Direct Product of Free Groups as the Fundamental Group of the Complement of a Union of Lines

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1. Introduction

It is well known that the fundamental group of the complement of a complex projective algebraic curve depends on the position of its singularities [4; 6; 9]. Let $\Sigma \subset CP^2$ be a union of projective lines and let $G = \pi_1(CP^2 \setminus \Sigma)$. We ask under what conditions will G be independent of the position of the singularities of Σ. The purpose of this paper is to give such a condition. First, we define a topological invariant $\beta(\Sigma)$ for Σ. To describe β, we introduce a graph Γ that lies on the arrangement of lines Σ and connects higher singularities (multiplicity ≥ 3) of Σ. This graph in general has more than one component and is not uniquely defined. However, we show that the homotopy type of Γ is independent of our choice, and we define $\beta(\Sigma)$ to be the first Betti number of Γ. In Section 3, we prove the following theorem.

THEOREM 1. *If $\beta(\Sigma) = 0$, then $G = \pi_1(CP^2 \setminus \Sigma)$ is independent of the position of the singularities and G is a direct product of free groups.*

In Section 4, we study the fundamental group of the complement of an arrangement of six lines. An arrangement of six lines can have at most four higher singularities. In case an arrangement of six lines has three or four higher singularities, all these higher singularities must be triple points. In Section 4 we also show the following.

THEOREM 2. *For an arrangement of at most six lines, G does not depend on the position of the singularities.*

Theorems 1 and 2 together imply that: *if two arrangements of lines have the same number of lines and the same local topology, and if their complements have nonisomorphic global fundamental groups, then they must both have at least seven lines and three higher singularities.* In [4], the author gave an example of a pair of arrangements of seven lines where both have three triple points and twelve double points and where their complements have nonisomorphic global fundamental groups. We see here that in this example both the number of lines and higher singularities (and their multiplicities) are smallest possible.

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2. Definition of $\beta(\Sigma)$

Let $\Sigma = L_1 \cup \cdots \cup L_n \subset CP^2$ be a union of n distinct projective lines, let D be the set of all double points, and let S be the set of all singularities of Σ of multiplicity ≥ 3. For each line L_i, $i = 1, \ldots, n$, let $S_i = L_i \cap S$, let the number of points of S_i be t_i, and let $S_i = \{a_1^i, \ldots, a_{t_i}^i\}$. If $S_i = \emptyset$ then $t_i = 0$. For each $j = 1, \ldots, t_i - 1$, choose a simple arc (i.e., an arc without self-intersection) $A_j^i \subset L_i \setminus D$ to connect a_j^i and a_{j+1}^i, and require that the interiors of A_l^i and A_s^i have empty intersection for $l \neq s$. Note that $A_i = A_1^i \cup \cdots \cup A_{t_i-1}^i$ is itself a simple arc and that $A_i \subset L_i$ goes through all points of S_i and avoids double points on L_i. In case $t_i \leq 1$, we let $A_i = \emptyset$. Let $\Gamma = S \cup A_1 \cup \cdots \cup A_n$, and note that Γ is a graph that lies on Σ. We call S the set of vertices and $\{A_j^i,\ i = 1, \ldots, n;\ j = 1, \ldots, t_i - 1\}$ the set of edges of Γ. Call Γ an S-graph. We can have different choices for A_i and ordering on S_i, so Γ is in general not uniquely defined. Let us now show the following lemma.

LEMMA 2.1. *Let Γ and Γ^* be S-graphs on Σ. Then Γ and Γ^* are homotopy equivalent.*

Proof. Let $\Gamma = S \cup A_1 \cup \cdots \cup A_n$ and $\Gamma^* = S \cup A_1^* \cup \cdots \cup A_n^*$ with $A_i, A_i^* \subset L_i$ be two S-graphs on Σ. Let $\Gamma = \Gamma_0$, and for $j = 1, \ldots, n$ let $\Gamma_j = S \cup A_1^* \cup \cdots \cup A_j^* \cup A_{j+1} \cup A_n$. Note that Γ_j is an S-graph. For $j = 1, \ldots, n$, Γ_{j-1} and Γ_j have the same chosen arc on L_i except when $i = j$. If $t_j \leq 1$ then $A_j = A_j^* = \emptyset$ and hence we have $\Gamma_{j-1} = \Gamma_j$. If $t_j > 1$, then A_j and A_j^* are both simple arcs on L_j. Choose a point b on S_j. Note that b is a deformation retract of both A_j and A_j^*, so retractions of A_j and A_j^* to the point b extend to a homotopy equivalence of Γ_j and Γ_j^* to the same graph. Hence Γ_j and Γ_j^* are homotopy equivalent. Inductively, this shows that Γ and Γ^* are homotopy equivalent. □

LEMMA 2.2. *Let $\Sigma_1, \Sigma_2 \subset CP^2$ be arrangements of lines and suppose that there is a homeomorphism $f\colon \Sigma_1 \to \Sigma_2$. Then S-graphs on Σ_1 and Σ_2 have the same homotopy type.*

Proof. Let $f\colon \Sigma_1 \to \Sigma_2$ be a homeomorphism. The assertion follows since f and f^{-1} map line to line, m-fold point to m-fold point, and S-graph to S-graph. □

Let X be a topological space that is homeomorphic to Σ, and let $f\colon \Sigma \to X$ be a homeomorphism. Then the image $f(L_i)$, $i = 1, \ldots, n$, is homeomorphic to S^2 and the image of an m-fold point lies on the image of m lines. Hence the concepts of m-fold point and S-graph can be carried over to X via the homeomorphism f. This shows that the homotopy type of an S-graph on Σ depends only on the topological type of Σ. Since the homotopy type of a finite connected graph is decided by the first Betti number of the graph, it follows that the homotopy type of an S-graph is decided by the number of components of the graph and by the Betti numbers of corresponding components. Choose an S-graph Γ on Σ, and let

$\beta(\Sigma) = \text{rank}\, H_1(\Gamma) = b_1(\Gamma)$ be the first Betti number of Γ. By Lemma 2.1, β is independent of the S-graph chosen and depends only on Σ.

LEMMA 2.3. *Let $\Sigma_1, \Sigma_2 \subset CP^2$ be arrangements of lines, and suppose that there is a homeomorphism $f\colon \Sigma_1 \to \Sigma_2$. Then $\beta(\Sigma_1) = \beta(\Sigma_2)$.*

Proof. This a corollary of Lemma 2.2. □

LEMMA 2.4. *Suppose that $\Sigma, \Sigma^* \subset CP^2$ are two arrangements of lines that intersect in nodes only. Then*

$$\beta(\Sigma \cup \Sigma^*) = \beta(\Sigma) + \beta(\Sigma^*).$$

Proof. An S-graph on $\Sigma \cup \Sigma^*$ is a disjoint union of an S-graph on Σ and an S-graph on Σ^*, and the first Betti number is additive with respect to disjoint union. □

LEMMA 2.5. *Let $\Sigma \subset CP^2$ be an arrangement of lines. Then $\beta(\Sigma) = 0$ if and only if S-graphs on Σ are disjoint unions of trees.*

Proof. Let Γ be an S-graph on Σ. Then $\beta(\Sigma) = 0 \iff b_1(\Gamma) = 0 \iff \Gamma$ is a union of disjoint trees. □

Let us recall some basic facts about a tree. The degree of a vertex v of a graph is the number of edges incident with v. A vertex v is isolated if $\deg(v) = 0$, and v is an endpoint if $\deg(v) = 1$. For a tree that has at least one edge, an endpoint always exists.

3. $\pi_1(CP^2 \setminus \Sigma)$ for Arrangements of Lines with $\beta(\Sigma) = 0$

In this section, we compute G for arrangements of lines with $\beta(\Sigma) = 0$. The following result of Oka and Sakamoto [7] is crucial to our calculation.

THEOREM 3.1. *Let C_1, C_2 be two distinct algebraic curves in the complex affine plane C^2 of degree n_1, n_2, respectively. Suppose that C_1 and C_2 intersect at $n_1 n_2$ distinct points. Then $\pi_1(C^2 \setminus C_1 \cup C_2) \cong \pi_1(C^2 \setminus C_1) \oplus \pi_1(C^2 \setminus C_2)$.*

For arrangements of lines, let us show the following lemma.

LEMMA 3.1. *Let $\Sigma_i \subset CP^2$, $i = 1, 2$, be an arrangement of n_i lines, and let $\Sigma = L \cup \Sigma_1 \cup \Sigma_2 \subset CP^2$ be an arrangement of $1 + n_1 + n_2$ lines. Let $C^2 = CP^2 \setminus L$, and suppose that Σ_1 and Σ_2 intersect at $n_1 n_2$ distinct points in C^2. Then*

$$\pi_1(CP^2 \setminus \Sigma) \cong \pi_1(CP^2 \setminus \Sigma_1 \cup L) \oplus \pi_1(CP^2 \setminus \Sigma_2 \cup L).$$

Proof. Let $C_1 = C^2 \cap \Sigma_1$ and $C_2 = C^2 \cap \Sigma_2$. Since C_1 and C_2 intersect at $n_1 n_2$ distinct points, by Theorem 3.1 we have

$$\begin{aligned}\pi_1(CP^2 \setminus \Sigma) &\cong \pi_1(C^2 \setminus C_1 \cup C_2) \cong \pi_1(C^2 \setminus C_1) \oplus \pi_1(C^2 \setminus C_2)\\ &\cong \pi_1(CP^2 \setminus \Sigma_1 \cup L) \oplus \pi_1(CP^2 \setminus \Sigma_2 \cup L).\end{aligned}$$ □

Denote a free group of rank t by F_t, a free Abelian group of rank r by A_r, and the multiplicity of a point P on Σ by $m(P)$.

THEOREM 3.2. *Let Σ be an arrangement of n lines, and let $S = \{a_1, a_2, \dots, a_k\}$ be the set of all singularities of Σ with multiplicity ≥ 3. Suppose that $\beta(\Sigma) = 0$; then*

$$\pi_1(CP^2 \setminus \Sigma) \cong A_r \oplus F_{m(a_1)-1} \oplus \cdots \oplus F_{m(a_k)-1},$$

where $r = n + k - 1 - m(a_1) - \cdots - m(a_k)$.

Proof. We proceed by induction on k. Suppose that $k = 0$; then Σ is n lines in general position and we have $G \cong A_{n-1}$ [9]. Suppose that our assertion is true for $k = s \geq 0$, and assume that Σ has $s+1$ higher singularities. Let Γ be an S-graph on Σ. By our assumption, Γ is a union of disjoint trees with $s+1$ vertices. There are two cases.

(i) *Γ has an isolated vertex.* Let a_1 be an isolated vertex of Γ, and note that a line that goes through a_1 will not go through any other point of S. Let $L_1, \dots, L_{m(a_1)}$ be lines that go through a_1.

(ii) *Γ has no isolated vertex.* Then Γ has at least one edge and hence has an endpoint. Let a_1 be an endpoint of Γ, and let L_1 be the line that contains the edge connecting a_1 and another vertex of Γ. Let $L_1, L_2, \dots, L_{m(a_1)}$ be lines that go through a_1.

In either of these two cases, $\Sigma \setminus L_1 \subset CP^2 \setminus L_1$ splits into two components and one of these components is $m(a_1) - 1$ parallel lines. These two components intersect in $CP^2 \setminus L_1$ in $(m(a_1) - 1)(n - m(a_1))$ points. Let $\Sigma_1 = L_1 \cup \cdots \cup L_{a_1}$ and $\Sigma_2 = L_1 \cup L_{m(a_1)+1} \cup L_{m(a_1)+2} \cup \cdots \cup L_n$. Note that Σ_2 has s higher singularities and $\beta(\Sigma_2) = 0$. By Lemma 3.1, we have

$$\begin{aligned}\pi_1(CP^2 \setminus \Sigma) &\cong \pi_1(CP^2 \setminus \Sigma_1) \oplus \pi_1(CP^2 \setminus \Sigma_2)\\ &\cong F_{m(a_1)-1} \oplus A_r \oplus F_{m(a_2)-1} \oplus \cdots \oplus F_{m(a_{s+1})-1}\\ &\cong A_r \oplus F_{m(a_1)-1} \oplus F_{m(a_2)-1} \oplus \cdots \oplus F_{m(a_{s+1})-1},\end{aligned}$$

where

$$\begin{aligned}r &= (n - (m(a_1) - 1)) + s - 1 - m(a_2) - \cdots - m(a_{s+1})\\ &= n + (s+1) - 1 - m(a_1) - \cdots - m(a_{s+1}).\end{aligned}$$

□

Let us give examples of two classes of arrangements of lines with $\beta(\Sigma) = 0$.

(i) Suppose that in Σ there is a line L that goes through the set S of all higher singularities. Then any other line of Σ can go through at most one higher singularity. Hence an S-graph on Σ is a simple arc on L that goes through all points of S, and $\beta(\Sigma) = 0$. For this class of arrangement, G is known (see, e.g. [4]) and Theorem 3.2 is a generalization of [4, Cor. 2.1].

(ii) Assume that Σ has two higher singularities. Then these two singular points either lie together on a line of the arrangement or they do not. In the former case, an S-graph on Σ is a simple arc with two vertices. In the latter case, an S-graph on Σ is a union of two isolated points. In both cases, $\beta(\Sigma) = 0$. We sum this up with the following corollary.

COROLLARY 3.1. *Let Σ be an arrangement of n distinct complex projective lines in CP^2 such that Σ has two distinct singular points a_1, a_2 with multiplicities ≥ 3. Let $r = n + 1 - m(a_1) - m(a_2)$. We then have*

$$G = \pi_1(CP^2 \setminus \Sigma) \cong A_r \oplus F_{m(a_1)-1} \oplus F_{m(a_2)-1}.$$

An arrangement Σ of at most five lines has at most two higher singularities. Hence Corollary 3.1 shows that, for an arrangement of at most five lines, G does not depend on the position of the singularities.

4. Arrangements of Six Lines with Three or Four Triple Points

For a real arrangement, a presentation of the fundamental group of its complement in CP^2 can be obtained as in [8]. In this section, we show that: (i) for any arrangement of six lines with three triple points, G is isomorphic to the fundamental group of the complement of W_1 whose configuration is given by

$$xy(x-y)(x+y-3z)(x+3y-3z)(3x+2y-6z) = 0; \qquad (4.1)$$

and (ii) for any arrangement of six lines that has four triple points, G is isomorphic to the fundamental group of the complement of W_2 whose configuration is given by

$$xy(x-y)(x+y-6z)(x+2y-6z)(2x+y-6) = 0. \qquad (4.2)$$

I

First, we consider an arrangement of six lines with four triple points. Such an arrangement is completely determined by the coordinates of its triple points. Let these four points be P_1, P_2, P_3, P_4. No three of these four points are collinear, for if this happens then the arrangement will have at least seven lines. There are six choices of pairs of points among these four points, and each pair of points gives us a line; their union gives us the arrangement.

LEMMA 4.1. *Let $\Sigma_1, \Sigma_2 \subset CP^2$ be arrangements of six lines with four triple points. Then there is a projective transformation T such that $T(\Sigma_1) = \Sigma_2$.*

Proof. Let A_1, A_2, A_3, A_4 be triple points of Σ_1, and let B_1, B_2, B_3, B_4 be triple points of Σ_2. Since no three points are collinear in each set of triple points, there is a unique projective transformation T of CP^2 such that $T(A_i) = B_i$. Because a projective transformation preserves projective lines, we have $T(\Sigma_1) = \Sigma_2$. □

The arrangement W_2 given by equation (4.2) has four triple points. By Lemma 4.1, we have our next corollary.

COROLLARY 4.1. *If $\Sigma \subset CP^2$ is an arrangement of six lines with four triple points, then*

$$G_2 = \pi_1(CP^2 \setminus \Sigma) \cong \pi_1(CP^2 \setminus W_2).$$

Proof. Let T be a projective transformation such that $T(\Sigma) = W_2$. Then T serves as a homeomorphism of a pair that maps CP^2 to itself and Σ to W_2. Hence we have a homeomorphism from $CP^2 \setminus \Sigma$ to $CP^2 \setminus W_2$, and this induces an isomorphism between $\pi_1(CP^2 \setminus \Sigma)$ and $\pi_1(CP^2 \setminus W_2)$. □

II

Let Σ be an arrangement of six lines with three triple points.

LEMMA 4.2. *Let $\Sigma \subset CP^2$ be an arrangement of six lines with three triple points. By performing a finite sequence of smooth equisingular deformations, we can deform Σ to W_1.*

Proof. Let $L_1, \dots, L_6$ be the lines of Σ. Note that Σ has six double points. Let O_1, O_2, O_3 be triple points and $D_1, \dots, D_6$ double points of Σ. By counting the number of lines, we see that there are three lines, say L_1, L_2, L_3, such that L_1 goes through O_2, O_3, L_2 goes through O_1, O_3, and L_3 goes through O_1, O_2. Let the third line that goes through O_1, O_2, O_3 be L_4, L_5, L_6, and let the defining linear equation of $L_1, L_2, \dots, L_6$ be $F_1, F_2, \dots, F_6$, respectively. Let the point of intersection of (i) L_4 and L_1 be D_1, (ii) L_5 and L_2 be D_2, (iii) L_6 and L_3 be D_3, (iv) L_5 and L_6 be D_4, (v) L_4 and L_6 be D_5, (vi) L_4 and L_5 be D_6.

If we know coordinates of (say) D_4, D_5, O_1, O_2, O_3 in CP^2, then their coordinates determine Σ. However, if we know the coordinates of only four of these points then we cannot determine the arrangement.

Assume that the coordinates of D_4, D_5, O_1, O_2 are fixed. Using the coordinates of O_1, O_2, we obtain the linear equation F_3 of L_3. Using the coordinates of D_4, D_5, we obtain the equation F_6 of L_6. Using the coordinates of O_1, D_5, we obtain the equation F_4 of L_4. Using the coordinates of O_2, D_4, we obtain the equation F_5 of L_5. Using F_4 and F_5, we obtain the coordinates of D_6. Equations F_3 and F_6 together determine the coordinates of D_3. To determine equations F_1, F_2 and the coordinates of D_1, D_2, we need to know the coordinates of O_3; by writing equations down directly, we see that all coefficients of F_1 and F_2, as well as the homogeneous coordinates of D_1 and D_2, are polynomial functions of coordinates of O_3.

Consider the arrangement W_1 given by equation (4.1), where coordinates of the triple points are $E_1 = [0, 3, 1]$, $E_2 = [3, 0, 1]$, and $E_3 = [0, 0, 1]$; coordinates of two double points are $E_4 = [\frac{3}{4}, \frac{3}{4}, 1]$ and $E_5 = [\frac{6}{5}, \frac{6}{5}, 1]$. Note that no three points of E_1, E_2, E_4, E_5 are collinear. There is a projective transformation T^* such that $T^*(D_4) = E_4$, $T^*(D_5) = E_5$, $T^*(O_1) = E_1$, $T^*(O_2) = E_2$, and $T^*(\Sigma)$ is an arrangement of six lines with three triple points E_1, E_2 and $T^*(O_3)$. Denote $V = T^*(O_3)$, and note that V lies on the line given by $x - y = 0$.

Fix D_4, D_5, O_1, O_2 and move O_3 in L_4 slightly (hence Σ must deform accordingly) if necessary. We assume that the coordinates of V are $[b, b, 1]$ and that b is not real. Let $C^2 = \{[x, y, 1] \in CP^2\}$. Connect V and E_3 by a real line segment $J = \{A_t = [tb, tb, 1] \mid t \in [0, 1]\}$ in C^2. Note that $A_0 = [0, 0, 1] = E_3$ and $A_1 = V$. Let Σ_t be the vanishing set of

$$F_t(x, y, z) = (x + y - 3z)(x - y)(x + 3y - 3z)(3x + 2y - 6z) \\ \times [(3 - tb)x + tby - 3tbz][tbx + (3 - tb)y - 3tbz].$$

Because b is not real, one sees that Σ_t is an arrangement of six distinct lines for each $t \in [0, 1]$.

Coordinates of triple points of Σ_t are $[0, 3, 1]$, $[3, 0, 1]$, and $a_1(t) = [tb, tb, 1]$. Coordinates of double points are $[\frac{3}{4}, \frac{3}{4}, 1]$, $[\frac{6}{5}, \frac{6}{5}, 1]$, $[\frac{3}{2}, \frac{3}{2}, 1]$, $[\frac{12}{7}, \frac{3}{7}, 1]$, $a_2(t) = [6tb, 9 - 6tb, 9 - 4tb]$, and $a_3(t) = [18 - 12tb, 3tb, 9 - 5tb]$. Note that Σ_t may have more than three triple points only if $a_2(t) = a_3(t)$, but if this happens then Σ_t will have fewer than six distinct lines, which is not possible. Since $F_t, a_1(t), a_2(t), a_3(t)$ depend smoothly on t, the family $\{F_t, t \in [0, 1]\}$ gives us a smooth equisingular deformation from $T^*(\Sigma)$ to W_1. The Lie group $PGL(2, C)$ is connected, so we can deform Σ to $T^*(\Sigma)$ via a smooth equisingular deformation. Hence we have a finite sequence of smooth equisingular deformations that carries Σ to W_1. □

COROLLARY 4.2. *If $\Sigma \subset CP^2$ is an arrangement of six lines with three triple points, then*

$$G_1 = \pi_1(CP^2 \setminus \Sigma) \cong \pi_1(CP^2 \setminus W_1).$$

Proof. If we perform a smooth equisingular deformation on an algebraic curve $C \subset CP^2$, then, up to isomorphism of groups, $\pi_1(CP^2 \setminus C)$ is unchanged [3]. Hence the combined effect of a finite sequence of equisingular deformations that deform Σ to W_1 will leave $\pi_1(CP^2 \setminus \Sigma)$ unchanged up to isomorphism. Hence $\pi_1(CP^2 \setminus \Sigma) \cong G_1$. □

Corollaries 3.1, 4.1, and 4.2 together imply our final result.

THEOREM 4.1. *Let Σ be an arrangement of at most six lines. Then $G = \pi_1(CP^2 \setminus \Sigma)$ does not depend on the position of the singularities.*

5. Some Arrangements of Seven Lines

Let Σ_1, Σ_2 be two arrangements of n lines in CP^2 that have same local topology. Suppose that their complements have nonisomorphic fundamental groups. By Corollary 3.1 and Theorem 4.1, $n \geq 7$ and each arrangement must have at least three higher singularities. In [4], we gave the following pair of arrangements of lines:

$$y(x - y)(x + y)(2x - y - 2z)(2x + y - 2z) \\ \times (3x - y - 6z)(3x + y - 6z) = 0, \tag{5.1}$$

$$xy(x - y)(x + y - 2z)(x - 2y - 2z) \\ \times (2x + y - 2z)(3x - y - 9z) = 0. \tag{5.2}$$

Both arrangements have seven lines, three triple points, and twelve double points; their complements have nonisomorphic fundamental groups. In [4], we showed that: (i) for the arrangement of lines V_1 given by (5.1), we have $\pi_1(CP^2 \setminus V_1) \cong F_2 \oplus F_2 \oplus F_2$, where F_2 is a free group of rank 2; and (ii) for the arrangement V_2 given by (5.2), $\pi_1(CP^2 \setminus V_2)$ has a nontrivial center. These two facts together imply that $\pi_1(CP^2 \setminus V_1)$ and $\pi_1(CP^2 \setminus V_2)$ are not isomorphic. By Corollary 3.1 and Theorem 4.1, we see that—among all pairs of arrangements of lines that have same local topology but have complements with nonisomorphic fundamental groups—the pair V_1 and V_2 have the smallest possible number of lines and singularities. Moreover, multiplicities of higher singularities of V_1 and V_2 are also lowest possible.

The arrangement V_1 has the property that all three higher singularities lie on one line of V_1. We point out that there is another arrangement of seven lines with three triple points and twelve double points such that $\pi_1(CP^2 \setminus \Sigma) \cong F_2 \oplus F_2 \oplus F_2$, and that these three triple points do not lie on one line of the arrangement. Consider the arrangement $V_3 = L \cup U_1 \cup U_2$ of seven lines, where (i) L is given by $x = 0$, (ii) U_1 is given by $y(x-y)(2x-y-2z)(2x+y-2z) = 0$, and (iii) U_2 is given by $(x-2y+8z)(x+2y-8z) = 0$. This arrangement has three triple points and twelve double points; coordinates of the triple points are $[0, 4, 1]$, $[0, 0, 1]$, and $[1, 0, 1]$. No line of V_3 contains all these three triple points. However, an S-graph on V_3 that connects these three triple points is composed of two arcs that lie on the X axis and the Y axis, respectively, and the graph has the form of the capital letter V. Hence $\beta(V_3) = 0$, so

$$\pi_1(CP^2 \setminus V_3) \cong F_2 \oplus F_2 \oplus F_2.$$

To end this paper, we ask: Is the condition that $\beta(\Sigma) = 0$ also a necessary one for $G = \pi_1(CP^2 \setminus \Sigma)$ to be a direct product of free groups? We surmise that this question has an affirmative answer.

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