Essential Surfaces and Tameness of Covers

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Suppose that M is a closed orientable 3-manifold. An *essential surface* S in M is a π_1-injective map of a closed surface S to M. Throughout this paper, we will restrict ourselves to the case of S being orientable. The surface cover M_S of S is said to be *topologically tame* if it is homeomorphic to $S \times R$. Equivalently, there is a compactification of the surface cover homeomorphic to $S \times [-1, 1]$. In this paper we establish tameness of surface covers for two natural classes of essential surfaces. The first class we call *topologically finite*. The defining properties of such surfaces are conditions that are easily seen to be satisfied by quasi-Fuchsian surfaces in hyperbolic 3-manifolds (cf. [RS1]). Using geometric techniques, it is straightforward to check that quasi-Fuchsian surfaces in a hyperbolic 3-manifold have topologically tame surface covers. In fact, M_S can be compactified by adding the quotient of the domain of discontinuity by the action of $f_*(\pi_1(S))$ at infinity (see e.g. [Th]). We give an alternate topological proof of tameness by using three important properties of geometrically finite surfaces. The proof is reminiscent of an argument in [HRS] that establishes this tameness for the case when S is a torus.

The second class of essential surfaces that we address are called *strongly filling*. *A filling surface* S is essential and satisfies the condition that every noncontractible loop in the 3-manifold always intersects every surface in the homotopy class of S. This immediately implies that the complementary regions of S are all simply connected when a least area representative surface is picked in the homotopy class of S, regardless of the choice of metric. In order to make this into a property that ensures tameness, we need a notion of strongly filling for an essential surface S. Strongly filling for S means that, in the universal cover of M, all pairs of points (that are sufficiently far apart) are separated by many disjoint planes lying over S. See Section 2. For quasi-Fuchsian surfaces in hyperbolic 3-manifolds, we show that it is sufficient to assume that every geodesic line has endpoints separated by at least one plane in the preimage of S. If S is totally geodesic in the hyperbolic case, we establish that strongly filling and filling are equivalent. Finally, we will show that if all essential surfaces are either strongly filling or topologically finite, then topological tameness is always true not only of the surface cover but also of any cover with finitely generated and freely indecomposable fundamental group. It is a well-known result of Simon [Si] that this is true for Haken 3-manifolds.

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Topologically Finite Surfaces

Suppose that M is a closed, orientable, irreducible 3-manifold and that $f : S \to M$ is an essential surface in M. Let H denote $f_*(\pi_1(S))$. Following [FHS], we will assume that f is a small perturbation of a least-area map, as one always exists in a given homotopy class of maps from S to M. This perturbation is chosen so that f is in general position and, in the universal covering of M, the complete inverse image of $f(S)$ consists of embedded planes meeting along lines. By an abuse of notation we will just say that f is "least area", rather than continually referring to the necessity of choosing a small perturbation. We will often consider the intermediate cover M_S associated to H. We let Π_S denote the collection of lifts of S to M_S. Each such lift consists of a map from some cover S' of S to M_S; we will say that such a lift is a plane if S' is a plane and the map to M_S does not factor through any covering maps. In Π_S we have a natural compact embedded surface, the lift of the map f to M_S, which we will denote S_c. We say that S is *topologically finite* if it satisfies the following three conditions.

(1) For every conjugate H^g of H, $H^g \cap H$ is finitely generated.
(2) There are finitely many lifts in Π_S that are not planes.
(3) There is a finite cover M'_S of M_S in which every lift of S is embedded.

It is not hard to see that quasi-Fuchsian surfaces in 3-manifolds satisfy these three properties (see [RS1]).

Our aim in this section is the following.

THEOREM 1.1. *If S is a topologically finite essential surface, then M_S is topologically tame.*

As in the work of Simon [Si], we will make use of the following algebraic result due to Cohen.

LEMMA 1.2. *Suppose G is finitely generated and splits as a graph of groups where all the edge groups are finitely generated. Then all the vertex groups are finitely generated.*

It follows from Tucker's criterion for tameness (see [T]) that M_S is topologically tame if and only if some finite cover of M_S is topologically tame. So, replacing M_S by M'_S, we may assume that all the lifts of S to M_S are embeddings.

The main technical lemma is the following.

LEMMA 1.3. *Suppose $\mathcal{P} \subset \Pi_S$ is some collection of lifts of S to M_S. Then*

(1) *Every component of $M_S \setminus \bigcup_{P \in \mathcal{P}} P$ has finitely generated fundamental group.*
(2) *All but finitely many components of $M_S \setminus \bigcup_{P \in \mathcal{P}} P$ are simply connected.*

Proof. We enumerate $\mathcal{P} = \{P_1, P_2, \ldots\}$. Let $\mathcal{S}_i = \{P_1, \ldots, P_i\}$ and let $\mathcal{C}_i$ denote the collection of components of $M_S \setminus \bigcup_{P \in \mathcal{S}_i} P$. Finally, let $\mathcal{C}$ denote the set of components of $M_S \setminus \bigcup_{P \in \mathcal{P}} P$. We first prove the claim for each $\mathcal{S}_i$. Note that, by assumption, only finitely many P_j are not planes. Thus we suppose that $P_1, \ldots, P_n$ are not planes, and that P_m is a plane, for any $m > n$.

Let $\mathcal{L}$ denote the curves of intersection of the elements of $\mathcal{P}$. As in [FHS], one sees that only finitely many elements of $\mathcal{L}$ are closed curves, while the rest are lines. Otherwise, since there are only finitely many lifts that are not planes but are of finite type, one could find a pair of double curves bounding annuli on two different lifts, a possibility ruled out by an area swap argument.

We prove the claim for $\mathcal{S}_i$ by induction on i. For $i = 1$ the claim follows from Lemma 1.2. For $i > 1$, the elements of $\mathcal{C}_i$ are obtained from those of $\mathcal{C}_{i-1}$ by cutting a given component of $M_S \setminus \bigcup_{P \in \mathcal{S}_{i-1}} P$ along a collection of surfaces, all of which are obtained from P_i cut along double curves and lines in $\mathcal{L}$. Thus, if $N \in \mathcal{C}_{i-1}$, one gets a splitting of $\pi_1(N)$ as a graph of groups, where the vertex groups are fundamental groups of elements of $\mathcal{C}_i$ and the edge groups are all finitely generated. It follows from Lemma 1.2 that all the vertex groups are finitely generated, establishing part 1 of the claim for $\mathcal{S}_i$. Since only finitely many of the edge groups are nontrivial, Grushko's theorem ensures that only finitely many of the vertex groups are nontrivial; together with the induction hypothesis, this establishes part 2 of the claim for $\mathcal{S}_i$.

We now need to establish the claim for the possibly infinite set $\mathcal{P}$. The elements of $\mathcal{C}$ are obtained from the elements of $\mathcal{C}_n$ by cutting along subsurfaces of the planar lifts $P_{n+1}, \ldots$. Thus, the fundamental group of every element of $\mathcal{C}_n$ splits as a graph of groups, where the edge groups are trivial and the vertex groups are fundamental groups of elements of $\mathcal{C}$. It follows from Grushko's theorem that finitely many of these vertex groups are nontrivial and that they are all finitely generated. Since the claim is true for $\mathcal{S}_n$, only finitely many of the elements of $\mathcal{C}_n$ have nontrivial fundamental group. It then follows that the claim is true for $\mathcal{C}$. □

Proof of Theorem 1.1. We proceed in a fashion similar to the proof of Theorem 2.1 in [HRS]. Let Π_S denote the collection of lifts of S to M_S. It suffices to engulf each compact region X in a product region (i.e., a homeomorphic copy of $S \times I$ in M_S for which inclusion is a homotopy equivalence). We first expand X so that it is a compact connected 3-manifold whose boundary is incompressible to the outside. Moreover, we can arrange that $\pi_1(X)$ surjects onto $\pi_1(M_S)$ by including S_c in X. Let $\{P_1, ..., P_n\}$ denote the collection of lifts intersecting X and let $\Pi_i = \Pi_S \setminus \{P_1, \ldots, P_i\}$. If M is Haken then the result follows directly from Simon's theorem. If M is not Haken then, by [HRS], the closures of the complementary regions of S in M are π_1-injective handlebodies. It then follows from Simon's theorem that the closures of all the components of $M_S \setminus \bigcup_{P \in \Pi_S} P$ are almost compact. Now one can successively apply Simon's theorem to obtain that the closure of each complementary component of $M_S \setminus \bigcup_{P \in \Pi_i} P$ is almost compact. Next consider the component N of $M_S \setminus \bigcup_{P \in \Pi_n} P$ containing X. This N is homeomorphic to $S \times I$ with a closed set from the boundary removed. One can thus find a product region containing X as required. □

Strongly Filling Surfaces

As before, we assume that M is a closed, orientable, irreducible 3-manifold. A surface S may have a finite number of components.

DEFINITION. An essential surface S immersed in a 3-manifold M satisfies the *k-plane property* for some positive integer k if, when S is represented as a least area map and given any collection of k planes lying over S in the universal covering of M, there exists a disjoint pair.

Recall that in [RS1] it is shown that any quasi-Fuchsian surface in a hyperbolic 3-manifold satisfies the k-plane property for some k. In [RW], an example is given of an immersed π_1-injective surface in a graph manifold that fails to have the k-plane property for any k.

DEFINITION. An essential surface $f : S \to M$ is *filling* if every surface in the homotopy class of f meets every loop in M that is homotopically nontrivial.

Note that if a least area representative g of the homotopy class of some essential surface f is chosen for some metric on M, then by [HRS] every complementary region of $M \setminus g(S)$ is a π_1-injective handlebody. So, if f is filling then all the complementary regions of $g(S)$ are balls—that is, a least area representative of the surface is (geometrically) filling.

We would like to thank Ian Leary for correcting an earlier definition of filling that did not achieve the correct property. Note that one can always homotope f so that there are complementary regions that are not balls yet their images in $\pi_1(M)$ are trivial. Assume that f is least area, so that all the preimages of $f(S)$ in the universal cover $\tilde{M}$ of M are embedded planes. An important special case of filling is if, in $\tilde{M}$, each nontrivial loop of M is homotopic to a curve that lifts to a line meeting some plane lying over $f(S)$ in a single point. In this case, any homotopy of f cannot remove this essential intersection. For example, if $f(S)$ is hyperbolically totally geodesic and is filling then it has this property. For in this case, each nontrivial loop can be homotoped to a geodesic curve that lifts to a geodesic line meeting the planes over $f(S)$ in at most single points, or the line lies in some plane. Indeed, if the loop is homotopic into S then its geodesic representative lies in S. But then it is easy to see that the filling property implies that all the complementary regions of the double curves of $f(S)$ pulled back to S are disks. Hence, in the universal covering of M, the loop lifts to a line in some plane over S that meets other planes in single points. Note we can define that a proper line L in $\tilde{M}$ has ends separated by a plane $\tilde{P}$ lying over S if, in the surface cover M_S, choosing $\tilde{P}$ to map to S_c entails that L maps to a line going from one end of M_S to the other.

Based on the discussion in the previous paragraph, we want to define the concept of strongly filling surfaces. To measure distances, we introduce the word metric d on the Cayley graph $\mathcal{G}$ of $\pi_1(M)$, embedded in the universal covering $\tilde{M}$ of M. There is a choice of generators of $\pi_1(M)$ involved in the construction, and we need to ensure that the definition is independent of this choice. The key idea is that if two vertices in the Cayley graph are far enough apart then they lie on opposite sides of some plane $\tilde{P}$ projecting to S in M. Choose some base point x_0 in $\tilde{M}$ and let $x_1, x_2, \ldots$ be all the translates of the base point by the covering translations. Choose a generating set for $\pi_1(M)$ and use this to define a Cayley graph $\mathcal{G}$ for $\pi_1(M)$ with vertices $x_0, x_1, x_2, \ldots,$ one for each element of $\pi_1(M)$ and with edges labeled by the generating set. Then the word metric d is defined on pairs of vertices and it measures the minimal number of edges in any path between the vertices. Notice that any two choices of generating sets give path metrics d and d' such that there exist constants k_1 and k_2 satisfying, for any pair of vertices x_i and x_j, $k_1 d'(x_i, x_j) < d(x_i, x_j) < k_2 d'(x_i, x_j)$.

Definition. Suppose that S is an essential surface in M with the k-plane property for some k. Then S is *strongly filling* if—for any positive integer n, for a suitable choice of a large enough constant α and any generating set for $\pi_1(M)$, and for any two points x_i and x_j in $\tilde{M}$—if $d(x_i, x_j) > \alpha$ then there are at least n disjoint planes $\tilde{P}_1, \ldots, \tilde{P}_k$ lying over S and separating x_i and x_j.

Clearly the constant α depends on a number of choices, including the base point, a metric on M, and a least area representative of S. By the previous paragraph, we see that changing the choice of generators and metric to d' only changes α to a new constant. Given a homotopy from S to any other least area map S', there is an α sufficiently large that the separation property of translations of the base point for the new surface S' is retained. This follows because the distance between S and S' is bounded and there is a lower bound for the distance between disjoint planes lying over S. So, if n is chosen large enough, then there will still be many disjoint copies of planes lying over S' and separating x_i and x_j. (Note that homotopic disjoint least area planes remain disjoint, by [FHS].) Similarly, we see that changing the base point again only moves all the translates by a fixed distance and, the same argument shows that, by taking n large enough, the separation property still works. Hence the definition depends only on the homotopy class of S, as required.

We investigate a number of issues concerning essential strongly filling surfaces elsewhere, including the construction of classes of examples satisfying the strongly filling property. The definition we have adopted is indeed strong and yet is natural, as we can show that a 3-manifold has such a surface if and only if its fundamental group acts freely on a finite-dimensional cubed complex with a metric of nonpositive curvature. Also, there are other (equivalent) formulations of strongly filling surfaces that are easier to check in practice (cf. [RS2]).

We observe some properties of the two conditions—filling and strongly filling.

Lemma 2.1.

(1) *Any strongly filling essential surface is filling.*
(2) *A totally geodesic surface immersed in a closed hyperbolic* 3*-manifold that is filling is also strongly filling.*

(3) *Assume that S is a least area quasi-Fuchsian surface in a closed hyperbolic 3-manifold M. Then S is strongly filling if and only if every geodesic line in the universal cover of M has ends on either side of some plane P lying over S.*

REMARKS. (1) For a totally geodesic surface S in a hyperbolic 3-manifold, if the complementary domains are balls then the surface is filling. The reason is that any least area map S' homotopic to S will have the same property. This is because any essential loop can be homotoped to a geodesic that lifts to a geodesic line in the universal covering. Then the totally geodesic planes lying over S will separate the ends of this line and therefore the same is true after any homotopy. Therefore no essential loop can be disjoint from S'.

(2) The conditions of the third part of Lemma 2.1 imply that the collection of limit quasi-circles of S separate any two points on the sphere at infinity S_∞ of hyperbolic 3-space, which is the universal cover of M. Any quasi-Fuchsian surface has two domains of discontinuity, which are the complementary regions of the limit quasi-circle in S_∞. Therefore, the topology on S_∞ induced by taking the domains of discontinuity as basic open sets is just the usual topology, in this case (see the proof of Lemma 2.1). This is also equivalent to the condition of strongly filling. The weaker assumption—that only lines which cover circles in M have ends separated by planes covering S—clearly implies the filling property.

(3) The proof of the third assertion makes it clear that, for a hyperbolic 3-manifold and a quasi-Fuchsian surface, the strongly filling property could be given by assuming that, there is a constant c such that translates of the base point that are more than distance c apart are separated by a single plane lying over S. We leave this to the reader as an exercise. In [RS2], we will complete this line of investigation by using Lemma 2.1 to show that, for quasi-Fuchsian surfaces, filling is enough to establish strongly filling.

Proof of Lemma 2.1. Suppose that S is strongly filling but not filling. Then we can assume that there is an essential loop C in M that is disjoint from S. Now S lifts to a line L in the universal covering of M, which does not meet any of the planes lying over S. Choose a base point x_0 on L. Now there is a sequence of translates of x_0 along L, and it is easy to see that the distance in the Cayley graph between these translates and x_0 becomes arbitrarily large. Hence we can find such a point x_1 with $d(x_0, x_1) > \alpha$. This implies that x_0 and x_1 are separated by some plane P lying over S and so clearly P meets L, giving a contradiction.

For the second assertion, if M is closed and hyperbolic and S is totally geodesic and filling, then any geodesic line L in the universal cover $\tilde{M}$ of M must cross at least one plane lying over S. For suppose L is disjoint from all such planes. Let Π denote all the planes lying over S in $\tilde{M}$. Then some closure R of a component of $\tilde{M} \setminus \Pi$ must be noncompact, since it contains L. But by [HRS], the closures of the complementary regions of S in M are π_1-injective handlebodies. One of these closures is covered by R and so must contain an essential loop missing S. Hence we conclude that the line L must have crossed some plane lying over S.

To complete the proof of the second assertion, it is convenient to go on to the third assertion. It is obvious that, once we have completed the third part, the second assertion follows as well, using the discussion in the foregoing paragraph. We can start with a least area quasi-Fuchsian surface S in a closed hyperbolic 3-manifold M and a geodesic line L in the universal covering. Assume the condition of the third part—that is, such a line has ends on either side of some plane P lying over S. Notice that this is the same as saying that the limit quasi-circle $\Lambda(P)$ for P has complementary domains U and U' in S_∞, containing the endpoints a and b (respectively) of L. We therefore see that, if all such complementary domains are taken as a basis for the open sets of a topology on S_∞, then this topology is Hausdorff. (See Remark (2) following Lemma 2.1.)

Suppose now that the strongly filling property fails. Then we have a sequence of base points x^n and translates x'^n of these so that $d(x^n, x'^n) > n$, but there is at most a bounded number of disjoint planes lying over S that separate x^n from x'^n for any n. By choosing appropriate subsequences, we can arrange for x^n to converge to a point a in $\tilde{M}$ and for x'^n to converge to b on S_∞. By assumption, there is a plane $\tilde{P}$ lying over S with limit quasi-circle $\Lambda(\tilde{P})$ separating b from any other point d on S_∞. The collection of all such planes clearly contains some that have limit quasi-circles $\Lambda(\tilde{P})$ with arbitrarily small diameter. Hence we can choose a sequence of planes $\tilde{P}_i$ with nested complementary domains U_i for $\Lambda(\tilde{P}_i)$ containing b so that $U_i \subset U_j$ for $i < j$ and $\bigcap U_i = \{b\}$. It is now straightforward to show that, if D_i is the complementary domain of $\tilde{P}_i$ in $\tilde{M}$ with U_i in its closure, then $\bigcap D_i = \emptyset$. We conclude that a is not in all D_i for i sufficiently large and so $\tilde{P}_i$ separates a from b for all such i.

Moreover since the limit quasi-circles of the $\tilde{P}_i$ are disjoint and converging to $\{b\}$, it is easy to check by the least area property [FHS] that a subsequence of the planes $\tilde{P}_i$ are all disjoint. In fact, the least area planes $\tilde{P}_i$ must lie in the convex hulls of their limit quasi-circles $\Lambda(\tilde{P}_i)$. These convex hulls will also converge to $\{b\}$, and it is easy to see the convex hulls have a disjoint subsequence. If we simply separate two such limit quasi-circles by a round circle on S_∞, then the convex hulls cannot intersect.

However, since x^n converges to a and x'^n converges to b, it is clear that for n sufficiently large, x^n and x'^n are on opposite sides of $\tilde{P}_i$ for arbitrarily many i. This gives a contradiction, which completes the proof of the strongly filling property.

To complete the proof of Lemma 2.1 we must show that the strongly filling property implies the condition of the third part of the lemma. Notice first that, by [RS1], there are at most $k-1$ planes lying over S whose limit circles can contain any given point a on the sphere S_∞ in the universal covering $\tilde{M}$. For the surface satisfies the k-plane property, and the arguments in [RS1] show that a collection of more than k planes has a disjoint pair with disjoint limit circles.

Let a and b be any two disjoint points on S_∞. We need to find some plane lying over S that separates these two points. Choose a set of translations of the base point x_n, for n any integer, such that x_n converges to a as $n \to -\infty$ and x_n converges to b as $n \to +\infty$. Moreover, we can arrange that the geodesic ray from

x_{-n} to a makes an angle at a with S_∞ that is within ε of $\pi/2$, and likewise for x_n and b. By strong filling, we can find an arbitrarily large collection of disjoint planes lying over S that separate x_n and x_{-n} for n large. Suppose that none of these planes separates a and b. We can find a subsequence P_i of these planes with limit circles $\Lambda(P_i)$ converging to a point c on S_∞. If c is neither a nor b, we can find a small round circle C centered at c that is disjoint from these limit circles for i large enough. Moreover, the planes P_i are then disjoint from the totally geodesic plane P with limit circle C, since least area planes lie in the convex hull of their limit sets. But C can be chosen small enough that P separates neither a and b nor, by extension, x_n and x_{-n} for n large. Hence we contradict the assumption that P_i separates x_n and x_{-n} for n large.

Finally, assume that $c = a$ (say). Then $\Lambda(P_i)$ converges to a as $i \to \infty$. It remains to show that the domain of discontinuity of P_i, which is converging to a, actually contains a. For i large it is obvious that this domain does not contain b and so we would have separated a and b as required. Suppose this never happens. By assumption, we can find an arbitrarily large number of disjoint planes amongst the P_i that separate x_n and x_{-n} for n large. The limit circles of these planes are shrinking to a but a is not contained in the small domain of discontinuity. The ratio between the diameters of these limit circles must be arbitrarily large, since there are arbitrarily many of them. It is easy then to see that the angle between the geodesic ray from x_{-n} to a must make a very small angle to S_∞, contrary to assumption. In fact, there must be a large ratio between the distance of the smallest limit circle to a and its diameter. There is a small round circle C enclosing this limit circle but not a. The totally geodesic plane P with limit circle C has x_{-n} separated from a. The distance from a to C is large compared to the diameter of C and so the angle conclusion follows immediately. This completes the proof of Lemma 2.1. □

THEOREM 2.2. *Suppose that $f: S \to M$ is an essential strongly filling surface. Then M_S is tame.*

THEOREM 2.3. *Assume that any essential surface in M is either topologically finite or strongly filling. Then any cover of M with finitely generated and freely indecomposable fundamental group is tame.*

REMARKS. (1) It follows from a theorem of Simon's that Theorem 2.3 holds for Haken manifolds.

(2) A nice class of 3-manifolds satisfying the hypotheses of Theorems 1.1 and 2.2 is those admitting a cubulation of nonpositive curvature (see [AR1]). It is also not difficult to verify that, if the manifold is atoroidal, then the collection of canonical essential surfaces in such 3-manifolds are strongly filling, topologically finite, and satisfy the 4-plane property.

(3) Note that these results apply in the case of geometrically infinite surfaces, where the intersections of conjugates of the surface subgroup are infinitely generated.

(4) Both Theorems 2.2 and 2.3 need only part of the definition of strongly filling, as is clear from the following proofs. However, we felt that the full condition of strongly filling is a natural property for essential surfaces to satisfy.

Proof of Theorem 2.2. The strategy is to show that any compact set X in M_S can be engulfed in a product region homeomorphic to $S \times I$. We may arrange easily that X is connected and contains the compact lift S_c of S to M_S. Clearly, since X is compact, it meets finitely many lifts of S to M_S. Consider the remaining lifts of S in M_S. We will show that the closure R of the component of the complement of these lifts that contains X is compact and π_1-injective. In particular, the region R will also contain S_c and hence $\pi_1(R)$ is isomorphic to $\pi_1(S)$. So, by standard results it follows that R is homeomorphic to $S \times I$.

We will work in the universal covering $\tilde{M}$ of M for convenience. Let $\tilde{P}$ be a choice of plane lying over S_c. Now there is a unique lift $\tilde{X}$ of X to $\tilde{M}$ that contains $\tilde{P}$. We will examine the family Π_X of planes in $\tilde{M}$ that project to S and are disjoint from $\tilde{X}$. Our aim is to show that the closure $\tilde{R}$ of the component of $\tilde{M} \setminus \bigcup \Pi_X$ that contains $\tilde{X}$, projects to a compact set R in M_S containing S_c.

Suppose on the contrary that R is not compact. Choose a base point x_0 in $\tilde{M}$ and some Riemannian metric D on M_S that is a lift of a metric on M. Let δ denote the diameter of M with this metric. Then a sequence of points y_i in R can be chosen with the following property: there exists a translate x_i of the base point that projects to z_i in M_S such that $D(y_i, z_i) < \delta$ and $D(y_i, S_c) > i$ for each i.

Now, since $D(y_i, S_c) > i$, it follows immediately that $d(x_0, x_i) \to \infty$ as $i \to \infty$. Thus, by our assumption of strongly filling, there will be an arbitrarily large number of disjoint planes separating x_0 and x_i for i large enough. In particular, by the k-plane property, we can achieve that arbitrarily many of these planes are disjoint from $\tilde{P}$. However, there is a lower bound for Riemannian distance between disjoint planes in $\tilde{M}$, so these planes can be found lying arbitrarily far from $\tilde{P}$. On the other hand, since X is compact, all points in $\tilde{X}$ are a bounded distance from $\tilde{P}$. Therefore we find that some of these planes do not meet $\tilde{X}$. But then this contradicts our assumption that $\tilde{R}$ is a closure of a component of $\tilde{M} \setminus \bigcup \Pi_X$, since we now have found planes of Π_X that separate points of $\tilde{R}$. This completes the proof that $\tilde{R}$ projects to a compact set R in M_S.

Finally, to show that R is π_1-injective, we use [HRS]. Any closure R of a component of the complement of a collection of lifts of S to M_S must be π_1-injective, since its lift R' to the universal cover of M is simply connected by Lemma 1.2 of [HRS]. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. By [S], the cover N of M corresponding to a finitely generated subgroup has a compact core K. The boundary of K can be chosen to be incompressible in the complement of K. Each component Z of the closure of the complement of K is then homotopy equivalent to $S \times \mathbb{R}^+$ for some closed orientable incompressible surface S in the boundary of K. We can now use the results of Theorems 1.1 and 2.2 to conclude that the surface cover corresponding to S (i.e., M_S) is tame and so homeomorphic to $S \times \mathbb{R}$. Then we can lift Z

homeomorphically to M_S. This shows that Z is homeomorphic to $S \times \mathbb{R}^+$ and so N is tame as required. $\square$

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