Positively Curved 4-Manifolds and the Nonnegativity of Isotropic Curvatures

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1. Introduction

It is a classical problem in Riemannian geometry to study the topology of compact 4-manifolds that admit a metric of positive sectional curvature. So far, such manifolds have only been understood under additional assumptions, and the Hopf conjecture remains unsolved: Does $S^2 \times S^2$ admit a positively curved Riemannian metric? Among these assumptions we find the *nonnegativity of the isotropic curvature*. This beautiful concept of curvature was introduced by Micallef and Moore in [13], and plays a role in the study of second variation of area of minimal surfaces that is similar to the role played by the sectional curvature in the study of geodesics.

In dimension 4, the nonnegativity of the isotropic curvature is equivalent to the nonnegativity of the Weitzenböck operator, which we will denote by F. Further, if the manifold M is oriented, then F commutes with the Hodge-star, and we can consider two components—the self-dual and anti-self-dual of F—with respect to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. The special feature of positive sectional curvature in dimension 4 is that, for each point of M, either F^+ or F^- is positive. It is then natural to investigate conditions that will imply that one of the components is positive (or nonnegative) for all points of M. Then, using the classical Bochner technique, we can conclude that M is definite.

Recall that, by Synge's theorem, a compact oriented 4-manifold with positive sectional curvature is simply connected. By the results of Donaldson [5] and Freedman [6], if such a manifold is definite then it is homeomorphic to S^4, or to the connected sum $\mathbf{CP}^2 \sharp \cdots \sharp \mathbf{CP}^2$, ($b_2$ times) if the second Betti number $b_2 > 0$.

Our first result in this paper is the following theorem.

THEOREM 1. *Let M be a compact, connected, oriented 4-manifold with positive sectional curvature. If each point of M has a nonpositive isotropic curvature, then M is definite. Moreover, if the Laplacian of the Weyl tensor W satisfies $\Delta W = 0$, then M is half-conformally flat (but not conformally flat). In this case, if the scalar curvature is constant then M is the complex projective plane $\mathbf{CP}^2$ with its standard metric.*

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It is known that locally irreducible 4-manifolds with nonnegative isotropic curvatures are also definite (see [16]). Then, in view of Theorem 1, we search for conditions that imply the nonnegativity of at least one of the components of F on a manifold M that has points with positive isotropic curvature and points for which some isotropic curvature is negative. Some such conditions are well-known. These conditions are in terms of the Weyl tensor and the scalar curvature S because, in dimension 4, F does not depend on the traceless Ricci tensor. We mention first that if M is orientable and $S \geq 0$, and if for some choice of orientation $W^- = 0$, then $F^- \geq 0$. In fact, this statement can be generalized to $\|W^-\|^2 \leq S^2/24$. It is interesting to observe that, if the sectional curvatures are nonnegative $(K \geq 0)$, the same conclusion can be obtained by finding lower bounds for $\|W^-\|$ in terms of S. For instance, suppose that we order the eigenvalues of $W^\pm$ such that $W_1^\pm \leq W_2^\pm \leq W_3^\pm$. Then $W_1^\pm \leq 0$. If $(W_2^-)^2 + (W_3^-)^2 \geq 5S^2/54$ and $K \geq 0$, then $F^+ \geq 0$. Further, if for both components we have $(W_2^\pm)^2 + (W_3^\pm)^2 \geq 5S^2/54$, then $F \geq 0$. This is proved by showing that the above inequality implies that, for each point of M, at least one of the components of the Weyl tensor satisfies $\|W^\pm\|^2 \geq -(W_1^\pm)S/2$. Therefore, in our next result we weaken the condition $(W_2^\pm)^2 + (W_3^\pm)^2 \geq 5S^2/54$ to $\|W^-\|^2 \geq -(W_1^-)S/2$. Before stating our next result, we point out that—at the points where Theorem 2(b) is not verified—the eigenvalues of W satisfy the inequality in (c). We then assume that this inequality holds for every point of M. With respect to the above order for the eigenvalues of $W^\pm$, our next result is as follows.

THEOREM 2. *Let M be an oriented 4-manifold with positive sectional curvature.*

(a) *If $\|W^-\|^2 \geq -(W_1^-)S/2$, then either $F^+ \geq 0$ or $W^- = 0$.*
(b) *If all sectional curvatures satisfy $K \geq S/16$, then $F \geq 0$.*
(c) *If $W_3^+ + W_1^- \geq S/12$, then either $F^+ \geq 0$ or $F^- > 0$.*

Moreover, in each case, if M is compact then M is definite.

Theorems 1 and 2 are proved in Section 4. In Section 5, we study compact 4-manifolds with harmonic Weyl tensor for which $F \geq 0$ or one of its components is nonnegative. Our first result in Section 5 extends a theorem of Polombo [17], who proved that if $\Delta W^+ = 0$ and the isotropic curvature is positive then the manifold M is conformally flat.

THEOREM 3. *Let M be a compact oriented 4-manifold with nonnegative isotropic curvature. If $\Delta W^+ = 0$, then one of the following holds.*

(a) *M is conformally flat with nonnegative scalar curvature.*
(b) *The universal cover is a Riemannian product of $N_1 \times N_2$, where N_1 is homeomorphic to the sphere S^2.*
(c) *M is biholomorphic to the complex projective space* $\mathbf{CP}^2$.

Notice that in cases (a) and (c) of Theorem 3 we can find a metric with harmonic curvature and whose isotropic curvatures are nonnegative. In fact, in case (a), the solution of the Yamabe conjecture by Schoen [18] implies that there exists a conformal metric with constant and nonnegative scalar curvature and hence such a metric has harmonic curvature [4, Prop. 3.17]. In case (c), we consider the standard metric in $\mathbf{CP}^2$.

Compact 4-manifolds with harmonic curvature and nonnegative isotropic curvature were completely classified in [14, Thm. 4.4]. Recall that the curvature tensor R is harmonic if and only if each of the components R_+^+, R_+^-, R_-^+, R_-^- with respect to the decomposition $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ is harmonic. However, by using Theorem 3 we obtain (in the following corollary) the same conclusions of [14, Thm. 4.4] but with weaker hypotheses. It is a consequence of the classification obtained in Corollary 1 that the harmonicity of W^+ and R_-^+ implies that either the curvature is harmonic or there exists a conformal metric (in the case of conformally flat manifolds) with harmonic curvature.

COROLLARY 1. *Let M be a compact oriented 4-manifold with nonnegative isotropic curvature. Suppose $\Delta W^+ = 0 = \Delta R_-^+$. Then either M is conformally flat or isometric to $\mathbf{CP}^2$ or the universal cover is isometric to $S^2 \times N_2$, where S^2 and N_2 have constant curvature.*

We then study compact 4-manifolds with harmonic Weyl tensor under the conditions of Theorem 2. Since the assumption in (b) implies nonnegative isotropic curvatures, we obtain in this case that *M is either conformally equivalent to S^4 or isometric to $\mathbf{CP}^2$*. If we have nonnegative sectional curvatures, we can also conclude that *M is covered by $\mathbf{R}^4$ or $S^3 \times \mathbf{R}$ or $S^2 \times S^2$* with their standard metrics. Under the assumption in (a), more possibilities can occur. A complete classification is the following theorem.

THEOREM 4. *Let M be a compact oriented 4-manifold with nonnegative sectional curvatures. If $\Delta W = 0$ and $\|W^-\|^2 \geq -(W_1^-)S/2$, then one of the following holds.*

(a) *M is conformally equivalent to S^4 or covered by $\mathbf{R}^4$ or $S^3 \times \mathbf{R}$ with their standard metrics.*
(b) *M is covered by $S^2 \times S^2$, where S^2 has constant curvature.*
(c) *M is isometric to $\mathbf{CP}^2$.*
(d) *M is anti-self-dual and negative definite.*
(e) *M is self-dual and the scalar curvature is not constant.*

The same classification is obtained under assumption (c) of Theorem 2. Also, in Section 5 we have more results for the case $\Delta W^+ = 0$. If $F^+ \geq 0$ and $\Delta W^+ = 0$, a result in [14] implies that either $W^+ = 0$ or the universal cover is Kähler with constant positive scalar curvature. Using Berger's classification for holonomy groups, we provide an understanding of the possible cases of this theorem in [14]. We then apply the classification in the cases of positive and nonnegative sectional curvatures.

2. The Weitzenböck Formula

Let M be an oriented Riemannian manifold of dimension 4, and let Λ^2 denote the bundle of exterior 2-forms and $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ the eigenspace splitting for the Hodge $\star$-operator. The 2-forms in Λ^2_+ are called *self-dual* and in Λ^2_-, *anti-self-dual.*

The Riemann curvature tensor defines a symmetric operator $\mathcal{R}\colon \Lambda^2 \to \Lambda^2$ given by

$$\mathcal{R}(e_{ij}) = \frac{1}{2}\sum_{k,l} R_{ijlk}e_{kl},$$

where $\{e_i\}$ is a local orthonormal basis of 1-forms, e_{ij} denotes the 2-form $e_i \wedge e_j$, and $R_{ijlk} = \langle R(e_i, e_j)e_l, e_k\rangle$. The operator $\mathcal{R}$ can be decomposed as

$$\mathcal{R} = \mathcal{R}^+_+ + \mathcal{R}^+_- + \mathcal{R}^-_+ + \mathcal{R}^-_-$$

with respect to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. This decomposition gives the irreducible components of $\mathcal{R}$ (see [19]). They are: trace $\mathcal{R}^+_+ =$ trace $\mathcal{R}^-_- = S/4$, where S is the scalar curvature; the two components $\mathcal{R}^+_-$ and $\mathcal{R}^-_+$ of the traceless Ricci tensor; and the two components of the Weyl tensor $W^+ = \mathcal{R}^+_+ - S/12$ and $W^- = \mathcal{R}^-_- - S/12$. A manifold is *conformally flat* if $W = 0$. It is said to be *half-conformally flat* if either $W^+ = 0$ or $W^- = 0$. An oriented manifold is *self-dual* if $W^- = 0$. It is clear that, on a half-conformally flat manifold, self-duality is a property that depends on the orientation.

The *signature* of an oriented compact 4-manifold is given in terms of the Weyl tensor by

$$\tau = \frac{1}{12\pi^2}\int_M \|W^+\|^2 - \|W^-\|^2\, dV.$$

Let $F\colon \Lambda^2(T_xM) \to \Lambda^2(T_xM)$ be the *Weitzenböck operator* given by

$$\begin{aligned}\langle F(e_{ij}), e_{kl}\rangle &= \mathrm{Ric}(e_i, e_k)\delta_{jl} + \mathrm{Ric}(e_j, e_l)\delta_{ik} - \mathrm{Ric}(e_i, e_l)\delta_{jk}\\ &\quad - \mathrm{Ric}(e_j, e_k)\delta_{il} - 2R_{ijlk},\end{aligned}$$

where Ric denotes the Ricci curvature. This operator satisfies the well known *Weitzenböck formula,* that is, $\Delta\omega = -\mathrm{div}\,\nabla\omega + F(\omega)$. Moreover, F is a symmetric operator and Λ^2_+ and Λ^2_- are F-invariant. Then $\star F = F\star$, and for each point of M we find a normal form for F (as in [19] for R). Since this normal form will be used in the rest of this article, we repeat the arguments used in [19].

PROPOSITION 2.1. *Let M be an oriented 4-manifold. Then, for each $x \in M$, there exists a positively oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_xM such that, relative to the corresponding basis $\{e_{12}, e_{34}, e_{13}, e_{42}, e_{14}, e_{23}\}$, F takes the form*

$$\begin{pmatrix} A_1 & 0 & 0\\ 0 & A_2 & 0\\ 0 & 0 & A_3\end{pmatrix},$$

where

$$A_i = \begin{pmatrix} \eta_i & \mu_i\\ \mu_i & \eta_i\end{pmatrix}.$$

Proof. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ be the orthonormal bases of eigenvectors of F^+ and F^-, respectively, and let r_i and s_i, $i = 1, 2, 3$, be the corresponding eigenvalues. Let us define the planes $P_i = (\alpha_i + \beta_i)/\sqrt{2}$ and $P_i^\perp = (\alpha_i - \beta_i)/\sqrt{2}$. Then $\{P_1, P_2, P_3, P_1^\perp, P_2^\perp, P_3^\perp\}$ is an orthonormal basis of $\Lambda^2(T_xM)$; moreover, $F(P_i) = \eta_i P_i + \mu_i P_i^\perp$ and $F(P_i^\perp) = \eta_i P_i^\perp + \mu_i P_i$, where $\eta_i = (r_i + s_i)/2$ and $\mu_i = (r_i - s_i)/2$. Since $\star P_i = P_i^\perp$, we also have that $P_i \wedge P_i = 0 = P_i^\perp \wedge P_i^\perp$, which in turn implies that P_i and $P_i^\perp$ are decomposable. We also have $P_1 \wedge P_2 = 0$ and hence $P_1 \cap P_2 \neq \{0\}$. Let $e_1 \in P_1 \cap P_2$ be a unit vector, and let e_2 and e_3 be such that $\{e_1, e_2\}$ and $\{e_1, e_3\}$ are oriented bases for P_1 and P_2, respectively. Choose e_4 to complete a positively oriented orthonormal basis of T_xM. Then $P_1 = e_1 \wedge e_2$, $P_2 = e_1 \wedge e_3$, and $e_1 \wedge e_4$ is either $\pm P_3$ or $\pm P_3^\perp$. The matrix of F relative to $\{e_{12}, e_{34}, e_{13}, e_{42}, e_{14}, e_{23}\}$ is of the above type. □

It follows from Proposition 2.1 that the self-dual 2-forms

$$\alpha_1 = \frac{\sqrt{2}}{2}(e_{12} + e_{34}), \quad \alpha_2 = \frac{\sqrt{2}}{2}(e_{13} - e_{24}), \quad \alpha_3 = \frac{\sqrt{2}}{2}(e_{14} + e_{23})$$

are the eigenvectors of F^+ with corresponding eigenvalues $r_i = \eta_i + \mu_i$, and that the anti-self-dual 2-forms

$$\beta_1 = \frac{\sqrt{2}}{2}(e_{12} - e_{34}), \quad \beta_2 = \frac{\sqrt{2}}{2}(e_{13} + e_{24}), \quad \beta_3 = \frac{\sqrt{2}}{2}(e_{14} - e_{23})$$

are the eigenvectors of F^- with corresponding eigenvalues $s_i = \eta_i - \mu_i$.

PROPOSITION 2.2. *Let $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis of Proposition 2.1. Then the 2-forms $\{\alpha_i\}$ are the eigenvectors of $\mathcal{R}_+^+$ and the 2-forms $\{\beta_i\}$ are the eigenvectors of $\mathcal{R}_-^-$. Moreover, if the corresponding eigenvalues are denoted by λ_i and φ_i, respectively, we have*

$$r_1 = 2(\lambda_2 + \lambda_3), \quad r_2 = 2(\lambda_1 + \lambda_3), \quad r_3 = 2(\lambda_1 + \lambda_2);$$
$$s_1 = 2(\varphi_2 + \varphi_3), \quad s_2 = 2(\varphi_1 + \varphi_3), \quad s_3 = 2(\varphi_1 + \varphi_2).$$

Proof. We will show that $\langle \mathcal{R}(\alpha_i), \alpha_j \rangle = 0$ and $\langle \mathcal{R}(\beta_i), \beta_j \rangle = 0$ for $i \neq j$. For simplicity, we will show that $\langle \mathcal{R}(\alpha_1), \alpha_2 \rangle = 0$; the others are proved in a similar manner. Since $\langle F(\alpha_1), \alpha_2 \rangle = 0$, we have

$$\langle F(e_{12}), e_{13} \rangle - \langle F(e_{12}), e_{24} \rangle + \langle F(e_{34}), e_{13} \rangle + \langle F(e_{34}), e_{24} \rangle = 0.$$

From the definition of F, we have that

$$\begin{aligned} 0 &= \mathrm{Ric}(e_2, e_3) - 2R_{1231} + \mathrm{Ric}(e_1, e_4) + 2R_{1242} \\ &\quad - \mathrm{Ric}(e_1, e_4) - 2R_{3431} - \mathrm{Ric}(e_2, e_3) + 2R_{3442} \\ &= -2R_{1231} + 2R_{1242} - 2R_{3431} + 2R_{3442} \\ &= -4\langle \mathcal{R}(\alpha_1), \alpha_2 \rangle. \end{aligned}$$

Now, the eigenvalues λ and φ are given by

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(K_{12} + K_{34}) - R_{1234}, & \varphi_1 &= \tfrac{1}{2}(K_{12} + K_{34}) + R_{1234}, \\ \lambda_2 &= \tfrac{1}{2}(K_{13} + K_{24}) + R_{1324}, & \varphi_2 &= \tfrac{1}{2}(K_{13} + K_{24}) - R_{1324}, \\ \lambda_3 &= \tfrac{1}{2}(K_{14} + K_{23}) - R_{1423}, & \varphi_3 &= \tfrac{1}{2}(K_{14} + K_{23}) + R_{1423}, \end{aligned} \tag{2.3}$$

where K_{ij} denotes the curvature of the plane $\{e_i, e_j\}$. On the other hand, from the definition of F we have

$$\begin{aligned} r_1 &= \langle F(\alpha_1), \alpha_1\rangle \\ &= \tfrac{1}{2}\big(\mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) + \mathrm{Ric}(e_3) + \mathrm{Ric}(e_4) - 2K_{12} - 2K_{34} + 4R_{1234}\big) \\ &= K_{13} + K_{24} + K_{14} + K_{23} + 2R_{1234}. \end{aligned}$$

Using the first Bianchi identity, we conclude that

$$r_1 = K_{13} + K_{24} + 2R_{1324} + K_{14} + K_{23} - 2R_{1423} = 2(\lambda_2 + \lambda_3).$$

Similarly, we obtain

$$\begin{aligned} s_1 &= K_{13} + K_{24} + K_{14} + K_{23} - 2R_{1234} = 2(\varphi_2 + \varphi_3), \\ r_2 &= K_{12} + K_{34} + K_{14} + K_{23} - 2R_{1324} = 2(\lambda_1 + \lambda_3), \\ s_2 &= K_{12} + K_{34} + K_{14} + K_{23} + 2R_{1324} = 2(\varphi_1 + \varphi_3), \\ r_3 &= K_{12} + K_{34} + K_{13} + K_{24} + 2R_{1423} = 2(\lambda_1 + \lambda_2), \\ s_3 &= K_{12} + K_{34} + K_{13} + K_{24} - 2R_{1423} = 2(\varphi_1 + \varphi_2). \end{aligned} \tag{2.4}$$

From these equations we conclude that $r_i + 2\lambda_i = s_i + 2\varphi_i = S/2$, where S is the scalar curvature. □

We can therefore state our next result as follows.

PROPOSITION 2.5. *The Weitzenböck operator is given in terms of the scalar curvature by*

$$F^+ = \frac{S}{2} - 2\mathcal{R}_+^+ = \frac{S}{3} - 2W^+, \qquad F^- = \frac{S}{2} - 2\mathcal{R}_-^- = \frac{S}{3} - 2W^-.$$

Now, let $\mathrm{Hom}(TM, TM) \to M$ be the bundle of the homomorphism of the tangent bundle TM. We denote the space of 2-forms with values in $\mathrm{Hom}(TM, TM)$ by $\Omega^2(\mathrm{Hom}(TM, TM))$. Notice that the curvature tensor R is in $\Omega^2(\mathrm{Hom}(TM, TM))$ and that, for V and W in T_xM, $R(V, W)$ is the skew-symmetric endomorphism of T_xM given by

$$R(V, W)e_j = \sum_{k=1}^{4}\langle R(V, W)e_j, e_k\rangle e_k = \sum_{k=1}^{4}\langle \mathcal{R}(V \wedge W), e_{kj}\rangle e_k.$$

Since a 2-form induces a skew-symmetric endomorphism of T_xM by the formula $V \wedge W(X) = \langle V, X\rangle W - \langle W, X\rangle V$, we conclude that $R(V, W) = -\mathcal{R}(V \wedge W)$.

There are naturally induced metrics on

$$\mathrm{Hom}(TM, TM) \quad \text{and} \quad \Omega^2(\mathrm{Hom}(TM, TM)).$$

For the Riemannian vector bundle $\mathrm{Hom}(TM, TM) \to M$ with connection ∇, the Weitzenböck formula applied to the 2-form R gives (see [10, p. 95]):

$$(\Delta R)(V, W) = (\nabla^*\nabla R)(V, W) + \rho(R)(V, W),$$

where

$$\rho(R)(V,W) = R(\mathrm{Ric}(V),W) + R(V,\mathrm{Ric}(W)) + (\tilde{\mathcal{R}}R)(V,W) + (\bar{\mathcal{R}}R)(V,W).$$

If $\{e_i\}$, $i = 1,\dots,4$, is an orthonormal basis of T_xM, then

$$\mathrm{Ric}(V) = \sum_{k=1}^{4} R(V,e_k)e_k,$$

$$(\tilde{\mathcal{R}}R)(V,W) = \sum_{k=1}^{4} R(e_k, R(V,W)e_k),$$

$$(\bar{\mathcal{R}}R)(V,W) = \sum_{k=1}^{4} 2[R(e_k,V), R(e_k,W)].$$

Let us consider the orthonormal basis of Proposition 2.1 and the 2-forms α_i and β_i as before. Recall that in SO(4) we have

$$[e_{ij}, e_{km}] = \delta_{im}e_{kj} + \delta_{jm}e_{ik} + \delta_{ik}e_{jm} + \delta_{jk}e_{mi},$$

which imply

$$[\alpha_1,\alpha_2] = \sqrt{2}\alpha_3, \quad [\alpha_2,\alpha_3] = \sqrt{2}\alpha_1, \quad [\alpha_3,\alpha_1] = \sqrt{2}\alpha_2,$$
$$[\beta_1,\beta_2] = \sqrt{2}\beta_3, \quad [\beta_2,\beta_3] = \sqrt{2}\beta_1, \quad [\beta_3,\beta_1] = \sqrt{2}\beta_2,$$

and $[\alpha_i,\beta_j] = 0$. If we denote by $-\rho(\mathcal{R})(e_{ij})$ the algebraic term in the Weitzenböck formula $\rho(R)(e_i,e_j)$, then a straightforward computation yields the following result.

PROPOSITION 2.6. *Let $\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\}$ (defined as before) be an orthonormal basis diagonalizing the symmetric operator F. Then the basis $\{\alpha_1,\alpha_2,\alpha_3\}$ diagonalizes the symmetric operator $\rho(R_+^+)$ with corresponding eigenvalues*

$$\rho_i = \frac{S}{2}\lambda_i - 2\lambda_i^2 - 4\lambda_j\lambda_k, \quad j,k \neq i.$$

Similarly, the basis $\{\beta_1,\beta_2,\beta_3\}$ diagonalizes the symmetric operator $\rho(R_-^-)$ with corresponding eigenvalues

$$\sigma_i = \frac{S}{2}\varphi_i - 2\varphi_i^2 - 4\varphi_j\varphi_k, \quad j,k \neq i.$$

It then follows that $\operatorname{trace}\rho(R_+^+) = \operatorname{trace}\rho(R_-^-) = 0$. *Moreover, $\rho(R_-^+)\colon \Lambda_+^2 \to \Lambda_-^2$ is given by*

$$\rho(R_-^+)(\alpha_i) = \left(\frac{S}{2} - 2\lambda_i\right)R_-^+(\alpha_i) - 2\sqrt{2}[R_-^+(\alpha_j), R_-^+(\alpha_k)]$$

and $\rho(R_+^-)\colon \Lambda_-^2 \to \Lambda_+^2$ is given by

$$\rho(R_+^-)(\beta_i) = \left(\frac{S}{2} - 2\varphi_i\right)R_+^-(\beta_i) - 2\sqrt{2}[R_+^-(\beta_j), R_+^-(\beta_k)].$$

3. Preliminary Lemmas

In this section we collect some preliminary formulas that will be used to prove our results. Let $\langle\ ,\ \rangle$ be the naturally induced inner product on the space of 2-forms $\Lambda^2(T_xM)$, and let $\|\ \|$ be its corresponding norm.

LEMMA 3.1. *Using the previous notation, we have*

(i) $\langle\rho(R_+^+), R_+^+\rangle = -18\det R_+^+ + \frac{S}{4}\big[\frac{S^2}{16} - \|R_+^+\|^2\big] = -18\det W^+ + \frac{S}{2}\|W^+\|^2$;
(ii) $\langle\rho(R_-^-), R_-^-\rangle = -18\det R_-^- + \frac{S}{4}\big[\frac{S^2}{16} - \|R_-^-\|^2\big] = -18\det W^- + \frac{S}{2}\|W^-\|^2$.

Proof. From Proposition 2.6 we have that

$$\langle\rho(R_+^+), R_+^+\rangle = -12\det R_+^+ - 2(\lambda_1^3+\lambda_2^3+\lambda_3^3) + \frac{S}{2}\|R_+^+\|^2.$$

Using that trace $R_+^+ = S/4$, we obtain

$$\lambda_1^3+\lambda_2^3+\lambda_3^3+3\lambda_1^2(\lambda_2+\lambda_3)+3\lambda_2^2(\lambda_1+\lambda_3)+3\lambda_3^2(\lambda_1+\lambda_2)+6\det R_+^+ = \frac{S^3}{64}.$$

Therefore,

$$2(\lambda_1^3+\lambda_2^3+\lambda_3^3) = 6\det R_+^+ - \frac{S^3}{64} + \frac{3S}{4}\|R_+^+\|^2,$$

and this gives the first equality in (i); the second is obtained by replacing R_+^+ with $W^+ + S/12$. In a similar way, we obtain (ii). □

LEMMA 3.2. *Let $W_i^\pm$, $i = 1,2,3$, be the eigenvalues of $W^\pm$. Then we have:*

(a) $(W_i^\pm)^2 \le \frac{2}{3}\|W^\pm\|^2$ *and* $\det W^\pm = W_i^\pm\big[(W_i^\pm)^2 - \frac{1}{2}\|W^\pm\|^2\big]$;
(b) *if* $\|W^\pm\|^2 \le S^2/24$ *then* $F^\pm \ge 0$; *and*
(c) *if* $F^+ \ge 0$ *(respectively* $F^- \ge 0$*) then* $\langle\rho(R_+^+), R_+^+\rangle \ge 0$ *(respectively* $\langle\rho(R_-^-), R_-^-\rangle \ge 0$*).*

Moreover, if $W^\pm \ne 0$, *then* $F^\pm > 0$ *implies*

$$\langle\rho(R_+^+), R_+^+\rangle > 0 \quad \textit{and} \quad \langle\rho(R_-^-), R_-^-\rangle > 0.$$

Proof. Using that trace $W^\pm = 0$, we obtain (a). Now, if $\|W^\pm\|^2 \le S^2/24$ then we have

$$(W_i^\pm)^2 \le \frac{2}{3}\|W^\pm\|^2 \le \frac{S^2}{36};$$

together with Proposition 2.5, this yields (b). It also follows from Proposition 2.5 that if $F^\pm \ge 0$ and $W^\pm \ge 0$ then $W_i^\pm \le S/6$. Substituting in the expression for the determinant, we have

$$\det W^\pm \le W_i^\pm\left[\frac{2}{3}\|W^\pm\|^2 - \frac{1}{2}\|W^\pm\|^2\right] \le \frac{S}{36}\|W^\pm\|^2,$$

which, when substituted into Lemma 3.1, concludes the proof. □

LEMMA 3.3. *Let us consider the Weyl tensor W as 2-form with values in*

$$\mathrm{Hom}(TM, TM).$$

Then we have

$$\langle \rho(R_+^+), R_+^+\rangle = \langle \rho(W^+), W^+\rangle,$$
$$\langle \rho(R_-^-), R_-^-\rangle = \langle \rho(W^-), W^-\rangle.$$

Proof. Computing the algebraic terms of the Weitzenböck formula for $\frac{S}{12}I$, we derive $\rho(\frac{S}{12}I) = 0$ and then

$$\langle \rho(R_+^+), R_+^+\rangle = \langle \rho(W^+), W^+\rangle - \langle \rho(R_+^+), \tfrac{S}{12}I\rangle.$$

But the second term is zero, since trace $\rho(R_+^+) = 0$. □

Now, since both Δ and F commute with the Hodge $\star$-operator, the Weitzenböck formula can be written as

$$\Delta\omega^\pm = -\operatorname{div}\nabla\omega^\pm + F(\omega^\pm).$$

If M is compact, then integrating by parts we obtain

$$(\Delta\omega^\pm, \omega^\pm) = (\nabla\omega^\pm, \omega^\pm) + \int_M \langle F(\omega^\pm), \omega^\pm\rangle\, dV. \tag{3.4}$$

Here (,) is the inner product on $\Lambda^2(M)$ given by

$$(\phi, \psi) = \int_M \langle \phi, \psi\rangle\, dV,$$

where dV is the volume form of M and $\langle\ ,\ \rangle$ is the naturally induced inner product on the space of 2-forms $\Lambda^2(T_xM)$.

Let $H^2(M; \mathbf{R})$ denote the *second de Rham cohomology group* of M. If M is compact, it follows from Hodge's theorem that $H^2(M;\mathbf{R})$ is isomorphic to the space of harmonic 2-forms denoted by $\mathcal{H}$; because $\star\Delta = \Delta\star$, we obtain the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. We set $b_2^\pm = \dim \mathcal{H}^\pm$. Therefore, the second Betti number $b_2 = b_2^+ + b_2^-$ and the signature $\tau = b_2^+ - b_2^-$. From (3.4), the following lemma is immediate.

LEMMA 3.5. *Let M be a compact oriented 4-manifold. Then we have:*

(i) *if $F^\pm \geq 0$, then a 2-form $\omega^\pm$ is harmonic if and only if it is parallel; and*
(ii) *if $F^\pm \geq 0$ and there is a point $p \in M$ such that $F^\pm(p) > 0$, then $b_2^\pm = 0$.*

4. 4-Manifolds with Negative Isotropic Curvature

Let $T_xM \otimes \mathbf{C}$ denote the complexified tangent space, and extend the Riemannian metric $\langle\ ,\ \rangle$ to a complex bilinear form (,). An element Z in $T_xM\otimes\mathbf{C}$ is said to be *isotropic* if $(Z, Z) = 0$. A 2-plane $\sigma \subset T_xM \otimes \mathbf{C}$ is *totally isotropic* if $(Z, Z) = 0$ for any $Z \in \sigma$. If σ is a totally isotropic 2-plane then there exists a basis $\{Z, W\}$ of σ such that

$$Z = e_i + \sqrt{-1}\,e_j \quad \text{and} \quad W = e_k + \sqrt{-1}\,e_m,$$

where e_i, e_j, e_m, e_k are orthonormal vectors of T_xM. Conversely, any two such vectors span a totally isotropic 2-plane. Let $\tilde{\mathcal{R}}$ denote the complex linear extension of the curvature operator $\mathcal{R}: \Lambda^2 \to \Lambda^2$.

DEFINITION 4.1. A Riemannian manifold is said to have *nonnegative isotropic curvature* if $\langle\tilde{\mathcal{R}}(Z \wedge W), (Z \wedge W)\rangle \geq 0$ whenever $\{Z, W\}$ is a totally isotropic 2-plane.

It follows from this definition that, for Z and W as above,

$$\langle\tilde{\mathcal{R}}(Z \wedge W), (Z \wedge W)\rangle = K_{ik} + K_{im} + K_{jk} + K_{jm} - 2\langle R(e_i, e_j)e_k, e_m\rangle \geq 0,$$

where K_{ik} denotes the sectional curvature of the plane spanned by e_i, e_k and R is the curvature tensor of M (see [13, p. 203]). Then, for a 4-manifold M, the nonnegativity of the isotropic curvature is equivalent to the nonnegativity of the Weitzenböck operator F (see (2.4)). Moreover, from Proposition 2.5 we see that the nonnegativity of the isotropic curvature implies the nonnegativity of the scalar curvature.

LEMMA 4.2. *Let M be an oriented 4-manifold with nonnegative sectional curvatures. If $\|W^+\|^2 \geq S^2/24$ then $\|W^-\|^2 \leq S^2/24$. Moreover, if the first inequality is strict then so is the second.*

Proof. It follows from trace $\mathcal{R}_+^+ =$ trace $\mathcal{R}_-^- = S/4$ that if $\mathcal{R}_+^+ \geq 0$ then $\|W^+\|^2 \leq S^2/24$ and if $\mathcal{R}_-^- \geq 0$ then $\|W^-\|^2 \leq S^2/24$. Therefore, if $\|W^+\|^2 \geq S^2/24$ then $\mathcal{R}_+^+$ has one nonpositive eigenvalue. We show now that nonnegative curvature implies $\mathcal{R}_-^- \geq 0$, which finishes the proof. For that, suppose that after reordering the basis $\{\alpha_i\}$ we have $\lambda_1 \leq \lambda_2 \leq \lambda_3$. If $\lambda_1 \leq 0$, then (2.3) implies that $R_{1234} \geq 0$ and hence $\varphi_1 \geq 0$. In order to show that $\varphi_2 \geq 0$ and $\varphi_3 \geq 0$, we consider the planes $P = (\alpha_1 + \beta_2)/\sqrt{2}$ and $P^\perp = (\alpha_1 - \beta_2)/\sqrt{2}$. The proof of Proposition 2.1 shows that there is an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of the tangent space such that $f_{12} = (\alpha_1 + \beta_2)/\sqrt{2}$ and $f_{34} = (\alpha_1 - \beta_2)/\sqrt{2}$. Hence,

$$K(f_1, f_2) + K(f_3, f_4) = \lambda_1 + \varphi_2 \geq 0,$$

and if $\lambda_1 \leq 0$ then $\varphi_2 \geq 0$. In a similar manner we show that $\varphi_3 \geq 0$. This proof also shows that if $\lambda_1 < 0$ then $\varphi_i > 0$ for $i = 1, 2, 3$. Moreover, if the sectional curvatures are positive then $\lambda_1 \leq 0$ implies that $\varphi_i > 0$ for $i = 1, 2, 3$. □

Proof of Theorem 1

If each point of M has a nonpositive isotropic curvature, then by Lemma 3.2 we conclude that at each point either $\|W^+\|^2 \geq S^2/24$ or $\|W^-\|^2 \geq S^2/24$. Let us suppose then that $\|W^+\|^2 \geq S^2/24$ at some point x. Then, from Lemma 4.2 it follows that at this point $\mathcal{R}_-^- > 0$. We claim that this implies $\mathcal{R}_-^- > 0$ for all points of M. In fact, since $\mathcal{R}_-^- > 0$ on an open neighborhood U of x, we still have $\|W^+\|^2 \geq S^2/24$ for every point of U; otherwise, all isotropic curvatures at these points would be positive. Then, by continuity, $\varphi_1 \geq 0$ and $\|W^+\|^2 \geq S^2/24$ on the boundary of U. If $\varphi_1 = 0$ then the proof of Lemma 4.2 implies that $\mathcal{R}_+^+ > 0$, contradicting $\|W^+\|^2 \geq S^2/24$. Then $\mathcal{R}_-^- > 0$ on the boundary of U, and this can be extended to all points of M. Therefore $\|W^-\|^2 < S^2/24$ for each point of M, and Lemmas 3.2 and 3.5 imply that $b_2^- = 0$ and hence M is positive definite.

If $\Delta W = 0$ then the Weitzenböck formula for W^- gives

$$\langle \Delta W^-, W^- \rangle = \tfrac{1}{2}\Delta(\|W^-\|^2) + \|\nabla W^-\|^2 + \langle \rho(W^-), W^- \rangle.$$

Integrating by parts, for a compact oriented manifold we obtain

$$\int_M \langle \Delta W^-, W^- \rangle \, dV = \int_M \|\nabla W^-\|^2 \, dV + \int_M \langle \rho(W^-), W^- \rangle \, dV.$$

Now Lemmas 3.3 and 3.2(c) imply that $\nabla W^- = 0$. It is well known that this fact implies that either $W^- = 0$ or M is locally Kähler—that is, Kähler on a double cover (see [3, p. 455, 16.75(ii)). We claim that the second case cannot occur. In fact, the second case would imply that one of the eigenvalues $W_i^- = S/6$, which in turn gives $W_j^- + W_k^- = -S/6$. Using Lemma 3.3 and Lemma 3.1, we conclude that $\det W^- = (S/36)\|W^-\|^2$. This implies that $W_j^- = W_k^- = -S/12$ and hence $\|W^-\|^2 = S^2/24$, contradicting that $\|W^+\|^2 \geq S^2/24$ by Lemma 4.2. Therefore $W^- = 0$ and, for this orientation, M is self-dual. Moreover, since $S > 0$ and $\|W^+\|^2 \geq S^2/24$, M cannot be conformally flat. In addition, if the scalar curvature is constant, since $\Delta W = 0$ we have that the curvature is harmonic. Because W^+ is not zero, the signature of M is nonzero. A result of Bourguignon [4] implies that M is Einstein. But a compact self-dual Einstein manifold with positive scalar curvature is either isometric to S^4 or to $\mathbf{CP}^2$ (see [7] or [9]). Since in our case M is not conformally flat, the only possible case is $\mathbf{CP}^2$. □

Remark 4.3. Notice that if we assume in Theorem 1 that one point has a negative isotropic curvature, then the curvature cannot be harmonic because the standard metric in $\mathbf{CP}^2$ has nonnegative isotropic curvature. We can also conclude that M is definite by supposing that $K \geq 0$ and that each point has a negative isotropic curvature. With these assumptions, for $\Delta W = 0$ we have that M is half-conformally flat and that the curvature is not harmonic. In fact, if $\Delta R = 0$ then S is constant. If $S = 0$ then, since $K \geq 0$, M is flat. If $S > 0$ then we conclude, as in the proof of Theorem 1, that M is isometric to $\mathbf{CP}^2$. In each case M has nonnegative isotropic curvatures.

Lemma 4.4. *Let $S \geq 0$ and suppose that the eigenvalues of $W^\pm$ are ordered such that $W_1^\pm \leq W_2^\pm \leq W_3^\pm$. Then we have:*

(a) *if* $(W_2^-)^2 + (W_3^-)^2 \geq 5S^2/144$ *and* $\|W^-\|^2 \leq S^2/24$, *then* $\|W^-\|^2 \geq -W_1^- S/2$;

(b) *if* $\|W^-\|^2 \geq -W_1^- S/2$ *and* $\varphi_1 \geq 0$, *then* $\rho(R_-^-) = 0$;

(c) *if* $\rho(R_-^-) = 0$ *then either* $W^- = 0$ *or* $\varphi_1 = 0$ *and* $\|W^-\|^2 = S^2/24$.

Proof. Suppose that $\|W^-\|^2 < -W_1^- S/2$ and $\|W^-\|^2 \leq S^2/24$. Using Lemma 3.2, the second inequality implies that $\det W^- \leq (S/36)\|W^-\|^2$ and hence $\det W^- < -W_1^- S/72$. Recall that $\det W^- = W_1^-\big[(W_i^-)^2 - \frac{1}{2}\|W^-\|^2\big]$ (Lemma 3.2(a)), which in turn implies in the previous inequality that $\|W^-\|^2 < 2(W_1^-)^2 + S^2/36$. This inequality, when added with $\|W^-\|^2 \leq S^2/24$, gives $(W_2^-)^2 + (W_3^-)^2 < 5S^2/144$, contradicting the assumption. This proves (a). Now it follows from Proposition 2.6 that the eigenvalues of $\rho(R_-^-)$ are given by $\sigma_i = (S/2)\varphi_i - 2\varphi_i^2 - 4\varphi_j\varphi_k$ and trace $\rho(R_-^-) = 0$. Substituting $\varphi_i = W_i^- + S/12$, we obtain

$$\sigma_i = -6(W_i^-)^2 + \frac{S}{2}W_i^- + 2\|W^-\|^2. \tag{4.5}$$

If $\varphi_1 \geq 0$ then $W_1^- \geq -S/12$; together with $\|W^-\|^2 \geq -W_1^- S/2$, this implies in (4.5) that $\sigma_1 \geq 0$. Using (4.5) again, $\sigma_1 \geq 0$ gives that

$$W_1^- \geq \frac{S}{24} - \frac{1}{2}\sqrt{\frac{4}{3}\|W^-\|^2 + \frac{S^2}{144}}.$$

Because we are supposing that $W_2^- \geq W_1^-$, the same inequality holds for W_2^-. Then we obtain that

$$W_3^- \leq -\frac{S}{12} + \sqrt{\frac{4}{3}\|W^-\|^2 + \frac{S^2}{144}}.$$

But $\varphi_1 \geq 0$ implies $\|W^-\|^2 \leq S^2/24$ and hence

$$\sqrt{\frac{4}{3}\|W^-\|^2 + \frac{S^2}{144}} < \frac{S}{4},$$

which gives

$$W_3^- \leq \frac{S}{24} + \frac{1}{2}\sqrt{\frac{4}{3}\|W^-\|^2 + \frac{S^2}{144}},$$

which in (4.5) implies $\sigma_3 \geq 0$. We have shown that all eigenvalues of $\rho(R_-^-)$ are nonnegative; since its trace is zero, we have proved (b). Now observe that

$$\sigma_i - \sigma_j = 6(\varphi_i - \varphi_j)\varphi_k.$$

Therefore, if all the σ_i are zero, then either $\varphi_i = 0$ for $i = 1, 2, 3$ or $\varphi_i = \varphi_j$. The first case implies $W^- = 0$. Moreover, from Lemma 3.1 we obtain that if $\rho(R_-^-) = 0$ then $\det W^- = (S/36)\|W^-\|^2$. Hence, in the second case, the only possibility is $\varphi_1 = \varphi_2 = 0$ and $\varphi_3 = S/4$, which gives $\|W^-\|^2 = S^2/24$ and so proves (c). □

PROPOSITION 4.6. *Let M be an oriented 4-manifold with nonnegative sectional curvature. If $(W_2^-)^2 + (W_3^-)^2 \geq 5S^2/144$ for all points of M, then $F^+ \geq 0$. Therefore, if $(W_2^\pm)^2 + (W_3^\pm)^2 \geq 5S^2/144$ then $F \geq 0$.*

Proof. Suppose that, for some point of M, F^+ has a negative eigenvalue. It follows from the proof of Lemma 4.2 that $\mathcal{R}_-^- > 0$ at this point. Since $\mathcal{R}_-^- > 0$ implies $\|W^-\|^2 < S^2/24$, from Lemma 4.4 we obtain that $W^- = 0$. But then $(W_2^-)^2 + (W_3^-)^2 \geq 5S^2/144$ implies $S = 0$, and since $K \geq 0$ we conclude that all sectional curvatures are zero at this point. This in turn implies $W^+ = 0$, which contradicts that F^+ has a negative eigenvalue. □

PROPOSITION 4.7. *Let M be an oriented 4-manifold with positive sectional curvature. If $\|W^-\|^2 \geq -W_1^- S/2$ for all points of M, then either $F^+ \geq 0$ or $W^- = 0$.*

Proof. If F^+ is not a nonnegative operator, then F^+ has a negative eigenvalue at some point of M. We again have $\mathcal{R}_-^- > 0$ at this point, and from Lemma 4.4(b) and (c) we conclude that $W^- = 0$, implying $\mathcal{R}_-^- = S/12$. We are supposing that the sectional curvatures are positive, so $\mathcal{R}_-^-$ is positive on an open neighborhood U of this point. Then we apply Lemma 4.4(b) and (c) once again to the points of U. We conclude that $W^- = 0$ for each point of U and hence on the boundary of U. This implies $\mathcal{R}_-^- > 0$ for all points on the boundary of U; continuing this procedure, we obtain that $W^- = 0$ for all points of M. □

The assumptions of Theorem 1 and Proposition 4.7 implied that $F^- > 0$ for all points of M. We now look for conditions which guarantee that one of the components of F is always nonnegative, even for cases where $\|W^\pm\|^2 < S^2/24$ for some point or where $W^\pm \neq 0$. First, observe that if F^+ has a nonpositive eigenvalue then, by Proposition 2.5, we have $W_3^+ \geq S/6$. If the sectional curvatures are nonnegative then from Lemma 4.2 we conclude that $\varphi_1 \geq 0$, implying $W_1^- \geq -S/12$. In this case we then have $W_3^+ + W_1^- \geq S/12$. Under this condition, we prove the following proposition.

PROPOSITION 4.8. *Let M be an oriented 4-manifold with positive sectional curvature. If $W_3^+ + W_1^- \geq S/12$, then either $F^+ \geq 0$ or $F^- > 0$.*

Proof. Let us suppose that F^+ has a negative eigenvalue at some x in M. Then, as before, on an open set U containing x we have $\|W^+\|^2 > S^2/24$ and $\mathcal{R}_-^- > 0$. Now, if y is a point on the boundary of U then we have $\|W^+\|^2 \geq S^2/24$ and $\mathcal{R}_-^- \geq 0$ at y. But if $\varphi_1 = 0$ then—since we have positive sectional curvatures—Lemma 4.2 implies that $\mathcal{R}_+^+ > 0$, which contradicts $\|W^+\|^2 \geq S^2/24$. Therefore we still have $\mathcal{R}_-^- > 0$ at y. If $\|W^+\|^2 \geq S^2/24$ on an open neighborhood of y where $\mathcal{R}_-^- > 0$, we repeat the argument to obtain the same conclusions. However, if $\|W^+\|^2 < S^2/24$, Lemma 3.2 implies that $W_3^+ < S/6$. Therefore, our assumption implies

$$S/6 + W_1^- > W_3^+ + W_1^- \geq S/12$$

and hence $W_1^- > -S/12$, that is, $\varphi_1 > 0$. Then $\mathcal{R}_-^- > 0$, since φ_1 is the smallest eigenvalue. We then conclude that $\|W^-\|^2 < S^2/24$ for every point of M and hence that $F^- > 0$. □

PROPOSITION 4.9. *Let M be an oriented 4-manifold with nonnegative sectional curvatures. If all sectional curvatures satisfy $K \geq S/16$ then $F \geq 0$.*

Proof. Suppose that, at some point, F^+ has a negative eigenvalue. Then we have $W_3^+ > S/6$. Again from Lemma 4.2 we obtain that $\varphi_1 > 0$ and hence $W_1^- > -S/12$. Then $W_3^+ + W_1^- > S/12$. With the notation of Section 2 and using Proposition 2.5, we conclude that $r_3 + s_1 < S/2$. As in the proof of Lemma 4.2, we consider planes $P = (\alpha_3 + \beta_1)/\sqrt{2}$ and $P^\perp = (\alpha_3 - \beta_1)/\sqrt{2}$; we obtain an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of the tangent space such that $f_{12} = (\alpha_3+\beta_1)/\sqrt{2}$ and $f_{34} = (\alpha_3 - \beta_1)/\sqrt{2}$. Hence, by the definition of F we have

$$r_3 + s_1 = 2[K(f_1, f_3) + K(f_1, f_4) + K(f_2, f_3) + K(f_2, f_4)] < S/2.$$

Hence S cannot be zero because this would imply that all sectional curvatures are zero, which contradicts (4.10). Moreover, (4.10) implies that the smallest curvature is less than $S/16$, contradicting our assumption. Thus F cannot have a negative eigenvalue, since the same proof could be applied to F^-. □

Proof of Theorem 2

The first part follows from the previous propositions, and we have that $F^+ \geq 0$ or $F^- > 0$ or $F \geq 0$. Now, suppose that M is compact. If $F^- > 0$, then Lemma 3.5 implies that $b_2^- = 0$ and M is positive definite. If $F^+ \geq 0$ and $b_2^+ = 0$ then M is negative definite. Now, if $F^+ \geq 0$ and $b_2^+ > 0$, from Lemma 3.5(a) we conclude that there exists a self-dual parallel 2-form. Then $\dim \operatorname{Ker} F^+ > 0$ and $\|W^+\|^2 = S^2/24$ for all points of M. Now Lemmas 4.2 and 3.2 imply that $F \geq 0$, that is, M has nonnegative isotropic curvatures. Since M is oriented, by Synge's theorem it is simply connected; because $b_2^+ > 0$, M is biholomorphic to $\mathbf{CP}^2$ by a result in [16]. Therefore M is definite. □

5. Nonnegatively Curved 4-Manifolds with $\Delta W = 0$

In this section we study first the case that the self-dual component $F^+ \geq 0$. In [14] the authors proved that if $\Delta W^+ = 0$ then either $W^+ = 0$ or the universal cover is Kähler with constant positive scalar curvature. A complete classification is the following.

THEOREM 5.1. *Let M be a compact oriented 4-manifold such that $\Delta W^+ = 0$ and $F^+ \geq 0$. Then one of the following holds.*

(a) *$W^+ = 0$ is identically zero and M is negative definite.*

(b) *M is an anti-self-dual Kähler manifold with $b_2^+ = 1$.*

(c) *M is a quotient of a* K3*-surface, either with a Ricci-flat* (*i.e. Calabi–Yau*) *metric or conformally flat isometrically covered by either $\mathbf{R}^4$ or $S^2 \times \mathbf{H}^2$, where S^2 is the standard sphere and $\mathbf{H}^2$ is the hyperbolic plane.* (A K3-surface is a complex surface with first Betti number $b_1 = 0$ and first Chern class $c_1 = 0$.)

(d) *M is locally Kähler* (*Kähler on a double isometric covering*) *and has constant positive scalar curvature. If M is Kähler then $b_2^+ = 1$.*

(e) *The universal cover has constant positive scalar curvature and is a Riemannian product $N_1 \times N_2$, where N_1 is homeomorphic to S^2.*

Proof. Integrating the Weitzenböck formula we obtain

$$\int_M \langle \Delta W^+, W^+ \rangle \, dV = \int_M \|\nabla W^+\|^2 \, dV + \int_M \langle \rho(W^+), W^+ \rangle \, dV.$$

It follows from Lemmas 3.3 and 3.2(c) that $\nabla W^+ = 0$. This in turn implies that either $W^+ = 0$ or $\|W^+\|^2 = S^2/24$ is constant and so is S. Next we consider all possibilities.

If there is a point in M for which $F^+(p) > 0$, then W^+ is identically zero. In fact, since $\|W^+\|$ is constant, if $W^+(p)$ is not zero then by Lemmas 3.2(c) and 3.3 we would have $\langle \rho(W^+), W^+ \rangle > 0$, contradicting the Weitzenböck formula for W^+, since $\Delta W^+ = 0$. Therefore $W^+(p) = 0$ and W^+ is identically zero. Now the fact that $F^+(p) > 0$ implies, by Lemma 3.5, that $b_2^+ = 0$; this is case (a).

If $\dim \operatorname{Ker} F^+ > 0$ for every point in M, it follows from the first part of this proof that $\|W^+\|^2 = S^2/24$ and S is constant. If $S = 0$, we conclude again that W^+ is identically zero and $F^+ = 0$. If $b_2^+ = 0$, we are again in case (a). If $b_2^+ > 0$ then, by Lemma 3.5, there exists a self-dual harmonic 2-form that is parallel. Thus M is an anti-self-dual Kähler manifold with the natural orientation. Here we have three possibilities as follows.

Case 1: The universal cover $\tilde{M}$ is an isometric product $N \times \mathbf{R}$. Consider an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that e_4 is tangent to $\mathbf{R}$. Now the formulas (2.4) together with the fact that $F^+ = 0$ give $F^- = 0 = W^-$ and M is conformally flat. Moreover, the eigenvalues of F are the Ricci curvatures and hence $\tilde{M}$ is Ricci flat; being conformally flat, $\tilde{M} = \mathbf{R}^4$, and this is one of the cases in (c).

Case 2: The universal cover $\tilde{M}$ is an isometric product $N_1 \times N_2$ and N_i is 2-dimensional. Consider an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that e_1 and e_2 are tangent to N_1 and e_3 and e_4 are tangent to N_2. From the formulas (2.4) and the fact that $F^+ = 0$, we obtain again that $\tilde{M}$ is conformally flat and so is M. Now, [15, Prop. 5.2] yields another case in (c).

Case 3: The restricted holonomy group G of M is irreducible. Recall that in this case Berger [2] proved that either M is locally symmetric or the only possibilities for G are $SU(2)$ or $U(2)$, since M is Kähler. Because we are supposing that M is locally irreducible, if M is locally symmetric then M is Einstein and hence Ricci-flat. In this case we conclude that M is flat. If $G = SU(2)$, Berger also proved that the metric is Ricci-flat; then $\tilde{M}$ has the Calabi–Yau metric (see [21]), yielding the remaining case in (c). If $G = U(2)$ then this is a holonomy group. This case will imply $b_2^+ = 1$, and we will be in case (b). In fact, Lemma 3.5 implies that each self-dual harmonic 2-form is parallel. Since $\mathbf{CP}^2$ has the same holonomy, each such form gives rise to a parallel and hence harmonic 2-form on $\mathbf{CP}^2$, by the holonomy principle. Because $b_2(\mathbf{CP}^2) = 1$, we have that $b_2^+ = 1$.

If $S > 0$ and $\dim \operatorname{Ker} F^+ > 0$ for every point in M, we can suppose (considering a double cover, if necessary) that M is Kähler. Then we proceed as above, studying the three possible cases. In Case 1, the formulas (2.4) would imply that N has nonnegative Ricci curvatures. The topological classification of compact 3-manifolds with nonnegative Ricci curvature in [8] implies that N is homeomorphic to S^3, contradicting that $b_2 > 0$. In Case 2 we have $K_{12} + K_{34} = S > 0$ and S is constant. Therefore, if each N_i has a point with nonpositive curvature, this contradicts $S > 0$. Then, say, N_1 has positive curvature, yielding (e). Now, in

Case 2 the only possibility for the restricted holonomy group is $U(2)$, which gives case (d) by the holonomy principle. □

REMARK 5.2. In case (b), there is more to be said if the first Betti number $b_1 = 0$. LeBrun observed in [11] that, with such assumptions, M is diffeomorphic to $\mathbf{CP}^2 \sharp m\overline{\mathbf{CP}^2}$ for $m > 9$, where $m\overline{\mathbf{CP}^2}$ is $\mathbf{CP}^2$ with the conjugate orientation and m points blown up.

COROLLARY 5.3. *Let M be a compact oriented 4-manifold such that $\Delta W^+ = 0$, $F^+ \geq 0$, and $K \geq 0$. Then one of the following holds*:

(a) $W^+ = 0$ *is identically zero and M is negative definite*;
(b) *M is flat*;
(c) *M is biholomorphic to* $\mathbf{CP}^2$;
(d) *M has constant positive scalar curvature and the universal cover is a Riemannian product $N_1 \times N_2$, where each N_i is homeomorphic to S^2.*

Proof. In (b) and (c) of Theorem 5.1, M is Ricci-flat, implying (b) of the Corollary. In cases (d) and (e) of the theorem we have $\|W^+\|^2 = S^2/24$. Now, Lemma 4.2 implies that $\|W^-\|^2 \leq S^2/24$ and from Lemma 3.2(c) we conclude that M has nonnegative isotropic curvature. If M is locally reducible then we are in case (d) of the Corollary. If M is locally irreducible and Kähler then it follows from [14, Thm. 2.1(b)] that M is simply connected, and [16, Thm. 1] implies that M is biholomorphic to $\mathbf{CP}^2$. This implies, in (d) of Theorem 5.1, that M itself is Kähler; by the foregoing, M would otherwise be covered by $\mathbf{CP}^2$, which is clearly a contradiction. □

Proof of Theorem 3

Now we add the assumption that $F^- \geq 0$ to the cases studied in Theorem 5.1. From Lemma 3.5 we obtain that if $F^+ > 0$ at some point p; then $b_2^+ = 0$. If $b_2^- = 0$ then the signature of M is zero. Since in this case we have $W^+ = 0$, we conclude that M is conformally flat and S is positive at p. If $b_2^- > 0$, we use again Lemma 3.5 to conclude that M is a Kähler manifold in the conjugate orientation. Reversing orientation, we get that M is isometric to $\mathbf{CP}^2$ with its standard metric (see [16, Prop. 3.8]). If $\dim \operatorname{Ker} F^+ > 0$ for every point in M and $S = 0$, since $F \geq 0$, we again have $W = 0$. For the remaining cases we can suppose, passing to a double cover, that M is Kähler. If the universal cover of M splits isometrically, we are in case (b) of the theorem. If the universal cover is not a product then $b_2 = 1$ and $S > 0$. In this case, M is biholomorphic to $\mathbf{CP}^2$.

Before we prove Corollary 1, we consider a Kähler manifold with the natural orientation and constant scalar curvature. In this case $\Delta W^+ = 0$. Then we prove the following proposition.

PROPOSITION 5.4. *Let M be a compact oriented Kähler manifold with constant scalar curvature. If $\Delta R_-^+ = 0$ then either M is Einstein or is covered by the isometric product of two surfaces of constant curvature. It follows then that the Ricci tensor is parallel and therefore $\Delta R = 0$.*

REMARK 5.5. Matsushima and Tanno (see [12; 20]) proved that a Kähler manifold with $\Delta W = 0$ must have parallel Ricci tensor. Here, in dimension 4, we can substitute the harmonicity of W^- by the harmonicity of R_-^+.

Proof. Let ω be the Kähler form in Λ_+^2. Since ω is parallel, we have that ω is an eigenvector of F^+ with corresponding eigenvalue zero. On an open dense set U of M, where W^+ has eigenvalues of constant multiplicity, we can find local sections e_1, e_2, e_3, e_4 of TM such that at $x \in U$ the basis $\{e_i(x)\}$ is the orthonormal basis of Proposition 2.1. Without loss of generality, we can suppose that $\omega = \alpha_1$. Then $\mathcal{R}(\alpha_2) = \mathcal{R}(\alpha_3) = 0$ and, with the notation used in (2.3), we have $\lambda_1 = S/4$ and $\lambda_2 = \lambda_3 = 0$. From (2.6) we conclude that $\rho(R_-^+) = 0$. Let μ be the Ricci form of M given by $\mu(X, Y) = \mathrm{Ric}(X, JY)$, where J is the complex structure given by ω. We then have

$$\mathrm{Ric}(e_1, e_2) = \mathrm{Ric}(e_3, e_4) = 0, \quad \mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_2, e_2),$$
$$\mathrm{Ric}(e_3, e_3) = \mathrm{Ric}(e_4, e_4).$$

Moreover, since F^+ has an eigenvalue of multiplicity 2, we can find a suitable linear combination of α_2 and α_3 such that $\{e_i\}$ is a basis of eigenvectors for the Ricci curvature operator. Therefore the only nonnull entry in the matrix of R_-^+ with respect to the bases $\{\alpha_i\}$ and $\{\beta_i\}$ is $\langle \mathcal{R}(\alpha_1), \beta_1 \rangle$. Let us denote this number by R_{11}. Now, since $\Delta R_-^+ = 0$ and $\rho(R_-^+) = 0$, the Weitzenböck formula integrated over M gives that $\nabla R_-^+ = 0$. This implies that R_{11} is constant on the dense set U and hence on M. If R_{11} is identically zero then M is Einstein. If R_{11} is never zero then we have the local equation

$$(\nabla_X R_-^+)(\alpha_1) = X(R_{11})\beta_1 + R_{11}\nabla_X \beta_1 - R_-^+(\nabla_X \alpha_1) = 0,$$

which implies that $\nabla_X \beta_1 = 0$, since R_{11} is constant and $R_-^+(\nabla_X \alpha_1) = 0$, because $\nabla_X \alpha_1$ is orthogonal to α_1. The existence of this local parallel section in Λ_-^2 implies that $\varphi_1 = S/4$ and $\varphi_2 = \varphi_3 = 0$ (since $\lambda_1 = S/4$ and $\lambda_2 = \lambda_3 = 0$). Thus M is locally a product of two surfaces, the constant scalar curvature is given by $S = K_{12} + K_{34}$, and—since $2R_{11} = K_{12} - K_{34}$ is constant—we have that M is covered by the isometric product of two surfaces of constant curvature. □

Proof of Corollary 1

We apply Proposition 5.4 to the possible cases of Theorem 3. In case (b), M is covered by a Riemannian product of two surfaces and hence, by Proposition 5.4, the surfaces have constant curvature. If M is Einstein then $R_-^+ = R_+^- = 0$. Since $F \geq 0$, Lemma 3.2(c) implies that $\langle \rho(R_+^+), R_+^+ \rangle \geq 0$ and $\langle \rho(R_-^-), R_-^- \rangle \geq 0$. Now the Weitzenböck formula gives that $\nabla R_+^+ = 0$ and $\nabla R_-^- = 0$, implying that M is locally symmetric. Thus, if M is locally irreducible, M is isometric to S^4 or to $\mathbf{CP}^2$.

THEOREM 5.6. *Let M be a compact oriented 4-manifold with nonnegative sectional curvatures. If all sectional curvatures satisfy $K \geq S/16$ and $\Delta W = 0$ then*

M is conformally flat, or is isometric to $\mathbf{CP}^2$*, or is covered by* $S^2 \times S^2$*, where* S^2 *has constant curvature.*

Proof. Recall that Proposition 4.9 implies that M has nonnegative isotropic curvature. Since now we have $\Delta W^{\pm} = 0$, this implies that (i) $W = 0$, or (ii) $\|W^{\pm}\|^2 = S^2/24$, or (iii) one of the components of W (say, W^-) is zero and $\|W^+\|^2 = S^2/24$. In the second case and third cases we have that S is constant, and without loss of generality we can assume that M is Kähler. If $\|W^{\pm}\|^2 = S^2/24$ then the signature is zero. In this case M is locally reducible; otherwise, [14, Thm. 2.1] would imply that $b_2(M) = 1$. Hence M is covered by $S^2 \times S^2$, and since the curvature of M is harmonic, S^2 has constant curvature. In the latter case, if $S = 0$ then M is flat. If $S > 0$ then M is biholomorphic to $\mathbf{CP}^2$, and since S is constant and $\Delta W = 0$, M is Einstein and thus isometric to $\mathbf{CP}^2$. □

Proof of Theorem 4

If $F^+ \geq 0$, we conclude either that W^+ is identically zero or S is constant and $\|W^+\|^2 = S^2/24$ for all points of M. The second case implies by Lemmas 4.2 and 3.2 that $F^- \geq 0$, and the Weitzenböck formula gives that either W^- is identically zero or $\|W^-\|^2 = S^2/24$. Therefore M has nonnegative isotropic curvatures. If $\|W^-\|^2 = S^2/24$ then, as before, we conclude that either M is flat or covered by $S^2 \times S^2$, where S^2 has its standard metric. If $W^- = 0$ and $S = 0$ then we obtain again that M is flat. If $W^- = 0$ and $S > 0$ then, since S is constant, the curvature is harmonic. It follows from Bourguignon's theorem [4] that M is an Einstein manifold, because the signature is nonzero. Since the isotropic curvatures are nonnegative, M is isometric to $\mathbf{CP}^2$. For the case where W^+ is identically zero, if $W^- = 0$ then M is conformally flat and hence either conformally equivalent to S^4 or covered by $\mathbf{R}^4$ or $S^3 \times \mathbf{R}$, where S^3 has constant curvature (see [15, Thm. 1]). If $W^- \neq 0$, we conclude from Corollary 5.3(a) that M is negative definite.

If F^+ has a negative eigenvalue at some point and hence on a neighborhood U of this point, then $\|W^+\|^2 > S^2/24$ for each point in U. From Lemma 4.4(b) and (c) we have that either $W^- = 0$ or $\|W^-\|^2 = S^2/24$ on U. But the second assertion contradicts $\|W^+\|^2 > S^2/24$ by Lemma 4.2. Therefore $W^- = 0$ on an open set and so W^- is identically zero by [1], since it satisfies $\Delta W^- = 0$. In this case S cannot be constant, because this would imply that the curvature is harmonic; since M is self-dual, it is Einstein by [4]. If $S = 0$ then M is flat. If $S > 0$ then, by the results in [7] and [9], M is isometric to $\mathbf{CP}^2$. Either case contradicts that F^+ has a negative eigenvalue. □

REMARK 5.7. If a nonnegatively curved compact oriented 4-manifold satisfies condition (c) of Theorem 2 and if $\Delta W = 0$, we can say more when $W^+ = 0$. In $W_3^+ + W_1^- \geq S/12$, $W^+ = 0$ implies $W^- = 0$ and $S = 0$ and therefore M is flat.

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