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Singular Integrals Related to Homogeneous Mappings

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1. Introduction

Let $n, m \in \mathbf{N}$ and $d = (d_1, \dots, d_m) \in \mathbf{R}^m$. Define the family of dilations $\{\delta_t\}_{t>0}$ on $\mathbf{R}^m$ by

$$\delta_t(x_1, \dots, x_m) = (t^{d_1}x_1, \dots, t^{d_m}x_m). \tag{1}$$

We say that $\Phi\colon \mathbf{R}^n \to \mathbf{R}^m$ is a (nonisotropic) homogeneous mapping of degree d if

$$\Phi(ty) = \delta_t(\Phi(y)) \tag{2}$$

holds for all $t > 0$ and $y \in \mathbf{R}^n$.

Let $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$ that is equipped with the normalized Lebesgue measure $d\sigma$. For a Calderón–Zygmund kernel on $\mathbf{R}^n$,

$$K(y) = \frac{\Omega(y)}{|y|^n}, \tag{3}$$

where Ω is homogeneous of degree 0 and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(y)\, d\sigma(y) = 0; \tag{4}$$

we define the singular integral operator $T_{\Omega,\Phi}$ on $\mathbf{R}^m$ by

$$(T_{\Omega,\Phi} f)(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x - \Phi(y))K(y)\, dy \tag{5}$$

for $x \in \mathbf{R}^m$.

The operators defined in (5) have their roots in the classical Calderón–Zygmund operators

$$(T_{\Omega,I} f)(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x - y)K(y)\, dy, \tag{6}$$

which corresponds to $n = m$, $d = (1, \dots, 1)$, and $\Phi = I = \mathrm{id}_{R^n \to R^n}$. In their fundamental work on the theory of singular integrals, Calderón and Zygmund [1] proved that the operators $T_{\Omega,I}$ in (6) are bounded on L^p for $1 < p < \infty$ if $\Omega \in L \log^+ L(\mathbf{S}^{n-1})$. Their result is nearly optimal in the sense that the space $L \log^+ L(\mathbf{S}^{n-1})$ cannot be replaced by any other Orlicz space $L^\phi(\mathbf{S}^{n-1})$ with a ϕ that is increasing and satisfies

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$$\lim_{t\to\infty}\frac{\phi(t)}{t\ln t}=0$$

(e.g., $\phi(t)=t(\ln t)^{1-\varepsilon}$).

In the ensuing development, an improvement of the result of Calderón and Zygmund was obtained by Connet [4] and Ricci and Weiss [12], who proved independently that the L^p boundedness of $T_{\Omega,I}$ still holds if $\Omega\in H^1(\mathbf{S}^{n-1})$. Here, $H^1(\mathbf{S}^{n-1})$ denotes the Hardy space on the unit sphere and contains $L\log^+ L(\mathbf{S}^{n-1})$ as a proper subspace. Their result can be stated as follows.

THEOREM A. *Let $\Omega\in H^1(\mathbf{S}^{n-1})$ satisfy* (4), *and let I denote the identity mapping from $\mathbf{R}^n$ to itself. Then, for every $p\in(1,\infty)$, there exists a $C_p>0$ such that*

$$\|T_{\Omega,I}f\|_{L^p(R^n)}\le C_p\|f\|_{L^p(R^n)}. \tag{7}$$

Fan, Guo, and Pan [6] have studied the L^p boundedness of singular integrals along homogeneous surfaces with rough kernels.

THEOREM B [6]. *Let $\Omega\in H^1(\mathbf{S}^{n-1})$ satisfy* (4). *Let $\phi\colon\mathbf{R}^n\to\mathbf{R}$ be homogeneous of degree h with $h>0$ and $\Phi(y)=(y,\phi(y))$. Let $T_{\Omega,\Phi}$ be defined as in* (5). *If $\phi\big|_{\mathbf{S}^{n-1}}$ is real-analytic then, for every $p\in(1,\infty)$, there exists a $C_p>0$ such that*

$$\|T_{\Omega,\Phi}f\|_{L^p(R^{n+1})}\le C_p\|f\|_{L^p(R^{n+1})}. \tag{8}$$

One of our main results in this paper is the following theorem.

THEOREM 1. *Let $\Omega\in H^1(\mathbf{S}^{n-1})$ satisfy* (4). *Let $\Phi\colon\mathbf{R}^n\to\mathbf{R}^m$ be a homogeneous mapping of degree $d=(d_1,\dots,d_m)$ with $d_1,\dots,d_m>0$. Let $T_{\Omega,\Phi}$ be defined as in* (5). *If $\Phi\big|_{\mathbf{S}^{n-1}}$ is real-analytic then, for every $p\in(1,\infty)$, there exists a $C_p>0$ such that*

$$\|T_{\Omega,\Phi}f\|_{L^p(R^m)}\le C_p\|f\|_{L^p(R^m)}. \tag{9}$$

REMARKS. (1) The function $T_{\Omega,\Phi}f$ is defined initially for f in a dense subset of $L^p(\mathbf{R}^n)$, say $\mathcal{S}(\mathbf{R}^n)$. Once (9) is established for all $f\in\mathcal{S}(\mathbf{R}^n)$, the operator $T_{\Omega,\Phi}$ can be extended to the full $L^p(\mathbf{R}^n)$ in the usual manner.

(2) Theorem B can now be seen as a special case of Theorem 1 by setting $m=n+1$, $\Phi(y)=(y,\phi(y))$, and $d=(1,\dots,1,h)$. It should be mentioned that the case $h<0$ was also addressed in [6]. We shall treat the case $d=(d_1,\dots,d_m)\notin(\mathbf{R}_+)^m$ in Theorem 2.

(3) One may combine Theorem 1 with the method described in [15, Chap. XI, Sec. 2.4] to obtain the L^p boundedness of $T_{\Omega,\Phi}$ when Φ is a polynomial (or generalized polynomial) mapping and $\Omega\in H^1(\mathbf{S}^{n-1})$. See also [7] and [14].

The main tools used in this paper come from Duoandikoetxea and Rubio de Francia [5], Fan and Pan [7], and Ricci and Stein [10]. The paper is organized as follows. A few definitions and lemmas will be recalled or proved in Section 2. Section 3 contains the proof of Theorem 1. Section 4 discusses extensions of Theorem 1 by allowing the d_j to be negative.

Throughout the rest of this paper, the letter C will stand for a constant but not necessarily the same one in each occurrence.

2. Some Definitions and Lemmas

The Hardy space $H^1(\mathbf{S}^{n-1})$ has many equivalent definitions, one of which is given in terms of the following radial maximal operator on $\mathbf{S}^{n-1}$:

$$P^+ \colon f \to \sup_{t\in[0,1)} \left| \int_{\mathbf{S}^{n-1}} P_{tx}(y) f(y)\, d\sigma \right|,$$

where $P_u(y) = (1-|u|^2)/|y-u|^n$.

Definition 2.1. An integrable function f on $\mathbf{S}^{n-1}$ is in the space $H^1(\mathbf{S}^{n-1})$ if $\|f\|_{\mathbf{S}^{n-1}} = \|P^+ f\|_{L^1(\mathbf{S}^{n-1})} < \infty$.

Next we shall recall the atomic decomposition for $H^1(\mathbf{S}^{n-1})$. For $z \in \mathbf{S}^{n-1}$ and $r > 0$, we let $D(z,r) = \mathbf{S}^{n-1} \cap \{y \in \mathbf{R}^n : |y-z| < r\}$.

Definition 2.2. A function $a \colon \mathbf{S}^{n-1} \to \mathbf{C}$ is an H^1 atom if the following are satisfied:

(i) $\operatorname{supp}(a) \subseteq D(z,r)$ for some $z \in \mathbf{S}^{n-1}$ and $r > 0$;
(ii) $\|a\|_\infty \le r^{-(n-1)}$;
(iii) $\int_{D(z,r)} a(y)\, d\sigma(y) = 0$.

The following result is well known (see e.g. [2; 3]).

Lemma 2.3. *If $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies* (4), *then there exist $\{c_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and H^1 atoms $\{a_j\}_{j\in\mathbf{N}}$ such that*

$$\Omega = \sum_{j=1}^{\infty} c_j a_j \quad \textit{and} \quad \|\Omega\|_{H^1(\mathbf{S}^{n-1})} \approx \sum_{j=1}^{\infty} |c_j|.$$

In part of our analysis we shall encounter oscillatory integrals with generalized polynomials as their phase functions. Thus we need the following lemma of van der Corput type, which was proved by Ricci and Stein [10].

Lemma 2.4. *Let $n \in N$, $\mu_1, \dots, \mu_n \in \mathbf{R}$, and $d_1, \dots, d_n$ be distinct positive real numbers. Let $\varepsilon = \min\{1/d_1, 1/n\}$. Then there exists a positive constant C, independent of $\{\mu_j\}$, such that*

$$\left| \int_\delta^\tau e^{i(\mu_1 t^{d_1} + \dots + \mu_n t^{d_n})} \psi(t)\, dt \right| \le C|\mu_1|^{-\varepsilon} \left(|\psi(\tau)| + \int_\delta^\tau |\psi'(t)|\, dt \right) \tag{10}$$

holds for $0 \le \delta < \tau \le 1$ and $\psi \in C^1([0,1])$.

Lemma 2.5. *Let $h_1, \dots, h_l > 0$ be distinct and*

$$Q(t,u) = t^{h_1} \sum_{|\alpha| \le s} a_\alpha u^\alpha + \sum_{j=2}^{n} t^{h_j} w_j(u),$$

where $t \in \mathbf{R}$, $u = (u_1, \dots, u_{n-1}) \in \mathbf{R}^{n-1}$, $\alpha \in (\mathbf{N} \cup \{0\})^{n-1}$, $a_\alpha \in \mathbf{R}$, and $w_j(\cdot)$ are real-valued. Let $r > 0$ and $b(u)$ be a measurable function on $[-r,r]^{n-1}$ that *satisfies* $\|b\|_\infty \le r^{-(n-1)}$. *Then there exist positive constants* C *and* γ *independent of* $\{a_\alpha\}, \{w_j(\cdot)\}, b(\cdot), r$ *such that*

$$\int_0^1 \left| \int_{[-r,r]^{n-1}} e^{iQ(t,u)} b(u)\, du \right| dt \le C\Big(r^s \sum_{|\alpha|=s} |a_\alpha| \Big)^{-\gamma}. \tag{11}$$

Proof. We shall establish (11) by employing the ideas in [6] in conjunction with Lemma 2.4. By using a dilation and a rotation (if necessary) we may assume that $r = 1$ and

$$|a_{(s,0,\dots,0)}| \approx \sum_{|\alpha|=s} |a_\alpha|. \tag{12}$$

Let $\eta = (u_2, \dots, u_{n-1})$ and $R(u) = \sum_{|\alpha|\le s} a_\alpha u^\alpha$. Then

$$\begin{aligned} &\int_0^1 \left| \int_{[-1,1]^{n-1}} e^{iQ(t,u)} b(u)\, du \right| dt \\ &\quad \le \int_{[-1,1]^{n-2}} \Big(\int_0^1 \left| \int_{-1}^1 e^{iQ(t,u_1,\eta)} b(u_1,\eta)\, du_1 \right|^2 dt \Big)^{1/2} d\eta \\ &\quad \le \int_{[-1,1]^{n-2}} \Big(\int_{-1}^1 \int_{-1}^1 I(u_1, v_1, \eta)\, du_1\, dv_1 \Big)^{1/2} d\eta, \end{aligned}$$

where

$$\begin{aligned} I(u_1, v_1, \eta) &= \left| \int_0^1 e^{i[Q(t,u_1,\eta) - Q(t,v_1,\eta)]}\, dt \right| \\ &\le C |R(u_1,\eta) - R(v_1,\eta)|^{-2\gamma} \end{aligned}$$

with $2\gamma = \min\{1/h_1, 1/l, 1/(s+1)\}$. Since

$$R(u_1, \eta) = a_{(s,0,\dots,0)} u_1^s + \text{lower powers in } u_1,$$

we may apply an inequality proved by Ricci and Stein [10, p. 182] and (12) to obtain

$$\int_{-1}^1 I(u_1, v_1, \eta)\, du_1 \le C \Big(\sum_{|\alpha|=s} |a_\alpha| \Big)^{-2\gamma}$$

uniformly in v_1 and η. Therefore,

$$\int_0^1 \left| \int_{[-1,1]^{n-1}} e^{iQ(t,u)} b(u)\, du \right| dt \le C \Big(\sum_{|\alpha|=s} |a_\alpha| \Big)^{-\gamma},$$

which proves (11). □

LEMMA 2.6. *For* $j \in \{1, 2\}$, *let* U_j *be a domain in* $\mathbf{R}^{n_j}$ *and* K_j *a compact subset of* U_j. *Let* $h(\cdot,\cdot)$ *be a real-analytic function on* $U_1 \times U_2$ *such that* $h(\cdot, z)$ *is a nonzero function for every* $z \in U_2$. *Then there exists a positive number* $\delta = \delta(h, K_1, K_2)$ *such that*

$$\sup_{z\in K_2} \int_{K_1} |h(w,z)|^{-\delta}\, dw < \infty. \tag{13}$$

Proof. By the compactness of K_1 and K_2 it suffices to show that, for every $(w_0, z_0) \in U_1 \times U_2$, there exist positive numbers $r = r(w_0, z_0)$, $s = s(w_0, z_0)$, and $\delta = \delta(w_0, z_0)$ such that

$$\sup_{|z-z_0|<s} \int_{|w-w_0|<r} |h(w,z)|^{-\delta}\, dw < \infty. \tag{14}$$

Clearly we may assume that $(w_0, z_0) = (0, 0)$ and $h(0, 0) = 0$. Since $h(\cdot, 0)$ is not identically zero, there exists a $k \in \mathbf{N}$ such that

$$\frac{\partial^\beta h(w,0)}{\partial w^\beta}\Big|_{w=0} = 0$$

for every multi-index $\beta = (\beta_1, \ldots, \beta_{n_1})$ satisfying $|\beta| < k$ and

$$\frac{\partial^\alpha h(w,0)}{\partial w^\alpha}\Big|_{w=0} \neq 0$$

for some $\alpha = (\alpha_1, \ldots, \alpha_{n_1})$ with $|\alpha| = k$. Thus, there exists a unit vector η such that

$$(\eta \cdot \nabla_w)^k h(w,0)\big|_{w=0} \neq 0.$$

By using a rotation (in the w variable) if necessary, we may assume that $\eta = (1, 0, \ldots, 0)$. Then

$$\frac{\partial^l h}{\partial w_1^l}(0,0) = 0$$

for $0 \leq l \leq k-1$ and

$$\frac{\partial^k h}{\partial w_1^k}(0,0) \neq 0.$$

Let $\tilde{w} = (w_2, \ldots, w_{n_1})$. By the Malgrange preparation theorem [8] there exist $\rho > 0$ and smooth functions $c(w,z), b_0(\tilde{w}, z), \ldots, b_{k-1}(\tilde{w}, z)$ on $\{|w| < \rho\} \times \{|z| < \rho\}$ such that

$$h(w,z) = c(w,z)[w_1^k + b_{k-1}(\tilde{w}, z) w_1^{k-1} + \cdots + b_0(\tilde{w}, z)]$$

and $c(w,z) \neq 0$ for $|w|, |z| < \rho$. Therefore,

$$\sup_{|\tilde{w}|, |z| < \rho/2} \int_{-\rho/2}^{\rho/2} |h(w,z)|^{-1/(k+1)}\, dw_1 < \infty.$$

By selecting $r = s = \rho/2$ and $\delta = 1/(k+1)$, we see that (14) holds. This completes the proof of (13). $\square$

We shall rely heavily on the following result in [6], which was established on the basis of earlier work by Duoandikoetxea and Rubio de Francia.

LEMMA 2.7. *Let $q > 1$, $l, m \in \mathbf{N}$, and $\{\sigma_{s,k} : 1 \leq s \leq l+1$ and $k \in \mathbf{Z}\}$ be a family of measures on $\mathbf{R}^m$ with $\sigma_{l+1,k} = 0$ for every $k \in \mathbf{Z}$. Let $\{\alpha_{sj} : 1 \leq s \leq l$, $j = 1, 2\} \subset (0, \infty)$, $\{\lambda_s : 1 \leq s \leq l\} \subset (0, \infty)\backslash\{1\}$, $\{M_s : 1 \leq s \leq l\} \subset \mathbf{N}$, and $\{L^{(s)} : 1 \leq s \leq l\} \subset L(\mathbf{R}^m, \mathbf{R}^{M_s})$, where $L(\mathbf{R}^m, \mathbf{R}^{M_s})$ denotes the space of linear transformations from $\mathbf{R}^m$ into $\mathbf{R}^{M_s}$. Suppose that*:

(i) $\|\sigma_{s,k}\| \leq 1$ *for* $k \in \mathbf{Z}$ *and* $1 \leq s \leq l$;
(ii) $|\hat{\sigma}_{s,k}(\xi)| \leq C(\lambda_s^k |L^{(s)}\xi|)^{-\alpha_{s2}}$ *for* $\xi \in \mathbf{R}^m$, $k \in \mathbf{Z}$, *and* $1 \leq s \leq l$;
(iii) $|\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s+1,k}(\xi)| \leq C(\lambda_s^k |L^{(s)}\xi|)^{\alpha_{s1}}$ *for* $\xi \in \mathbf{R}^m$, $k \in \mathbf{Z}$, *and* $1 \leq s \leq l$;
(iv) *for* $1 \leq s \leq l$, σ_s^* *is a bounded operator on* $L^q(\mathbf{R}^m)$, *where* $\sigma_s^*(f) = \sup_{k \in \mathbf{Z}}(|\sigma_{s,k}| * |f|)$.

Then, for $2q/(q+1) < p < 2q/(q-1)$, *there exists a* $C_p > 0$ *independent of* $\{L^{(s)}\}_{s=1}^l$ *such that*

$$\Big\| \sum_{k \in \mathbf{Z}} \sigma_{1,k} * f \Big\|_{L^p(R^m)} \leq C_p \|f\|_{L^p(R^m)} \tag{15}$$

and

$$\bigg\| \Big(\sum_{k \in \mathbf{Z}} |\sigma_{1,k} * f|^2 \Big)^{1/2} \bigg\|_{L^p(R^m)} \leq C_p \|f\|_{L^p(R^m)} \tag{16}$$

hold for all $f \in L^p(\mathbf{R}^m)$.

We shall adopt some of the notation used in [6] and [7]. For $\Phi : \mathbf{R}^n \backslash \{0\} \to \mathbf{R}^m$, $\Omega : \mathbf{S}^{n-1} \to \mathbf{C}$, and $k \in \mathbf{Z}$ we use $\sigma_{\Omega,\Phi,k}$ to denote the measure given by

$$\int_{\mathbf{R}^m} f(x)\, d\sigma_{\Omega,\Phi,k} = \int_{2^k \leq |y| < 2^{k+1}} f(\Phi(y))\, \Omega\Big(\frac{y}{|y|}\Big) |y|^{-n}\, dy. \tag{17}$$

3. Proof of Theorem 1

In this section we shall let Φ be a homogeneous mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$ of degree $d = (d_1, \dots, d_m)$. We also assume the following:

(a) $d_1, \dots, d_m > 0$;
(b) $\Phi\big|_{\mathbf{S}^{n-1}}$ is real-analytic;
(c) there are $l, \tilde{l} \in \mathbf{N}$ such that $l \leq \tilde{l} \leq m$, $\{j : 1 \leq j \leq m$ and $d_j = d_1\} = \{1, \dots, \tilde{l}\}$, and $\{\Phi_1, \dots, \Phi_l\}$ forms a basis for $\operatorname{span}\{\Phi_1, \dots, \Phi_{\tilde{l}}\}$.

Conditions (a) and (b) imposed on Φ are exactly the same as given in Theorem 1, while (c) can be satisfied by a simple reordering of $\Phi_1, \dots, \Phi_m$ (if necessary) unless $\{\Phi_j : d_j = d_1\} = \{0\}$. For $\xi = (\xi_1, \dots, \xi_m) \in \mathbf{R}^m$ we shall let $\tilde{\xi} = (\xi_1, \dots, \xi_{\tilde{l}})$.

Under the foregoing assumptions, we have the following lemma.

LEMMA 3.1. *There exist* $\varepsilon, A > 0$ *and* $L \in L(\mathbf{R}^{\tilde{l}}, \mathbf{R}^l)$ *such that*

$$|\hat{\sigma}_{\Omega,\Phi,k}(\xi)| \leq A(2^{d_1 k} |L\tilde{\xi}|)^{-\varepsilon} \|\Omega\|_2 \tag{18}$$

holds whenever $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^m$, *and* $\Omega \in L^2(\mathbf{S}^{n-1})$.

Proof. By assumption there exists an $L = (L_1, \dots, L_l) \in L(\mathbf{R}^{\tilde{l}}, \mathbf{R}^l)$ such that

$$\sum_{j=1}^{\tilde{l}} \xi_j \Phi_j(y) = \sum_{s=1}^{l} (L_s \tilde{\xi}) \Phi_s(y). \tag{19}$$

Define $h : \mathbf{S}^{n-1} \times \mathbf{S}^{l-1} \to \mathbf{R}$ by

$$h(y,\omega)=\sum_{j=1}^{l}\omega_j\Phi_j(y),$$

where $y\in\mathbf{S}^{n-1}$ and $\omega=(\omega_1,\ldots,\omega_l)\in\mathbf{S}^{l-1}$. Since $\{\Phi_1,\ldots,\Phi_l\}$ is linearly independent, $h(\cdot,\omega)$ is a nonzero function for every $\omega\in\mathbf{S}^{l-1}$. By Lemma 2.6, there exists a $\delta_1>0$ such that

$$\sup_{\omega\in\mathbf{S}^{l-1}}\int_{\mathbf{S}^{n-1}}|h(y,\omega)|^{-\delta_1}\,d\sigma=A_1<\infty.$$

By letting $\varepsilon=\min\{1/d_1,1/m,\delta_1/2\}$ and using Lemma 2.4,

$$\begin{aligned}|\hat{\sigma}_{\Omega,\Phi,k}(\xi)|&=\Bigg|\int_{\mathbf{S}^{n-1}}\bigg(\int_1^2\exp\bigg\{i\bigg[2^{d_1k}\sum_{s=1}^{l}(L_s\tilde{\xi})\Phi_s(y)t^{d_1}\\&\qquad+\sum_{j=\tilde{l}+1}^{m}2^{d_jk}\xi_j\Phi_j(y)t^{d_j}\bigg]\bigg\}\frac{dt}{t}\bigg)\Omega(y)\,d\sigma(y)\Bigg|\\&\le A_2 2^{-\varepsilon d_1k}\int_{\mathbf{S}^{n-1}}|\Omega(y)|\left|\sum_{s=1}^{l}(L_s\tilde{\xi})\Phi_s(y)\right|^{-\varepsilon}d\sigma(y)\\&\le (A_1A_2)(2^{d_1k}|L\tilde{\xi}|)^{-\varepsilon}\|\Omega\|_{L^2(\mathbf{S}^{n-1})}.\end{aligned}$$

□

Proof of Theorem 1. Let $p\in(1,\infty)$. By the atomic decomposition of $H^1(\mathbf{S}^{n-1})$, it suffices to prove that

$$\|T_{\Omega,\Phi}f\|_{L^p(\mathbf{R}^m)}\le A_p\|f\|_{L^p(\mathbf{R}^m)}\tag{20}$$

when Ω satisfies

(i) $\mathrm{supp}(\Omega)\subseteq D(z_0,r)$ for some $z_0\in\mathbf{S}^{n-1}$ and $r\in(0,2]$;
(ii) $\|\Omega\|_\infty\le r^{-(n-1)}$;
(iii) $\int_{\mathbf{S}^{n-1}}\Omega(y)\,d\sigma(y)=0$.

We shall begin by assuming that Ω satisfies (i)–(iii) with $z_0=(0,\ldots,0,1)$ and $0<r<1/4$. An application of Lemma 3.1 gives

$$|\hat{\sigma}_{\Omega,\Phi,k}(\xi)|\le A(2^{d_1k}|L\tilde{\xi}|)^{-\varepsilon}r^{-(n-1)/2}.\tag{21}$$

For $1\le j\le\tilde{l}$ and $|u|=|(u_1,\ldots,u_{n-1})|<r$, let

$$\phi_j(u)=\Phi_j(u,(1-|u|^2)^{1/2}).$$

Let

$$a_{j\alpha}=\frac{1}{\alpha!}\frac{\partial^\alpha\phi_j}{\partial u^\alpha}(0)$$

for $1\le j\le\tilde{l}$ and $|\alpha|<M=[(n-1)/(2\varepsilon)]+1$.

Next we introduce the mappings $\Psi^{(1)},\ldots,\Psi^{(M+1)}:\mathbf{R}^n\backslash\{0\}\to\mathbf{R}^m$ by

$$\Psi^{(1)}(y)=\Phi(y)$$

and

$$\Psi^{(s)}(y) = \Big(|y|^{d_1} \sum_{|\alpha| \le M-s+1} a_{1\alpha}(y')^\alpha, \ldots, |y|^{d_1} \sum_{|\alpha| \le M-s+1} a_{\tilde{l}\alpha}(y')^\alpha, \\ \Phi_{\tilde{l}+1}(y), \ldots, \Phi_m(y) \Big)$$

for $1 < s \le M+1$, where $y = (y_1, \ldots, y_n) \ne 0$ and $y' = (y_1/|y|, \ldots, y_{n-1}/|y|)$. Thus, for all y satisfying $2^k \le |y| < 2^{k+1}$ and $y/|y| \in \operatorname{supp}(\Omega)$,

$$|\Psi^{(1)}(y) - \Psi^{(2)}(y)| \le C 2^{d_1 k} r^{(n-1)/(2\varepsilon)} \tag{22}$$

and

$$|\Psi^{(s)}(y) - \Psi^{(s+1)}(y)| \le C 2^{d_1 k} r^{M-s+1} \tag{23}$$

when $1 < s \le M$.

For $1 \le s \le M+1$ let $\sigma_{s,k} = \sigma_{\Psi^{(s)},\Omega,k}$. Let $L^{(1)}\xi = r^{(n-1)/(2\varepsilon)} L\tilde{\xi}$, where $L\tilde{\xi}$ is given as in Lemma 3.1. For $1 < s \le M$, let n_s denote the number of monomials $u^\alpha = u_1^{\alpha_1} \cdots u_{n-1}^{\alpha_{n-1}}$ of degree $|\alpha| = M - s + 1$ and let $\mathbf{R}^{n_s}$ be labeled by α (i.e., $\mathbf{R}^{n_s} = \{(x_\alpha)\}_{|\alpha| \le M-s+1}$). For $s = 2, \ldots, M$, define $L^{(s)} \in L(\mathbf{R}^m, \mathbf{R}^{n_s})$ by

$$L^{(s)}\xi = \left(r^{M-s+1} \sum_{j=1}^{\tilde{l}} a_{j\alpha} \xi_j \right)_{|\alpha| \le M-s+1}.$$

It follows from (19), (22), and (23) that

$$|\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s+1,k}(\xi)| \le C(2^{d_1 k} |L^{(s)}\xi|) \tag{24}$$

for $s = 1, \ldots, M$. We claim that

$$|\hat{\sigma}_{s,k}(\xi)| \le C(2^{d_1 k} |L^{(s)}\xi|)^{-\gamma_s} \tag{25}$$

holds for $1 \le s \le M$ and some positive exponents γ_s.

Clearly, for $s = 1$, (25) follows from (21) with the choice $\gamma_1 = \varepsilon$. For $1 < s \le M$, let

$$Q_k(t, y, \xi) = t^{d_1} \sum_{|\alpha| \le M-s+1} \left(\sum_{j=1}^{\tilde{l}} a_{j\alpha} \xi_j \right) (y')^\alpha + \sum_{j=\tilde{l}+1}^{m} t^{d_j} \xi_j \Phi_j(y).$$

Then it follows from Lemma 2.5 that, for some $\gamma_s > 0$,

$$\begin{aligned} |\hat{\sigma}_{s,k}(\xi)| &\le \int_1^2 \left| \int_{\mathbf{S}^{n-1}} e^{iQ_k(2^k t, y, \xi)} \Omega(y)\, d\sigma(y) \right| dt \\ &\le C\left(2^{d_1 k} r^{M-s+1} \sum_{|\alpha| = M-s+1} \left| \sum_{j=1}^{\tilde{l}} a_{j\alpha} \xi_j \right| \right)^{-\gamma_s} = C(2^{d_1 k} |L^{(s)}\xi|)^{-\gamma_s} \end{aligned}$$

holds for all $k \in \mathbf{Z}$. Thus (25) holds for $s = 1, \ldots, M$.

In addition to (24) and (25), we have

$$\|\sigma_{s,k}\| \le C \tag{26}$$

and

$$\left\| \sup_{k \in \mathbf{Z}} (|\sigma_{s,k}| * |f|) \right\|_p \le C_p \|f\|_p \tag{27}$$

for $1 \le s \le M$ and $1 < p < \infty$, the latter of which comes as a consequence of the L^p boundedness of the maximal operator

$$f \to \sup_{k\in\mathbf{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(x_1 - t^{d_1}, \ldots, x_m - t^{d_m})|\, dt$$

established in [11] and [16]. Observe that

$$\Psi^{(M+1)}(y) = (|y|^{d_1}\Phi_1(z_0), \ldots, |y|^{d_1}\Phi_{\tilde{l}}(z_0), \Phi_{\tilde{l}+1}(y), \ldots, \Phi_m(y)).$$

By repeating the preceding arguments we can find additional mappings

$$\Psi^{(M+2)}, \ldots, \Psi^{(N)}, \Psi^{(N+1)}$$

from $\mathbf{R}^n\backslash\{0\}$ to $\mathbf{R}^m$ such that

$$\Psi^{(N+1)}(y) = (|y|^{d_1}\Phi_1(z_0), \ldots, |y|^{d_m}\Phi_m(z_0)) \tag{28}$$

and (24)–(27) hold for $\sigma_{s,k} = \sigma_{\Psi^{(s)},\Omega,k}$ (with appropriate choices of $L^{(s)}$, γ_s and with d_1 replaced by the d_j), $M+1 < s \le N$. It follows from (28) and

$$\int_{\mathbf{S}^{n-1}} \Omega(y)\, d\sigma(y) = 0$$

that $\sigma_{N+1,k} = 0$ for all $k \in \mathbf{Z}$. Applying Lemma 2.7 then gives

$$\|T_{\Omega,\Phi} f\|_{L^p(\mathbf{R}^m)} = \left\| \sum_{k\in\mathbf{Z}} \sigma_{1,k} * f \right\|_{L^p(\mathbf{R}^m)} \le C_p \|f\|_{L^p(\mathbf{R}^m)}$$

for $1 < p < \infty$.

Finally, let us point out that the restriction $z_0 = (0, \ldots, 0, 1)$ can be lifted by using an appropriate rotation on $\mathbf{S}^{n-1}$. For $1/4 \le r \le 2$ the entire process can be greatly simplified because the factor $r^{-(n-1)/2}$ in (21) becomes harmless; we omit the details. This ends the proof of Theorem 1. □

4. Further Results

In this section we discuss how the boundedness result in Theorem 1 can be extended to cover the case where some or all of the d_j are negative.

The first step is to obtain the following variant of Lemma 2.4.

LEMMA 4.1. *Let $n \in N$, $\mu_1, \ldots, \mu_n \in \mathbf{R}$, and $d_1, \ldots, d_n$ be distinct nonzero real numbers. Then there exists a positive constant C independent of $\{\mu_j\}$ such that*

$$\left| \int_\alpha^\beta e^{i(\mu_1 t^{d_1} + \cdots + \mu_n t^{d_n})} \psi(t)\, dt \right| \le C|\mu_1|^{-1/n} \left(|\psi(\beta)| + \int_\alpha^\beta |\psi'(t)|\, dt \right)$$

holds for $1/2 \le \alpha < \beta \le 1$ and $\psi \in C^1([1/2, 1])$.

Lemma 4.1 can be verified by using the Ricci–Stein [10] arguments in Section 3. Combining Lemma 4.1 with the method used in Section 3 allows us to obtain the following.

THEOREM 2. *Let $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfy (4). Let $\Phi : \mathbf{R}^n \to \mathbf{R}^m$ be a homogeneous mapping of degree $d = (d_1, \ldots, d_m)$ with $d_j \ne 0$ for $1 \le j \le m$. Suppose $\Phi\big|_{\mathbf{S}^{n-1}}$*

is real-analytic and set $T_{\Omega,\Phi} f = \sum_{k\in\mathbf{Z}} \sigma_{\Omega,\Phi,k} * f$ *(in the sense of distribution). Then for every* $p \in (1,\infty)$ *there exists a* $C_p > 0$ *such that*

$$\|T_{\Omega,\Phi} f\|_{L^p(R^m)} \le C_p \|f\|_{L^p(R^m)}$$

for all $f \in \mathcal{S}(\mathbf{R}^m)$.

References

[1] A. P. Calderón and A. Zygmund, *On singular integrals,* Amer. J. Math. 78 (1956), 289–309.

[2] R. Coifman and G. Weiss, *Extension of Hardy spaces and their use in analysis,* Bull. Amer. Math. Soc. 83 (1977), 569–645.

[3] L. Colzani, *Hardy spaces on spheres,* Ph.D. dissertation, Washington University, St. Louis, MO, 1982.

[4] W. Connet, *Singular integrals near* L^1, Proc. Sympos. Pure Math., 35 (S. Wainger and G. Weiss, eds.), pp. 163–165, Amer. Math. Soc., Providence, RI, 1979.

[5] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates,* Invent. Math. 84 (1986), 541–561.

[6] D. Fan, K. Guo, and Y. Pan, L^p *estimates for singular integrals associated to homogeneous surfaces,* preprint.

[7] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math. 119 (1997), 799–839.

[8] L. Hörmander, *The analysis of linear partial differential operators, I,* Springer-Verlag, Berlin, 1983.

[9] D. Müller, *Singular kernels supported on homogeneous submanifolds,* J. Reine Angew. Math. 356 (1985), 90–118.

[10] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I: Oscillatory integrals,* J. Funct. Anal. 73 (1987), 179–194.

[11] ———, *Multiparameter singular integrals and maximal functions,* Ann. Inst. Fourier (Grenoble) 42 (1992), 637–670.

[12] F. Ricci and G. Weiss, *A characterization of* $H^1(\Sigma_{n-1})$, Proc. Sympos. Pure Math., 35 (S. Wainger and G. Weiss, eds.), pp. 289–294, Amer. Math. Soc., Providence, RI, 1979.

[13] E. M. Stein, *Oscillatory integrals in Fourier analysis,* Beijing lectures in harmonic analysis (Beijing, 1984), pp. 307–356, Princeton Univ. Press, Princeton, NJ, 1986.

[14] ———, *Problems in harmonic analysis related to curvature and oscillatory integrals,* Proc. Internat. Congr. Math., 1 (Berkeley, CA, 1986), pp. 196–221, Amer. Math. Soc., Providence, RI, 1987.

[15] ———, *Harmonic analysis: Real-variable methods, orthogonality and oscillatory integrals,* Princeton Univ. Press, Princeton, NJ, 1993.

[16] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature,* Bull. Amer. Math. Soc. 84 (1978), 1239–1295.

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