1}. We then continue the construction at the $(n+1)$th step with the original annulus A_0 centered at x_0 but now turning in the counterclockwise direction. Since we have found Case II in the clockwise direction, we cannot find Case II in the counterclockwise direction without repeating the situation of w_0 being on the concave side of the last nonseparating circular arc yet on the convex side of the arcs in the last S_{n-1} from Case I. Simple topological considerations rule out this possibility and we thus find a closed curve in Ω surrounding x_0 in at most N more steps.

It follows that there can be no point of density of $f^{-1}(E_{m,k})$ and that the harmonic measure of $E_{m,k}$ is therefore zero. The theorem is proved.

References

[1] L. V. Ahlfors, *Conformal invariants, Topics in geometric function theory,* McGraw-Hill, New York, 1973.

[2] A. Beurling, *Ensembles exceptionels,* Acta Math. 72 (1939), 1–13.

[3] J. Garnett and D. Marshall, *Harmonic measure,* Cambridge University Press (to appear).

[4] J. E. McMillan, *Boundary behavior of a conformal mapping,* Acta Math. 123 (1969), 43–67.

[5] ———, *On the boundary correspondence under conformal mapping,* Duke Math. J. 37 (1970), 725–739.

[6] M. D. O'Neill, *J. E. McMillan's area theorem*, Colloq. Math. 79 (1999), 229–234.

[7] M. D. O'Neill and R. E. Thurman, *A problem of McMillan on conformal mappings,* Pacific Math. J. (to appear).

[8] Ch. Pommerenke, *Boundary behavior of conformal maps,* Springer-Verlag, Berlin, 1992.

M. D. O'Neill
Department of Mathematics
Claremont McKenna College
Claremont, CA 91711

moneill@mckenna.edu

R. E. Thurman
Insightful Corp.
1700 Westlake Avenue N, Suite 500
Seattle, WA 98109

rthurman@insightful.com