The Codimension-1 Property in Bergman Spaces over Planar Regions

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1. Introduction

Let G be a bounded planar region containing the origin in the complex plane $\mathbb{C}$. For $1 \le p < \infty$, the Bergman space $L_a^p(G)$ consists of all analytic functions f in G with

$$\|f\|_p = \left(\int_G |f(z)|^p \, dA(z) \right)^{1/p} < \infty,$$

where dA denotes the Lebesgue measure on the complex plane.

Let ϕ be a smooth function with compact support. The Vitushkin localization operator T_ϕ is defined by

$$T_\phi f(z) = \int \frac{f(w) - f(z)}{w - z} \bar{\partial}\phi \, dA(w),$$

where f is a bounded function with compact support. Let $L^p(G)$ be the space of measurable functions that are zero off G, and let

$$\|f\|_p = \left(\int_G |f(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

The Bergman space $L_a^p(G)$ is a closed subspace of the Banach space $L^p(G)$. It is well known that the operator T_ϕ is a bounded linear operator on $L^p(G)$ and leaves $L_a^p(G)$ invariant.

Let $H^\infty(G)$ denote the Banach algebra generated by bounded analytic functions on G. A closed subspace M of $L_a^p(G)$ is an $H^\infty(G)$ *invariant subspace* if it is invariant under multiplication by each bounded analytic function on G. The dimension of M/zM is no less than 1 since zero is in G. An $H^\infty(G)$ invariant subspace M satisfies the *codimension-1 property* if the dimension of M/zM is 1. Let $Z(M)$ be the set of common zeros of functions in M. We say that M has the *division property* if $f(z)/(z-\lambda)$ is in M whenever $\lambda \in G \setminus Z(M)$ and $f \in M$ with $f(\lambda) = 0$. In [5] it was shown that the codimension-1 property is actually equivalent to the division property. For $f_1, f_2, \dots, f_n$ in $L_a^p(G)$, let $[f_1, f_2, \dots, f_n]$ denote the

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$H^\infty(G)$ invariant subspace generated by $f_1, f_2, \dots, f_n$, that is, the $L^p(G)$ closure of the set

$$\{p_1 f_1 + p_2 f_2 + \cdots + p_n f_n,\ p_1, p_2, \dots, p_n \in H^\infty(G)\}.$$

A point λ_0 on the boundary of G is called a *removable point for* $H^\infty(G)$ if every function in $H^\infty(G)$ extends an analytic function in a neighborhood of λ_0.

The structure of invariant subspaces of the Bergman spaces (in particular, over the unit disk) has attracted a lot of attention in recent years. For more information, we refer the reader to [1; 2; 3; 4; 5; 6] and the references therein. Though there are invariant subspaces that do not satisfy the codimension-1 property, such subspaces are difficult to construct (see [2], [3], and [4]). It is always easy to construct invariant subspaces with the codimension-1 property. For example, the invariant subspace $[f]$ ($[f]$ is called a *cyclic* invariant subspace with cyclic vector f) has the codimension-1 property for each f in $L^p_a(G)$. In some cases, such as with the Hardy spaces and certain weighted Dirichlet spaces on the unit disk, all invariant subspaces are cyclic and therefore have the codimension-1 property. Hence, it is interesting to know when an invariant subspace of the Bergman space has this property.

In [6], considering some local conditions of functions in an invariant subspace on the boundary, the second author obtained some sufficient conditions for the invariant subspace of the Bergman space on the unit disk to have the codimension-1 property. Recently, Aleman and Richter obtained in [1] some local integrability conditions on functions in an invariant subspace of the Bergman space on the unit disk—conditions that ensure the invariant subspace has the codimension-1 property. However, it seems that their method works only on the unit disk. In this paper, we continue our work along this line and generalize Aleman and Richter's result to the Bergman space over a general region of the complex plane. The main idea is to use the Vitushkin localization operator to localize functions in the invariant subspace and to show that the codimension-1 property depends only on the local behaviors of functions in the subspace. Even in the open unit disk case, our proof simplifies Aleman and Richter's original proof. The main results of the paper are the following.

THEOREM A. *Let V be an open disk with center $\lambda_0 \in \partial G$, and let $f \in L^p_a(G) \cap L^s(V \cap G, dA)$ and $g \in L^p_a(G) \cap L^{s'}(V \cap G, dA)$, where $1/s + 1/s' = 1/p$. Suppose that $H^\infty(G \cap V)$ is dense in both $L^s_a(G \cap V)$ and $L^{s'}_a(G \cap V)$, and that λ_0 is not a removable point for $H^\infty(G)$. Then the $H^\infty(G)$ invariant subspace $[f, g]$ has the codimension-*1 *property.*

Notice that the condition that $H^\infty(V \cap G)$ be dense in $L^p_a(G \cap V)$ is very weak since it holds if $\partial(V \cap G)$ satisfies some wild analytic capacity conditions (see [7]). The second assumption, that λ_0 is not a removable point for $H^\infty(G)$, is also wild since most planar regions satisfy the condition.

COROLLARY B. *Let V and λ_0 satisfy the conditions in Theorem A. Suppose that a function f in $L^p_a(G)$ is bounded on $V \cap G$. Then every $H^\infty(G)$ invariant subspace containing f has the codimension-*1 *property.*

2. Proofs of the Results

The following properties about the Vitushkin operator T_ϕ are well known.

LEMMA 1. *Let ϕ be a smooth function with support in V. The Vitushkin localization operator T_ϕ satisfies the following properties*:

(1) *the function $T_\phi f$ is analytic on $G \cup (\operatorname{supp}\phi)^c$ for each function f in $L^p_a(G)$*;
(2) *T_ϕ is a bounded linear operator on $L^p_a(G)$*;
(3) *if $f \in L^p_a(G) \cap L^s(G \cap V)$ where $s > p$, then for every compact set K*

$$\int_K \left(\int \left| \frac{f(z)}{z-\lambda} \bar{\partial}\phi(z) \right| dA(z) \right)^s dA(\lambda) < \infty.$$

LEMMA 2. *Let M be the $H^\infty(G)$ invariant subspace generated by f and g. Then M has the codimension-1 property if and only if there is a point $\lambda \in G$ with $f(\lambda) \neq 0$ and $g(\lambda) \neq 0$ such that*

$$\frac{f(z)g(\lambda) - g(z)f(\lambda)}{z-\lambda} \in M.$$

Proof. In [5], Richter showed that M has the codimension-1 property if and only if there exists a $\lambda \in G$ such that, for bounded analytic functions p and q, if $p(\lambda)f(\lambda) + q(\lambda)g(\lambda) = 0$ then

$$\frac{p(z)f(z) + q(z)g(z)}{z-\lambda} \in M.$$

However,

$$\begin{aligned}\frac{p(z)f(z) + q(z)g(z)}{z-\lambda} \\ = \frac{p(z)-p(\lambda)}{z-\lambda} f(z) + \frac{q(z)-q(\lambda)}{z-\lambda} g(z) + p(\lambda)\frac{f(z) - \frac{f(\lambda)}{g(\lambda)}g(z)}{z-\lambda}.\end{aligned}$$

This proves the lemma. □

Recall that V in Theorem A denotes the disk with radius δ centered at λ_0.

LEMMA 3. *Let $f \in L^p_a(G) \cap L^s(V \cap G, dA)$ and $g \in L^p_a(G) \cap L^{s'}(V \cap G, dA)$, where $1/s + 1/s' = 1/p$. Let ϕ be a smooth function with support in V. Then $(T_\phi f)g$ belongs to $[g]$.*

Proof. By Lemma 1, we know that there exists a $\delta_0 < \delta$ such that $T_\phi f \in L^s_a((G \cap V) \cup \{z : \delta_0 < |z - \lambda_0| < N\})$, where N is a constant greater than the diameter of G. Using that $H^\infty(G \cap V)$ is dense in $L^s_a(G \cap V)$ and that T_ϕ is a bounded operator on $L^s_a(G \cap V)$, we conclude there exists a sequence $\{p_n\} \subset H^\infty(G \cap V)$ such that $T_\phi p_n$ converges to $T_\phi f$ in $L^s(G \cap V, dA)$ and uniformly on $\bar{G} \setminus V$. Hence

$$\begin{aligned}\int_G &|(T_\phi p_n)g - (T_\phi f)g|^p\,dA\\ &\le \sup_{z\in\bar G\setminus V} |T_\phi p_n - T_\phi f|\|g\|_p + \|p_n - T_\phi f\|_{L^s_a(G\cap V)}\|g\|_{L^{s'}_a(G\cap V)}\\ &\to 0.\end{aligned}$$

Since $T_\phi p_n \in H^\infty(G)$, we see that $(T_\phi f)g \in [g]$. □

Define

$$H[f,g,h](\lambda) = \int \frac{f(z)g(\lambda) - g(z)f(\lambda)}{z-\lambda}\overline{h(z)}\,dA(z).$$

Proof of Theorem A. From Lemma 2, it suffices to show that the identity

$$H[f,g,h](\lambda) = 0$$

holds for $\lambda \in G\cap V$ and $h \perp M$ (Note that $H[f,g,h](\lambda)$ is zero off G). Let ϕ be a C^∞ function with support in V. It follows from Lemma 3 that

$$\int (T_\phi f)g\bar h\,dA = 0.$$

Using Fubini's theorem, we get

$$\begin{aligned}\int \phi f g\bar h\,dA &= \frac{1}{\pi}\int f(z)\overline{h(z)}\int \frac{g(\lambda)}{\lambda - z}\bar\partial\phi(\lambda)\,dA(\lambda)\,dA(z)\\ &= -\frac{1}{\pi}\int \bar\partial\phi(\lambda)g(\lambda)\int\frac{f(z)}{z-\lambda}\overline{h(z)}\,dA(z)\,dA(\lambda).\end{aligned}$$

Similarly, we have

$$\int \phi f g\bar h\,dA = -\frac{1}{\pi}\int \bar\partial\phi(\lambda)f(\lambda)\int\frac{g(z)}{z-\lambda}\overline{h(z)}\,dA(z)\,dA(\lambda).$$

Hence,

$$\int \bar\partial\phi(\lambda)H[f,g,h](\lambda)\,dA(\lambda) = 0.$$

Thus, by Weyl's lemma, $H[f,g,h](\lambda)$ equals an analytic function $A[f,g,h](\lambda)$ on V a.e. with respect to the area measure.

Claim. The function $A[f,g,h](\lambda)$ is the zero function.

For $F \in H^\infty(G)$ and $\lambda \in G$, we have

$$\frac{F(z) - F(\lambda)}{z-\lambda} \in H^\infty(G).$$

On the other hand,

$$\begin{aligned}&\frac{F(z)f(z)g(\lambda) - F(\lambda)f(\lambda)g(z)}{z-\lambda}\\ &\qquad = \frac{F(z)-F(\lambda)}{z-\lambda}f(z)g(\lambda) + F(\lambda)\frac{f(z)g(\lambda) - f(\lambda)g(z)}{z-\lambda}.\end{aligned}$$

Thus it follows that

$$H[Ff, g, h](\lambda) = F(\lambda)H[f, g, h](\lambda)$$

on G. Using the same argument for $H[f, g, h]$, one can show that there is an analytic function $A[Ff, g, h]$ on V such that $H[Ff, g, h]$ equals $A[Ff, g, h]$ almost everywhere on V. Therefore,

$$A[Ff, g, h](\lambda) = F(\lambda)A[f, g, h](\lambda)$$

on $G \cap V$. Now assume that $A[f, g, h]$ is not zero; then the function F extends a meromorphic function on V. Since F is bounded on G, we conclude that λ_0 is not a pole for $F(\lambda)$. Hence, λ_0 is a removable point for $H^\infty(G)$. This a contradiction. Thus, $H[f, g, h](\lambda) = 0$. □

Proof of Corollary B. It follows from Theorem A that the invariant subspace generated by f and g has the codimension-1 property. Now the corollary follows from [5].

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