The Grunsky Operator and the Schatten Ideals

Gavin L. Jones

1. Introduction

Let J be a bounded Jordan curve in the complex plane. It divides the Riemann sphere into two simply connected Jordan domains Ω, Ω^* with Riemann maps g, g_* from the unit disc U and the exterior of the closed disc U^*, which extend as homeomorphisms of the boundary. We study the Grunsky operator Γ_g (defined in the next section) and its relationship to the welding homeomorphism $h = g_*^{-1} \circ g$ of the unit circle to itself for certain classes of smooth quasicircles. We recall two theorems (which are equivalent).

Theorem 1.1 (Pommerenke [7]). *Let g be a conformal map of the unit disc to a simply connected region Ω. Then $\partial\Omega$ is a quasicircle if and only if the Grunsky operator Γ_g, acting on the Dirichlet space, has norm less than* 1.

Theorem 1.2 (Beurling and Ahlfors [3]). *Let J be a Jordan curve in the plane with welding h. Then J is a quasicircle if and only if the composition operator $V_h \colon f(z) \to f(h(z))$ is bounded on the Dirichlet space.*

These theorems are related by the idea of a conformal map acting as a composition operator. We will sketch this in Section 2, and in the remainder of the paper will prove the following two theorems.

Theorem 1.3. *Let g be a conformal map of the unit disc to the interior of a Jordan curve. The Grunsky operator lies in the pth Schatten ideal γ_p $(p \geq 1)$ of operators on the Dirichlet space if and only if $\log g' \in B_p$, the Besov space.*

Theorem 1.4. *Let J be a quasicircle with welding h. The commutator $[V_h, H]$ of V_h with the Hilbert transform H lies in γ_p if and only if $\log g' \in B_p$, where g is the conformal map to the interior.*

The proofs of the theorems will be straightforward applications of atomic decompositions of Bergman spaces and quasiconformal estimates, given our initial descriptions of the welding and the Grunsky operator. We note the obvious analogy

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2. Composition Operators

We start by working with conformal maps as composition operators. Let $A(S^1)$ denote those analytic functions analytic in a neighbourhood of S^1, and let $A(\Omega)$ denote the analytic functions in a region Ω. Consider the composition operator $W\colon f \to f(g^{-1}(\cdot))$ that acts from $A(S^1)$ to $A(U) + A(U^*)$ by first passing to $A(\Omega) + A(\Omega^*)$, under composition and the Cauchy integral, and then by the conformal isomorphisms to $A(U) + A(U^*)$. Working in the topology of locally uniform convergence, one needs only perform the Cauchy integral on smooth curves.

One checks that, with respect to analytic functions on U and U^*, W has a matrix of the form

$$\begin{pmatrix} I & \Gamma \\ 0 & \Delta \end{pmatrix}.$$

Now Γ is commonly referred to as the Grunsky operator, and Δ consists of the Faber transform [4] (i.e., composition by g^{-1} followed by analytic projection to Ω via the Cauchy integral), followed by the composition with g. In the same way, g_*^{-1} induces a map W_* with matrix

$$\begin{pmatrix} \Delta_* & 0 \\ \Gamma_* & I \end{pmatrix}.$$

We may study the operator

$$\Lambda_J = \begin{pmatrix} \Gamma & \Delta_* \\ \Delta & \Gamma_* \end{pmatrix} j,$$

where the operator j acts as $f(z) \to f(1/z)$ on $A(S^1)$ and the operator Λ_J acts on $A(U) + A(U_*)$. Grunsky, using an integral representation, proved that Λ_J is an isometry of $D(S^1)$, the Dirichlet space, if and only if J has zero area [7, Thm. 4.1]. We recall that

$$D(S^1) = \left\{ f(z) = \sum a_n z^n \in L^2(S^1) : |f|_D^2 = \sum |n||a_n|^2 \right\} < \infty.$$

This splits into D_+ and $D_- = D \ominus D_+$, the analytic and co-analytic parts with projections P^+ and P^-. We can now show equivalence of the first two theorems as follows. If J is a quasi-circle then the welding operator $f \to f \circ h$ is bounded on the Dirichlet space, by Theorem 1.2. But $V_h\colon f \to f \circ h$ has matrix

$$V_h = W^{-1} W_* = \begin{pmatrix} \Delta_* - \Gamma\Delta^{-1}\Gamma_* & -\Gamma\Delta^{-1} \\ \Delta^{-1}\Gamma_* & \Delta^{-1} \end{pmatrix}.$$

Hence, if J is a quasicircle then Δ is invertible. Since Λ_J is an isometry, the invertibility of Δ forces $|\Gamma| < 1$. Now, by Theorem 1.1, J is a quasicircle.

The action of the Hilbert transform is that of the matrix

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

so

$$[V_h, H] = 2\begin{pmatrix} 0 & \Gamma\Delta^{-1} \\ \Delta^{-1}\Gamma_* & 0 \end{pmatrix}.$$

Therefore, if Δ is invertible then conditions for compactness et cetera of the Grunsky operators are equivalent to conditions on the welding commutator. To prove the Schatten class criterion, we will look at the necessity in terms of the Grunsky operator in the next section, and the sufficiency will be studied from the viewpoint of welding.

3. The Grunsky Operator

In this section we prove the necessary part of Theorem 1.3. Grunsky defined the operator Γ, which appeared in the previous section, using the kernel

$$\log \frac{g(z) - g(\zeta)}{(z - \zeta)} = \sum_{k,l} b_{k,l} z^k \zeta^l.$$

This induces an operator Γ on the Dirichlet space D_+ (via the matrix $\{b_{k,l}\}$ with respect to the basis $\{z^n/n^{1/2}\}$). Hence by differentiating we obtain an operator $\Gamma_g = \Gamma j$ on the Bergman space $L^2_+(U)$ of square-integrable holomorphic functions on the unit disc:

$$\Gamma_g f(z) = \int S_g(z, \zeta) f(1/\zeta)\, dA(\zeta),$$

with

$$S_g(z, \zeta) = \frac{1}{(z - \zeta)^2} - \frac{g'(z) g'(\zeta)}{(g(z) - g(\zeta))^2}.$$

We note that $6S_g(z, z) = S_g(z)$ is the Schwarzian derivative of g at z and is related to the deviation from conformality [5]. Recall that Γ_g lies in the pth Schatten class γ_p $(p \geq 1)$ if and only if

$$\sum |\langle \Gamma_g e_n, e_n \rangle|^p < \infty$$

for any orthonormal basis e_n.

LEMMA 3.1. *Let g be a conformal map $g\colon U \to \Omega$, and let $p \geq 1$. If the Grunsky operator Γ_g lies in γ_p of the Dirichlet space, then*

$$\int_U |S_g(z, z)|^p (1 - |z|^2)^{2p-2}\, dm(z) < \infty.$$

We denote the hyperbolic area density $d\lambda(z) = dm(z)/(1 - |z|^2)^2$; the hyperbolic metric is ρ.

Proof. Pick $r > 0$ sufficiently small as well as a hyperbolic lattice $\{z_i\}$ in the unit disc so that $\rho(z_i, z_j) \geq r\delta_{i,j}$ and $\inf_{j \neq i} \rho(z_i, z_j) \leq 2r$. Let e_n denote the standard basis of l_2. Now define a map $A\colon l^2 \to L^2_+(U)$ by $Ae_n = k_{z_n}$, where $k_{z_n}(\zeta) = 1 - |z_n|^2/(1 - \overline{z_n}\zeta)^2$, the unit reproducing kernel for z_n in $L^2_+(U)$. From [9], it now follows that A is bounded and surjective for r sufficiently small; hence, if $\Gamma_g \in \gamma_p$ then $A^*\Gamma_g A \in \gamma_p$. But

$$\sum |\langle A^*\Gamma_g A e_n, e_n\rangle|^p < \infty$$

implies

$$\sum |\langle \Gamma_g k_{z_n}, k_{z_n}\rangle|^p = \sum |S_g(z_n, z_n)|^p (1-|z_n|^2)^{2p} < \infty.$$

From [6] we have that if $d\lambda_\alpha(z) = (1-|z|^2)^\alpha dm(z)$ for $\alpha > -1$ then

$$|f|^p_{L^p(d\lambda_{(\alpha)})} \approx \sum |f(z_n)|^p (1-|z_n|^2)^{\alpha+2}. \tag{$*$}$$

Applying this for $\alpha = 2p-2$, we have

$$|\Gamma_g|_{\gamma_p} \geq c_p \int_U |S_g(z,z)|^p (1-|z|^2)^{2p-2}\, dm(z),$$

proving the lemma. $\square$

However, Theorem 1.3 gave a criterion in terms of $\log g'$ rather than the Schwarzian derivative. The Besov spaces B_p for $p > 1$ are defined as those analytic functions in the disc f with

$$(1-|z|) f'(z) \in L^p(d\lambda).$$

The minimal space B_1 consists of f with $f'' \in L^1$.

LEMMA 3.2. *Let g be a conformal map of the unit disc. Then the Schwarzian derivative S_g lies in $L^p(d\lambda_{2p-2})$ if and only if* $\log g' \in B_p$ *for* $p \geq 1$.

Proof. Given a disc D_i, we write

$$|S_g|_{D_i} = \sup_{z \in D_i} |S_g(z)| d(z, \partial D_i)^2.$$

Now cover U by isometric hyperbolic discs D_i, having centers z_i, with at most finite multiplicity. From $(*)$ we have that $S_g \in L^p(d\lambda_{2p-2})$ implies $\sum |S_g|^p_{D_i} < \infty$.

Recall [5] that if g is a conformal map of the disc u to the complex plane with $|S_g|_U < 2$, then g extends to G a quasiconformal map of the plane with $G|_U = g$ and dilatation μ_G such that $|\mu_G|_\infty = \frac{1}{2}|S_g|_U$. Set $\phi_g = g''/g'$. From [5] we then have

$$|\phi_g(0)| \leq |S_g|_U.$$

On D_i (center z_i) this gives $|\phi_g(z_i)|(1-|z_i|^2) \leq C|S_g|_{D_i}$. Hence $S_G \in L^p(d\lambda_{2p-2})$ shows that $\sum[|\phi_g(z_i)|(1-|z_i|^2)]^p$ converges, and $(*)$ implies $\log g' \in B_p$ for $p > 1$. At $p = 1$, $S_g \in L^1(dm)$ gives $\phi_g \in L^2(dm)$. So $\phi_g' = S_g + \frac{1}{2}\phi_g^2 \in L^1(dm)$ and $\log g' \in B_1$.

Conversely, easy estimates show that

$$|S_g|_{L^p(d\lambda_{2p-2})} \leq 2|\log g'|_{B_p}. \qquad \square$$

We may also give the condition in terms of the dilatation μ of G as follows.

LEMMA 3.3 [2; 8]. *Let g be a conformal map of U to the interior of a Jordan curve J so that* $\limsup_{|z|\to 1}|S_g(z)|(1-|z|^2)^2 = 0$. *Then g extends to a quasiconformal map G of a neighborhood of the unit disc with dilatation*

$$\mu_G(1/\bar{z}) = -\tfrac{1}{2}(z/\bar{z})^2(1-|z|^2)^2 S_g(z).$$

This yields our next lemma.

LEMMA 3.4. *Let g be a conformal map of U to the interior of a Jordan curve J, and let $p \geq 1$. Then the operator $\Gamma_g \in \gamma_p$ only if there is a quasiconformal extension G of g with $\mu_G \in L^p(d\lambda)$.*

4. Welding and Sufficiency

In this section we conclude the proofs of Theorems 1.3 and 1.4 by showing that the Besov condition is sufficient for γ_p membership. We do this by simple quasiconformal calculations, and we will only compute the cases $p = 1, 2$ since the others follow by trivial applications of Holder's inequality.

To prove the sufficiency of Theorem 1.4 we need to show that if $\log g' \in B_p$ then $P^- V_h P^+ \in \gamma_p$. From Lemmas 3.2 and 3.3 we may assume that h is a quasisymmetric homeomorphism of the circle with quasiconformal extension H to the unit disc such that $\mu_H \in L^p(d\lambda)$, and with H smooth [5].

LEMMA 4.1. *Let f be a Dirichlet finite analytic function in the unit disc, and let h be a quasisymmetric homeomorphism of S^1. If H is a C^1 quasiconformal homeomorphism of the unit disc U with boundary map h, then*

$$|P^- V_h f|_D \leq |\bar{\partial} f \circ H|_{L^2}.$$

We identify f with its boundary values, recalling that the Poisson kernel P takes functions on S^1 to functions on the unit disc. The inner product on D is of the form

$$\langle f, g\rangle_D = \langle \partial f, \partial g\rangle_{L^2} + \langle \bar{\partial} f, \bar{\partial} g\rangle_{L^2}.$$

Now observe that Dirichlet finite functions in the unit disc split into three orthogonal subspaces, D_+, D_-, D_0. Hence $f \circ H = F_+ + F_- + F_0$ and we have

$$|P^- V_h f|_D = |\bar{\partial} F_-|_{L^2} \leq |\bar{\partial} f \circ H|_{L^2}.$$

We now need to control the γ_p norms of $P^- V_h$, recalling that (for $p \geq 2$) $B \in \gamma_p$ if and only if $\sum |Be_n|^p < \infty$ for all orthonormal bases e_n [9].

LEMMA 4.2. *Let H be a C^1 quasiconformal map of the unit disc. Then $P^- V_H$, the operator of composition with H followed by the antiholomorphic projection, lies in $\gamma_p(D_+, D_-)$ for $p \geq 2$ if*

$$\int |\mu_{H^{-1}}(z)|^p \, d\lambda < \infty.$$

From the previous lemma, $|P^- V_H e_n|_{L^2} \leq |\bar{\partial} e_n(H(\cdot))|_{L^2}$. Hence

$$|P^- V_H e_n|^2 \leq \int_U |\bar{\partial} e_n(H(z))|^2 \, dm(z)$$
$$= \int_U |\partial e_n(H(z)) \bar{\partial} H(z)|^2 \, dm(z)$$

and, setting $\zeta = H(z)$,

$$= \int_U |\partial e_n(\zeta)|^2 |\bar{\partial} H(H^{-1}\zeta)|^2 \, dm(z).$$

But $J_H(z) dm(z) = dm(\zeta)$ and $J_H(z) = |\partial H(z)|^2 - |\bar{\partial} H(z)|^2$, so

$$|P^- V_H e_n|^2 \leq \frac{1}{1 - |\mu|_\infty^2} \int_U |\partial e_n(\zeta)|^2 |\mu_{H^{-1}}(\zeta)|^2 \, dm(\zeta).$$

Thus

$$|P^- V_H e_n|^2 \leq 2K \int_U |\partial e_n(\zeta)|^2 |\mu_{H^{-1}}(\zeta)|^2 \, dm(\zeta).$$

Working in the Bergman space [9], we have

$$\sum |\partial e_n(z)|^2 = 1/(1 - |z|^2)^2,$$

so

$$\sum |P^- V_H e_n|^2 \leq 2K \int_U |\mu_{H^{-1}}(\zeta)|^2 \, d\lambda(\zeta).$$

This gives the $p = 2$ condition. To control $p \in (2, \infty)$, we apply Holder's inequality with exponents $p/2$ and $p/p - 2$, recalling that $\int |\partial e_n(\zeta)|^2 \, dm(\zeta) = 1$ since e_n are unit vectors.

We must work a little harder for $p \in [1, 2)$.

LEMMA 4.3. *Let H be a C^1 quasiconformal map of the unit disc. Then $P^- V_H$, the operator of composition with H followed by the antiholomorphic projection, lies in $\gamma_1(D_+, D_-)$ if*

$$\int |\mu_{H^{-1}}(z)| \, d\lambda < \infty.$$

From [9], it is sufficient to show that our condition forces

$$\sum |\langle P^- V_H e_n, \overline{e_n} \rangle| < \infty$$

for any orthonormal basis e_n of D_+. We then estimate

$$\begin{aligned}
&|\langle P^- V_H e_n, \overline{e_n} \rangle_D| \\
&\quad \leq \int |\partial e_n(H(z))| |\mu_H(z)| |\partial H(z)| |\partial e_n(z)| \, dm(z) \\
&\quad \leq \left[\int |\partial e_n(z)|^2 |\mu_H(z)| \, dm(z) \right]^{1/2} \\
&\qquad \times \left[\int |\partial e_n(H(z))|^2 |\mu_H(z)| |\partial H(z)|^2 \, dm(z) \right]^{1/2}
\end{aligned}$$

by the Cauchy–Schwarz inequality.

The second of these integrals, changing variables to $\zeta = H(z)$ for H, K quasi-conformal, gives

$$\leq \left[\int |\partial e_n(\zeta)||\mu_H(H^{-1}(\zeta))| J_H(H^{-1}(\zeta)) J_H^{-1}(H^{-1}(\zeta)) 2K \, dm(\zeta) \right]^{1/2}$$
$$\leq (2K)^{1/2} \left[\int |\partial e_n(\zeta)|^2 |\mu_{H^{-1}}(\zeta)| \, dm(\zeta) \right]^{1/2}.$$

Therefore,

$$\begin{aligned}
&\sum |\langle P^- V_H e_n, e_n \rangle| \\
&\quad \leq (2K)^{1/2} \sum \left[\int |\partial e_n(z)|^2 |\mu_H(z)| \, dm(z) \right]^{1/2} \\
&\qquad \times \left[\int |\partial e_n(z)|^2 |\mu_{H^{-1}}(z)| \, dm(z) \right]^{1/2} \\
&\quad \leq (2K)^{1/2} \left[\sum \int |\partial e_n(z)|^2 |\mu_H(z)| \, dm(z) \right]^{1/2} \\
&\qquad \times \left[\sum \int |\partial e_n(z)|^2 |\mu_{H^{-1}}(z)| \, dm(z) \right]^{1/2} \\
&\quad \leq (2K)^{1/2} \left[\int |\mu_H(z)| \, d\lambda(z) \right]^{1/2} \left[\int |\mu_{H^{-1}}(z)| \, d\lambda(z) \right]^{1/2}.
\end{aligned}$$

But our map H is a quasi-isometry of the hyperbolic metric, so

$$c' < |J_H(z)| d\lambda(z) < C'$$

for c', C' bounded by constants depending on $(1 - |\mu_H|_\infty)^{-2}$ [1]. Setting $z = H^{-1}(x)$, we thus have

$$\begin{aligned}
\left[\int |\mu_H(z)| \, d\lambda(z) \right]^{1/2} &= \left[\int |\mu_H(H^{-1}(x))| \, d\lambda(z) \right]^{1/2} \\
&\leq \left[\int C' |\mu_{H^{-1}}(x)| \, d\lambda(x) \right]^{1/2}.
\end{aligned}$$

This yields

$$|P^- V_H|_{\gamma_1} \leq C \int |\mu_{H^{-1}}(z)| \, d\lambda(z),$$

where C has a bound of order at worst $(1 - |\mu_H|_\infty)^{-3/2}$. To deal with $p \in (1, 2)$ we use Holder's inequality as before. We observe that $\mu_H \in L^p(d\lambda)$ if and only if $\mu_{H^{-1}} \in L^p(d\lambda)$.

Proofs of Theorem 1.3 and Theorem 1.4. The theorems stated at the outset now follow. For Theorem 1.3, Lemmas 3.1 and 3.2 prove necessity. Conversely, given a Jordan curve J with conformal mapping g from U to the interior, suppose $\log g' \in B_p$. Then J is a quasicircle, by Lemma 3.2 and 3.3. Therefore the operator Δ^{-1} is bounded, and $\Gamma_g \in \gamma_p$ if there is a quasiconformal extension G of g with dilatation

μ_G satisfying $\mu_G \in L^p(d\lambda)$, by Lemmas 4.1 and 4.2. Such an extension is given by Lemma 3.3. This proves Theorem 1.3.

In order to prove Theorem 1.4, first we note that if $[V_h, H] \in \gamma_p$ then $\Gamma_g \in \gamma_p$, and so $\log g' \in B_p$. Conversely, given J a Jordan curve and g a conformal map to its interior with $\log g' \in B_p$, we have (from Lemma 3.4) that g extends to a quasiconformal homeomorphism of the plane with $\mu_G \in L^p(d\lambda)$. Thus, the welding homeomorphism h of the unit circle has a quasiconformal extension H to the unit disc with $\mu_H \in L^p(d\lambda)$. Hence, from Lemmas 4.2 and 4.3, $P^- V_h P^+ \in \gamma_p$ and, running the same arguments, we obtain $P^+ V_h P^- \in \gamma_p$, proving the theorem.

From the argument we see that Theorems 1.3 and 1.4 remain true if $\log g' \in B_p$ is replaced by $S_g \in L^p(d\lambda_{2p-2})$, or by the condition that g admits a quasiconformal extension to the plane with $\mu_G \in L^p(d\lambda)$.

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Department of Mathematical Sciences
University of Liverpool
Liverpool L69 3BX
United Kingdom

gljones@liv.ac.uk