Polynomial Hulls of Sets in $\mathbb{C}^3$ Fibered over the Unit Circle

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Introduction

Let X be a compact subset of $\mathbb{C}^2$ lying over the unit circle $\mathbb{T}$; that is, letting $\pi(z, w) = z$ be the projection to the first coordinate, $\pi(z) \in \mathbb{T}$ for all $z \in X$. The fiber $X_z = \{ w \in \mathbb{C} : (z, w) \in X \}$ is identified with $\{z\} \times X_z = X \cap \pi^{-1}(z) \subseteq \mathbb{C}^2$. A number of authors [AW; S1; F] have discussed the polynomial hull $\hat{X}$ of X. The following definitive result was obtained by Slodkowski [S2].

THEOREM. *Suppose that each fiber X_z is connected and simply connected. Then $\hat{X} \setminus X$ is the union of graphs of H^∞ functions h whose boundary values $h^*(z)$ are contained in X_z for almost all $z \in \mathbb{T}$.*

Without the assumption that the fibers are connected, the conclusion no longer holds and in fact the hull may contain no analytic structure. This is the case in the example of Wermer [W] discussed below, where the fibers X_z are totally disconnected.

The H^∞ functions h of the theorem are sometimes referred to as analytic selection functions. In the context of a connected compact plane set, "simply connected" means having connected complement in $\mathbb{C}$; this is equivalent to being polynomially convex. When X is a compact subset of $\mathbb{C}^3$ lying over the unit circle, a statement like that of the above theorem—that the hull of X is composed of the graphs of ($\mathbb{C}^2$-valued) selection functions—is no longer true, at least when the fibers X_z are not linearly convex: this special case carries over to all dimensions [AW, S1]. Helton and Merino [HM] and Černe [C] have given examples of sets X in $\mathbb{C}^3$ with "nice" fibers X_z such that $\pi(\hat{X})$ is the closed unit disk but where no analytic selection functions exist. In their examples the reason that $\hat{X}$ covers the disk is that there is a 1-variety with boundary in X. In particular, $\hat{X}$ contains analytic structure. In general, however, as was first shown by Stolzenberg [St], polynomial hulls need not contain analytic structure. The purpose of this note is to construct a compact subset X of $\mathbb{C}^3$, lying over the unit circle, such that $\hat{X}$ covers the unit disk but contains no analytic structure and such that the fibers are topologically simple, like those in Slodkowski's theorem—in our case the polynomially convex fibers $X_z \subseteq \mathbb{C}^2$ will be contractible to a point in $\mathbb{C}^2$. It is of interest to note

Received February 16, 1996.
Michigan Math. J. 43 (1996).

that Slodkowski [S3] has recently obtained some positive results on polynomial hulls of sets $X \subseteq \mathbb{C}^3$ lying over the unit circle.

THEOREM 1. *There exists a compact subset X of $\mathbb{C}^3$ lying over $\{|z| = 1\}$ with $\pi(\hat{X}) = \{|z| \leq 1\}$ such that each fiber X_z is a contractible polynomially convex set and such that $\hat{X} \cap \{|z| < 1\}$ contains no analytic variety of positive dimension.*

That $\hat{X}$ contains no analytic variety of positive dimension is equivalent to the fact that every holomorphic map from the unit disk into $\hat{X}$ is constant.

To produce $X \subseteq \mathbb{C}^3$ we shall adapt a method of Wermer [W], who constructed a set in $\mathbb{C}^2$ over the unit circle whose polynomial hull covers the unit disk and which contains no analytic variety of positive dimension. A different variant of Wermer's example has been given by Levenberg [L]. First we "localize" Wermer's example in $\mathbb{C}^2$.

THEOREM 2. *There exists a compact subset Y of $\mathbb{C}^2$ with $\pi(Y) \subseteq \mathbb{T}$ such that:*

(i) $\pi(\hat{Y}) = \{|z| \leq 1\}$,

(ii) *for all closed proper subsets α of $\mathbb{T}$ and $M > 0$,*

$$\hat{Z} \setminus Z = \hat{Y} \setminus Y,$$

where $Z = Y \cup \{ (z, w) : z \in \alpha, |w| \leq M \}$; and

(iii) *$\hat{Y} \setminus Y$ contains no analytic variety of positive dimension.*

REMARK. Aside from (ii), this is Wermer's example. The main point here is that one can "fatten" the fibers (to disks of large radius) over an arbitrary proper closed subset α and still have a set $\hat{Z}$ with no analytic structure over $\{|z| < 1\}$. This is in contrast to the sets in Slodkowski's theorem: the hulls of these sets become strictly larger over the unit disk if the sets are "fattened" over a set α as in (ii).

We shall first indicate the construction for Theorem 2 and then apply it to the proof of Theorem 1.

1. Proof of Theorem 2

The construction of Y is a refinement of Wermer's construction. We shall adopt his notation, which we recall here for the convenience of the reader. By a change of scale in the z-variable, we replace the unit circle by $\{|z| = 1/2\}$; in particular, α is now a proper closed subset of $\{|z| = 1/2\}$. Let $a_1, a_2, \dots$ denote the points in the disk $|z| < 1/2$ both of whose coordinates are rational. For each j let B_j be the algebraic (2-valued) function $B_j = (z_1 - a_1)(z - a_2) \cdots (z - a_{j-1})\sqrt{z - a_j}$. Let $g_n = \sum_1^n c_j B_j$, where the c_j are positive numbers to be chosen below. Let $\{w_j(z)\}_{1 \leq j \leq 2^n}$ be the set of 2^n values of $g_n(z)$—these are not necessarily distinct. Let $\Sigma(c_1, c_2, \dots, c_n)$ be the intersection of the "graph" of g_n in $\mathbb{C}^2$ with $|z| \leq 1/2$; that is,

$$\Sigma(c_1, c_2, \dots, c_n) = \{ (z, w) \in \mathbb{C}^2 : |z| \leq 1/2,\ w = w_j(z) \text{ for } 1 \leq j \leq 2^n \}.$$

Let $\mathcal{B}$ be a countable basis for the topology of $\mathbb{T}$ consisting of nonempty open subsets of $\mathbb{T}$. Let $\{\alpha_n\}$ be a sequence of closed proper subsets of $\mathbb{T}$ such that, for each $W \in \mathcal{B}$, $\alpha_n = \mathbb{T} \setminus W$ for infinitely many n. This implies that for every closed proper subset α of $\mathbb{T}$, $\alpha \subseteq \alpha_n$ for infinitely many n. We put $E_n = \{ (z, w) \in \mathbb{C}^2 : z \in \alpha_n \text{ and } |w| \leq n \}$. Set $r_n = (1 - 1/n)(1/2)$, $n = 1, 2, \ldots$. Aside from (2+), the following lemma is taken directly from [W].

LEMMA 1. *There exists a sequence c_j $(j = 1, 2, \ldots)$ of positive constants with $c_1 = 1/10$ and $c_{j+1} \leq (1/10)c_j$, there exists a sequence of positive constants $\{\varepsilon_j\}$, and there exists a sequence of polynomials $\{P_j\}$ in z and w such that:*

$$\{P_j = 0, |z| \leq 1/2\} = \Sigma(c_1, c_2, \ldots, c_j),\ j = 1, 2, \ldots ; \tag{1}$$

$$\{P_{j+1} \leq \varepsilon_{j+1}, |z| \leq 1/2\} \subseteq \{P_j < \varepsilon_j, |z| \leq 1/2\},\ j = 1, 2, \ldots ; \tag{2}$$

$$\begin{aligned}(E_{j+1} \cup \{|P_{j+1}| \leq \varepsilon_{j+1}, |z| = 1/2\})^{\widehat{}} \cap \{|z| \leq r_j\} \\ \subseteq \{|P_j| < \varepsilon_j, |z| \leq r_j\},\ j = 1, 2, \ldots ; \textit{ and}\end{aligned} \tag{2+}$$

$$\begin{aligned}&\textit{if } |a| \leq 1/2 \textit{ and } |P_j(a, w)| \leq \varepsilon_j, \textit{ then there exists } w_j \textit{ with} \\ &P_j(a, w_j) = 0 \textit{ and } |w - w_j| < 1/j,\ j = 1, 2, \ldots .\end{aligned} \tag{3}$$

For the proof of Lemma 1 we shall need the following.

LEMMA 2. *Let $\delta_1 > 0$, $C > 0$, and $0 < r < 1/2$. Let α be a proper closed subset of $\{|z| = 1/2\}$ and set $E = \{ (z, w) \in \mathbb{C}^2 : z \in \alpha \text{ and } |w| \leq C \}$. Let Q_1 and Q_2 be polynomials in z and w that are monic in w. Suppose that $\{Q_2 = 0, |z| = 1/2\} \subseteq \{|Q_1| < \delta_1, |z| = 1/2\}$. Then there exists a $\delta_2 > 0$ such that*

$$(E \cup \{|Q_2| \leq \delta_2, |z| = 1/2\})^{\widehat{}} \cap \{|z| \leq r\} \subseteq \{|Q_1| < \delta_1, |z| \leq r\}.$$

Proof. Set $S_k = E \cup \{|Q_2| \leq 1/k, |z| = 1/2\}$, $k = 1, 2, \ldots$. Then $S_k \downarrow S$ where $S = E \cup \{Q_2 = 0, |z| = 1/2\}$. Hence $\hat{S}_k \downarrow \hat{S}$.

We claim that $\hat{S} = E \cup \{Q_2 = 0, |z| \leq 1/2\}$. Since E is polynomially convex, $\hat{S} \setminus S \subseteq \{|z| < 1/2\}$. Also $\{Q_2 = 0, |z| \leq 1/2\} \subseteq \hat{S}$ by the maximum principle, since $\{Q_2 = 0, |z| = 1/2\} \subseteq S$. Suppose that $(a, b) \in \hat{S}$ with $|a| < 1/2$. Let μ be a Jensen measure [B] for (a, b), supported on S. Then

$$\log|Q_2(a, b)| \leq \int_S \log|Q_2|\, d\mu.$$

The push-forward $\pi_*(\mu)$ is Poisson measure on $\{|z| = 1/2\}$ for the point a. Therefore $\mu(S \setminus E) = \pi_*(\mu)(\{|z| = 1/2\} \setminus \alpha) > 0$, because α is a proper subset of $\{|z| = 1/2\}$. Since $\log|Q_2| = -\infty$ on $S \setminus E$, we have $\log|Q_2(a, b)| = -\infty$; that is, $(a, b) \in \{Q_2 = 0, |z| \leq 1/2\}$. This yields the claim.

Since $\{Q_2 = 0, |z| = 1/2\} \subseteq \{|Q_1| < \delta_1, |z| = 1/2\}$, we have that $\{Q_2 = 0, |z| \leq 1/2\} \subseteq \{|Q_1| < \delta_1, |z| \leq 1/2\}$ and therefore $\hat{S} \cap \{|z| \leq r\} \subseteq \{|Q_1| < \delta_1, |z| \leq r\}$. Hence, for k sufficiently large, we have $\hat{S}_k \cap \{|z| \leq r\} \subseteq \{|Q_1| < \delta_1, |z| \leq r\}$. The lemma follows by taking $\delta_2 = 1/k$. □

Proof of Lemma 1 (after [W]; modified to achieve (2+)). For $j = 1$ take $c_1 = 1/10$, $\varepsilon_1 = 1/4$, and $P_1(z, w) = w^2 - (1/100)(z - a_1)$. Then (1) and (3) hold for $j = 1$; also $\{|P_1| \le \varepsilon_1, |z| \le 1/2\} \subseteq \{|w| < 1\}$. Assuming that c_j, ε_j, P_j have been chosen for $j = 1, 2, \dots, n$ so that (1), (2), (2+), and (3) hold, we shall choose $c_{n+1}, \varepsilon_{n+1}, P_{n+1}$.

For each positive c we define a polynomial

$$P_c(z, w) = \prod_{j=1}^{2^n} [(w - w_j(z))^2 - c^2(B_{n+1}(z))^2].$$

Then $\{P_c(z, w) = 0\} \cap \{|z| \le 1/2\} = \Sigma(c_1, c_2, \dots, c_n, c)$ and

$$P_c = P_n^2 + c^2 Q_1 + \cdots + (c^2)^{2^n} Q_{2^n},$$

where the Q_j are polynomials in z and w not depending on c.

We claim that for c sufficiently small,

$$\{|P_c| \le \varepsilon_n^2/2, |z| \le 1/2\} \subseteq \{|P_n| < \varepsilon_n, |z| \le 1/2\}. \tag{4}$$

By (2), for $j \le n$ we have

$$\{|P_c| \le \varepsilon_n^2/2, |z| \le 1/2\} \subseteq \{|w| < 1\} \tag{5}$$

for c sufficiently small. It follows that the sets $A_c \equiv \{|P_c| \le \varepsilon_n^2/2, |z| \le 1/2\}$ are compact and that $\{P_c\}$ converges uniformly on $\{|z| \le 1/2, |w| \le 1\}$ to P_n^2 as $c \downarrow 0$. Hence $A_c \subseteq \{|P_n| < \varepsilon_n, |z| \le 1/2\}$ for c sufficiently small—this is (4).

Now we fix $c > 0$ such that (4) holds and such that $c < (1/10)c_n$. We set $c_{n+1} = c$ and $P_{n+1} = P_c$. Then we choose ε_{n+1} such that:

(i) $\varepsilon_{n+1} < \varepsilon_n^2/2$;

(ii) $|P_{n+1}(z, w)| \le \varepsilon_{n+1}$ and $|z| \le 1/2$ implies that there exists w_{n+1} with $|P_{n+1}(z, w_{n+1}) = 0$ and $|w - w_{n+1}| < 1/(n + 1)$; and

(iii) $(E_{n+1} \cup \{|P_{n+1}| \le \varepsilon_{n+1}, |z| = 1/2\})^\frown \cap \{|z| \le r_n\} \subseteq \{|P_n| < \varepsilon_n, |z| \le r_n\}$.

To achieve (iii) we apply Lemma 2 with $\alpha = \alpha_{n+1}$, $C = n + 1$, $\delta_1 = \varepsilon_n$, $r = r_n$, $Q_1 = P_n$, and $Q_2 = P_{n+1}$. For Lemma 2 we must check that $\{P_c = 0, |z| = 1/2\} \subseteq \{|P_n| < \varepsilon_n, |z| = 1/2\}$. This follows from (4). Now Lemma 2 yields a δ_2. Choosing $\varepsilon_{n+1} \le \delta_2$ gives (iii).

We now have (1), (2), (2+), and (3) for $j = n + 1$. This gives Lemma 1. □

We now define, following [W],

$$X = \bigcap_{n=1}^{\infty} \{|P_n| \le \varepsilon_n, |z| \le 1/2\} \quad \text{and} \quad Y = X \cap \{|z| = 1/2\}.$$

Wermer shows that $X = \hat{Y}$ and that X contains no analytic varieties of positive dimension.

Now let α be a proper closed subset of $\{|z| = 1/2\}$ and let $M > 0$. Set $E = \alpha \times \{|w| \le M\}$ and let $Z = E \cup X$. To complete the proof of Theorem 2 we need to show that $\hat{Z} \setminus Z = \hat{Y} \setminus Y$. There exists a strictly increasing sequence of positive integers n_k such that $\alpha \subseteq \alpha_{n_k}$ and $n_k \ge M$ for all k. Hence $E \subseteq E_{n_k}$ for all k. Let

$0 < r < 1/2$. There exists a k_0 such that $r_{n_k-1} > r$ for $k \geq k_0$. Now, by (2+), for $k \geq k_0$ we have

$$\begin{aligned}\hat{Z} \cap \{|z| \leq r\} &\subseteq \hat{Z} \cap \{|z| \leq r_{n_k-1}\} \\ &\subseteq (E_{n_k} \cup \{|P_{n_k}| \leq \varepsilon_{n_k}, |z| = 1/2\})\widehat{\ } \cap \{|z| \leq r_{n_k-1}\} \\ &\subseteq \{|P_{n_k-1}| < \varepsilon_{n_k-1}, |z| < 1/2\}.\end{aligned}$$

Therefore

$$\hat{Z} \cap \{|z| \leq r\} \subseteq \bigcap_{k=k_0}^{\infty} [\{|P_{n_k-1}| \leq \varepsilon_{n_k-1}\} \cap \{|z| < 1/2\}] = X \cap \{|z| < 1/2\}.$$

Hence $\hat{Z} \cap \{|z| < 1/2\} = X \cap \{|z| < 1/2\}$ and so $\hat{Z} \setminus Z = X \setminus Y = \hat{Y} \setminus Y$. This gives Theorem 2. □

We remark, for use below, that for all z with $|z| = 1$, all fibers Z_z are polynomially convex; indeed, each Z_z is either Y_z or the closed unit disk and the fibers Y_z are polynomially convex by the construction.

2. Proof of Theorem 1

In $\mathbb{C}^3$ let z, w_1, w_2 be coordinates and let $\pi(z, w_1, w_2) = z$ be the projection to the first coordinate. Let X be a compact subset of $\mathbb{C}^3$ lying over $\{|z| = 1\}$. As before, $X_z = \{ (w_1, w_2) \in \mathbb{C}^2 : (z, w_1, w_2) \in X \}$.

LEMMA 3. *Suppose that Q is a closed subset of the unit circle of positive linear measure such that $w_2 = 0$ on the set X_z for all $z \in Q$. Then*

$$\hat{X} \cap \{|z| < 1\} \subseteq \{w_2 = 0\}.$$

Proof. Say $x = (z, p_1, p_2) \in \hat{X}$ with $|z| < 1$. Let μ be a Jensen representing measure [B] on X for evaluation at x for the polynomials. Then $\pi_*(d\mu) = dm_z \equiv (1/2\pi) P_z(\theta)\, d\theta$ on the unit circle $z = e^{i\theta}$, where $P_z(\theta)$ is the Poisson kernel. Hence

$$\log|p_2| \leq \int_X \log|w_2|\, d\mu = -\infty$$

since $\log|w_2| = -\infty$ on $Q_1 \equiv \pi^{-1}(Q) \cap X$ and $\mu(Q_1) = m_z(Q) > 0$. Hence $p_2 = 0$. □

Let Z be the subset of $\mathbb{C}^2$ given by Theorem 2 starting from a proper closed subset α of the unit circle with nonempty interior and taking $M = 1$. Choose a continuous real nonnegative function ϕ on $\mathbb{T}$, $\phi \not\equiv 0$, such that $\{ z \in \mathbb{T} : \phi(z) = 0 \}$ is a subset of α of positive linear measure. Define X as the image in $\mathbb{C}^3$ of $Z \times [0, 1]$ under the map $((z, w), t) \mapsto (z, (1-t)w, t\phi(z))$. In other words, X_z is the cone in $\mathbb{C}^2$ with vertex $(0, \phi(z))$ and base $Z_z \times \{0\} \subseteq \mathbb{C}^2$; hence the fiber X_z is contractible.

We have the following equality between subsets of $\mathbb{C}^3$:

$$X \cap \{w_2 = 0\} = Z \times \{0\}. \tag{6}$$

To see this, consider two cases. If $\phi(z) = 0$, then $z \in \alpha$ and clearly $X_z = \{(w_1, 0) : |w_1| \le 1\} = Z_z \times \{0\}$. If $\phi(z) \ne 0$, then $x = ((1-t)w, t\phi(z)) \in X_z \cap \{w_2 = 0\}$ (where $w \in Z_z$ and $0 \le t \le 1$) implies that $t = 0$ and so $x = (w, 0) \in Z_z \times \{0\}$; that is, $X_z \cap \{w_2 = 0\} = Z_z \times \{0\}$.

We claim that X_z is polynomially convex for all z with $|z| = 1$. For this we again consider the previous two cases. If $\phi(z) = 0$ then $X_z = \Delta \times \{0\} \subseteq \mathbb{C}^2$, where Δ is the closed unit disk, and so X_z is polynomially convex. Suppose next that $b \equiv \phi(z) > 0$ and let $x \in \widehat{X_z} \subseteq \mathbb{C}^2$. We have $X_z = \{((1-t)w, t\phi(z)) : 0 \le t \le 1, w \in Z_z\}$. Then $\pi_{w_2}(X_z) = [0, b]$. Hence $\pi_{w_2}(\widehat{X_z}) = [0, b]$. Therefore $s_0 \equiv w_2(x) \in [0, b]$ and x is in the polynomial hull of $X_z \cap \{w_2 = s_0\} = (s_1 \cdot Z_z) \times \{s_0\}$, where $s_1 = 1 - s_0/\phi(z)$ and $s_1 \cdot Z_z = \{s_1 z : z \in Z_z\}$ is polynomially convex since Z_z is polynomially convex. Thus $X_z \cap \{w_2 = s_0\}$ is polynomially convex, so $x \in X_z \cap \{w_2 = s_0\} \subseteq X_z$ and it follows that X_z is polynomially convex.

Let $W = \hat{X} \cap \{|z| < 1\}$. We claim that $W \subseteq \{w_2 = 0\}$. For this, by Lemma 3 it suffices to show that $w_2 = 0$ on X_z for all z, $|z| = 1$, such that $\phi(z) = 0$ (a set of positive measure). But $\phi(z) = 0$ implies that $X_z = \Delta \times \{0\}$, as observed above. So indeed $W \subseteq \{w_2 = 0\}$. Hence $K \subseteq \{w_2 = 0\}$, where we define $K \subseteq \mathbb{C}^3$ to be the intersection of $\{|z| = 1\}$ with the closure of W. By the local maximum modulus principle, $W \subseteq \hat{K}$. Moreover, $K \subseteq \hat{X}$ and therefore $K_z \subseteq (\hat{X})_z = \widehat{X_z} = X_z$, by the preceding paragraph. Thus $K \subseteq X$. Hence $K \subseteq X \cap \{w_2 = 0\} = Z \times \{0\}$, by (6). Therefore

$$\hat{X} \cap \{|z| < 1\} \subseteq \hat{K} \subseteq \hat{Z} \times \{0\}.$$

Since $Z \times \{0\} \subseteq X$, we conclude that

$$\hat{X} \cap \{|z| < 1\} = (\hat{Z} \cap \{|z| < 1\}) \times \{0\}.$$

Theorem 1 follows, since $\hat{Z}$ contains no analytic varieties of positive dimension. □

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