# Commuting Toeplitz Operators on the Bergman Space of an Annulus

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## Introduction

Let $\Omega$ be a domain in the complex plane $\mathbf{C}$, and let $L^2_a(\Omega)$ be the Bergman space consisting of those analytic functions on $\Omega$ that are square integrable on $\Omega$ with respect to area measure $dA$. Of particular interest are the cases $\Omega = D = \{z \in \mathbf{C}: |z| < 1\}$ and $\Omega = \mathcal{A} = \{z \in \mathbf{C}: R < |z| < 1\}$ for $0 < R < 1$. The Bergman space is a closed subspace of the Hilbert space $L^2(\Omega)$ of all square integrable complex-valued functions on $\Omega$, so there is an orthogonal projection $P$ from $L^2(\Omega)$ onto $L^2_a(\Omega)$. If $\varphi$ belongs to $L^\infty(\Omega)$, the Toeplitz operator with symbol $\varphi$, denoted $T_\varphi$, is a linear operator from $L^2_a(\Omega)$ to $L^2_a(\Omega)$ defined by $T_\varphi f = P(\varphi f)$. In [6] Axler and the author characterized commuting Toeplitz operators on $L^2_a(D)$ whose symbols are harmonic. A complex-valued function is harmonic on $\Omega$ if its Laplacian vanishes identically on $\Omega$. We proved that two Toeplitz operators with symbols harmonic on $D$ commute only in the obvious cases. In this paper we want to prove the analogous theorem for Toeplitz operators acting on $L^2_a(\mathcal{A})$, provided their symbols are in a certain subclass of functions harmonic in $\mathcal{A}$. It is well known that every function harmonic on $D$ is of the form $f + \bar{g}$, where $f$ and $g$ are analytic on $D$. On the other hand, the logarithmic conjugation theorem [5, p. 179] implies that every $u$ harmonic on $\mathcal{A}$ is of the form $u(z) = f(z) + \bar{g}(z) + c\log|z|$, where $f$ and $g$ are analytic on $\mathcal{A}$, $c \in \mathbf{C}$. Our commutativity theorem applies to harmonic symbols without the logarithmic terms. Namely, we have the following.

**Theorem 1.** *Suppose that $\varphi = f_1 + \bar{f}_2$ and $\psi = g_1 + \bar{g}_2$ are bounded harmonic functions on $\mathcal{A}$. Then $T_\varphi T_\psi = T_\psi T_\varphi$ if and only if:*

(i) *$\varphi$ and $\psi$ are both analytic on $\mathcal{A}$; or*
(ii) *$\bar{\varphi}$ and $\bar{\psi}$ are both analytic on $\mathcal{A}$; or*
(iii) *there exist constants $a, b \in \mathbf{C}$, not both 0, such that $a\varphi + b\psi$ is constant on $\mathcal{A}$.*

The main tool in the proof of the disk theorem was the automorphisms of the disk. However, the automorphisms of the annulus are very sparse and

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we must make use of the reproducing kernels of $L^2_a(\mathfrak{A})$ instead. The commutativity problem leads to a certain annulus mean value property of functions integrable on $\mathfrak{A}$. That property is the annulus analog of the well-known invariant mean value property. The problem of finding the connection between harmonicity and the invariant mean value property of functions has a long history. The study of the problem started with Furstenberg [10], in the context of a noncompact Riemann space and certain probability measure, and continued in the case of the unit disk and the unit ball of $\mathbf{C}^n$ with Lebesgue measure by Nagel and Rudin [12], Arazy, Fisher, and Peetre [2], Axler and the author [6], Engliš [9], and Ahern, Flores, and Rudin [1]. Recently, Arazy and Zhang [3] have extended the results of [1] to Cartan domains of rank $r$ in $\mathbf{C}^n$. Also, Furstenberg's result has been generalized in a recent paper by Ben Natan et al. [7].

Characterizing functions that have the annulus mean value property is still an open problem, but we are able to prove certain results in a special case, which is discussed in Section 1. These results suffice to prove Theorem 1, which we do in Section 2. The open problems seem to be difficult, because we are able to use the reproducing kernels for the annulus in their power series form only. The reproducing kernels could also be written in the closed form using the Weierstrassian $P$-function (see [8, p. 10]). For more information about the reproducing kernels of $L^2_a(\mathfrak{A})$ see [11], [15], and [16]. The invariant mean value property is closely connected to the Berezin transform. The Bergman space of the annulus and the corresponding Berezin transform have also been studied in the papers by Peetre [13; 14].

## 1. Mean Value Property

We will start with a lemma that will be used throughout the paper.

LEMMA 2. *Suppose that $u(z) = f(z) + \bar{g}(z) + c\log|z|$ is a harmonic function in $L^2(\mathfrak{A})$, with $f$ and $g$ analytic on $\mathfrak{A}$. Then $f$ and $g$ belong to $L^2_a(\mathfrak{A})$.*

*Proof.* Without loss of generality, we can assume that $u$ is a real-valued harmonic function. Hence $u = \operatorname{Re} F + c\log|z|$ for some analytic function $F$. If we express $F$ as a Laurent series $F(z) = \sum_{n=-\infty}^{\infty} 2a_n z^n$, then $u$ has the form

$$u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} (a_n r^n + \bar{a}_{-n} r^{-n}) e^{in\theta} + c\log r.$$

Since

$$\begin{aligned}\int_0^{2\pi} |u(re^{i\theta})|^2 \frac{d\theta}{2\pi} &= (|c|\log r + 2\operatorname{Re} a_0)^2 \\ &\quad + \sum_{|n|\geq 1}\sum_{|m|\geq 1}\int_0^{2\pi} (a_n r^n + \bar{a}_{-n} r^{-n})(\bar{a}_m r^m + a_{-m} r^{-m}) e^{i(n-m)\theta}\frac{d\theta}{2\pi},\end{aligned}$$

it follows that

$$\infty > \int_{\mathfrak{A}} |u(z)|^2 \, dA(z)$$
$$= k + 2\pi \int_R^1 \left[ \sum_{|n| \ge 1} (|a_n|^2 r^{2n} + |a_{-n}|^2 r^{-2n} + a_n a_{-n} + \bar{a}_n \bar{a}_{-n}) \right] r \, dr,$$

where $k$ is a positive constant. For an arbitrary $R < r < 1$,

$$\sum_{|n| \ge 1} (a_n a_{-n}) = \int_0^{2\pi} \left( \sum_{|n| \ge 1} a_n r^n e^{in\theta} \right) \left( \sum_{|m| \ge 1} a_{-m} r^{-m} e^{-im\theta} \right) \frac{d\theta}{2\pi}$$
$$= \frac{1}{4} \int_0^{2\pi} (F(re^{i\theta}) - F(0))^2 \frac{d\theta}{2\pi},$$

so that $\sum_{|n| \ge 1}(a_n a_{-n})$ converges, since $F$ is analytic on $\mathfrak{A}$. Thus

$$\int_R^1 \sum_{|n| \ge 1} |a_n|^2 r^{2n+1} \, dr < \infty,$$

which means that $F \in L_a^2(\mathfrak{A})$. □

REMARK. The proof of Lemma 2 shows that if $u$ bounded, then $f$ and $g$ are actually in the Hardy space $H^2(\mathfrak{A})$, that is, the space of analytic functions $h$ on $\mathfrak{A}$, with

$$\sup \int_0^{2\pi} |h(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty,$$

where the supremum is taken over $R < r < 1$.

For $w \in \mathfrak{A}$, the reproducing kernel $K_w$ is the unique function in $L_a^2(\mathfrak{A})$ such that $\langle f, K_w \rangle = f(w)$ for all $f \in L_a^2(\mathfrak{A})$. The inner product $\langle f, g \rangle$ is defined in the usual way as

$$\int_{\mathfrak{A}} f(z) \bar{g}(z) \, dA(z).$$

It is known that

$$K_w(z) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{n+1}{1 - R^{2(n+1)}} (\bar{w}z)^n - \frac{1}{2\pi \log R} (\bar{w}z)^{-1}.$$

Let $k_w = K_w / \|K_w\|$ denote the normalized reproducing kernel. Suppose $\varphi = f_1 + \bar{f}_2$ and $\psi = g_1 + \bar{g}_2$ are bounded harmonic functions, with $f$s and $g$s analytic on $\mathfrak{A}$. Then $T_\varphi T_\psi = T_\psi T_\varphi$ implies that

$$\langle T_\psi k_w, T_{\bar{\varphi}} k_w \rangle = \langle T_\varphi k_w, T_{\bar{\psi}} k_w \rangle \tag{1}$$

for all $w$. A simple calculation shows that $T_\psi k_w(z) = g_1(z) k_w(z) + \bar{g}_2(w) k_w(z)$ and $T_{\bar{\varphi}} k_w = f_2(z) k_w(z) + \bar{f}_1(w) k_w(z)$, so that the left side of (1) becomes

$$\int_{\mathfrak{A}} g_1(z) \bar{f}_2(z) |k_w(z)|^2 \, dA(z) + f_1(w) g_1(w) + f_1(w) \bar{g}_2(w) + \bar{f}_2(w) \bar{g}_2(w).$$

By interchanging $\varphi$ and $\psi$, a similar formula can be obtained for the right side of (1). Thus (1) becomes

$$\int_{\mathcal{A}} (f_1\bar{g}_2 - g_1\bar{f}_2)(z)|k_w(z)|^2\, dA(z) = (f_1\bar{g}_2 - g_1\bar{f}_2)(w) \tag{2}$$

for all $w \in \mathcal{A}$. For an integrable function $u$ we can introduce its Berezin transform $Bu$ as

$$(Bu)(w) = \int_{\mathcal{A}} u(z)|k_w(z)|^2\, dA(z).$$

By (2), our function $u = f_1\bar{g}_2 - g_1\bar{f}_2$ must be a fixed point of the Berezin transform. We will state this as the following proposition.

PROPOSITION 3. *Assume that $\varphi = f_1 + \bar{f}_2$ and $\psi = g_1 + \bar{g}_2$ are bounded harmonic functions on $\mathcal{A}$, and that $T_\varphi T_\psi = T_\psi T_\varphi$. Then the function $u = f_1\bar{g}_2 - g_1\bar{f}_2$ must satisfy $Bu = u$ in $\mathcal{A}$.*

We say that $u$ in the proposition has the annulus mean value property. A special case of the main theorem of [1] is that $Bu = u$ in $D$ implies that $u$ is harmonic on $D$ for all functions integrable on $D$. One may naturally wonder if the same is true for functions integrable on $\mathcal{A}$. Before we offer partial answers, we would like to know which harmonic functions do have the annulus mean value property.

PROPOSITION 4. *Suppose that $u$ is a harmonic function in $L^2(\mathcal{A})$. Then $Bu = u$ in $\mathcal{A}$ if and only if $u = f + \bar{g}$, with $f$ and $g$ analytic on $\mathcal{A}$.*

*Proof.* If $u(z) = f(z) + \bar{g}(z) \in L^2(\mathcal{A})$, then by Lemma 2 we can conclude that $Bf = f$ and $B\bar{g} = \bar{g}$, so the sufficiency follows. Patrick Ahern pointed out to us that $\log|z|$ does not have the annulus mean value property. If it did, then

$$\log|w| = B(\log|z|)(w) = \frac{1}{\|K_w\|^2}\left[\int_R^1 r\log r \int_0^{2\pi} |K_w(re^{i\theta})|^2\, d\theta\, dr\right]$$

and therefore $\log|w|$ is equal to the quotient of two Laurent series in $t \equiv |w|^2$. This would mean that $\log t$ has a meromorphic extension to all of $\mathcal{A}$, which is impossible. The necessity of Proposition 4 follows from this observation and Lemma 2. □

We would now like to study the fixed points of the Berezin transform on $\mathcal{A}$. We will assume that those fixed points are continuous functions on $\bar{\mathcal{A}}$.

PROPOSITION 5. *Suppose that $u \in C(\bar{\mathcal{A}})$. Then $Bu = u$ in $\mathcal{A}$ and*

$$\int_0^{2\pi} u(Re^{i\theta})\frac{d\theta}{2\pi} = \int_0^{2\pi} u(e^{i\theta})\frac{d\theta}{2\pi}$$

*if and only if $u = f + \bar{g}$, where $f$ and $g$ are analytic on $\mathcal{A}$.*

*Proof.* Suppose that the averages of $u$ on both circles are the same. Then the Poisson extension of $u|_{\partial\mathcal{A}}$ is a harmonic function $v$ on $\mathcal{A}$ without the logarithmic term (see e.g. [4, p. 254]). Let $v$ denote again its continuous extension to the boundary. Proposition 4 guarantees that $Bv = v$ in $\mathfrak{A}$, and so $B(u-v) = (u-v)$. Since $u-v=0$ on $\partial\mathfrak{A}$, a standard argument shows that $u = v$ on $\mathfrak{A}$, that is, $u = f+\bar{g}$ for some functions $f$ and $g$ analytic on $\mathfrak{A}$. Conversely, if $u = f+\bar{g} \in C(\bar{\mathfrak{A}})$, then $f$ and $g$ belong to $H^2(\mathfrak{A})$ by the remark following Lemma 2. Cauchy's theorem guarantees that the averages of $u$ on both boundaries are the same. □

Suppose now that $u \in C(\bar{\mathfrak{A}})$, but the averages are not equal. Let $L_n(z) = z^n$ for $n \in \mathbf{Z}$. Define

$$(u*L_n)(z) = \int_0^{2\pi} u(ze^{-i\phi})\, L_n(e^{i\phi})\frac{d\phi}{2\pi}$$

for $z \in \bar{\mathfrak{A}}$. An application of Fubini's theorem gives, for $n \neq 0$,

$$\mathrm{Av}_{\partial D_R}(u*L_n) = \mathrm{Av}_{\partial D}(u*L_n) = 0,$$

where $D_R = \{z \in \mathbf{C}: |z| < R\}$ and Av stands for Average. Since $u \in C(\bar{\mathfrak{A}})$, so is $u*L_n$.

Lemma 6. *If $u \in C(\bar{\mathfrak{A}})$ and $Bu = u$ in $\mathfrak{A}$, then $B(u*L_n) = u*L_n$ in $\mathfrak{A}$ for all $n \in \mathbf{Z}$.*

*Proof.* A simple calculation shows that

$$B(u*L_n)(z) = \int_0^{2\pi}\left[\int_{\mathfrak{A}} u(we^{-i\phi})|k_z(w)|^2\, dA(w)\right]L_n(e^{i\phi})\,\frac{d\phi}{2\pi}.$$

From the Laurent series expansion of $K_z(w)$, one sees that $K_{ze^{-i\phi}}(we^{-i\phi}) = K_z(w)$ and $\|K_z\|^2 = \|K_{ze^{-i\phi}}\|^2$. Hence

$$\int_{\mathfrak{A}} u(we^{-i\phi})|k_z(w)|^2\, dA(w) = u(ze^{-i\phi})$$

and Lemma 6 follows. □

Remark. We have actually proved a stronger assertion that for $u \in C(\bar{\mathfrak{A}})$,

$$B(u*L_n) = (Bu)*L_n.$$

Lemma 6 implies that $u*L_n$ satisfies the hypotheses of Proposition 5 for all $n \neq 0$ and consequently $u*L_n$ is harmonic on $\mathfrak{A}$. Hence, for all $n \neq 0$,

$$0 = \Delta(u*L_n)(z) = \int_0^{2\pi} \Delta[u(ze^{-i\phi})]e^{in\phi}\,\frac{d\phi}{2\pi} = \int_0^{2\pi} (\Delta u)(ze^{-i\phi})e^{in\phi}\,\frac{d\phi}{2\pi}.$$

We write $z = re^{i\theta}$, and a change of variable for $\theta-\phi$ shows that

$$\int_0^{2\pi} (\Delta u)(re^{i\phi})e^{in\phi}\,\frac{d\phi}{2\pi} = 0$$

for all $n \neq 0$. In other words, $\Delta u$ is a radial function on $\mathfrak{A}$.

Our goal is to show that $u$ is of the form (radial + harmonic). The proof, in fact, shows that this is the case in a more general situation—namely, for $\mathbf{R}^n$ ($n \geq 1$). The proof of the following lemma was communicated to the author by Sheldon Axler.

Let $S$ be the unit sphere in $\mathbf{R}^n$ and let $\sigma$ denote normalized surface area measure on $S$. For $u$ a function defined on $\mathcal{A}$, the radialization of $u$, denoted $\mathcal{R}(u)$, is the function on $\mathcal{A}$ defined by

$$\mathcal{R}(u)(x) = \int_S u(x\zeta)\, d\sigma(\zeta). \tag{3}$$

LEMMA 7. *If $u \in C^2(\mathcal{A})$, then $\Delta(\mathcal{R}(u)) = \mathcal{R}(\Delta u)$.*

*Proof.* Let $\mathcal{O}$ denote the orthogonal group of all orthogonal linear maps on $\mathbf{R}^n$; in other words, $\mathcal{O}$ is the set of all linear isometries on $\mathbf{R}^n$. Then $\mathcal{O}$ is a compact group. Let $\tau$ be the Haar measure (the rotation-invariant probability measure) on this group. Then, as is well known,

$$\mathcal{R}(u)(x) = \int_{\mathcal{O}} u(Tx)\, d\tau(T)$$

for all $x \in \mathcal{A}$ and $T \in \mathcal{O}$. Taking the Laplacian (with respect to $x$) of both sides of the equation above, we obtain

$$\begin{aligned} \Delta(\mathcal{R}(u))(x) &= \int_{\mathcal{O}} \Delta(u(Tx))\, d\tau(T) \\ &= \int_{\mathcal{O}} (\Delta u)(Tx)\, d\tau(T) \\ &= \mathcal{R}(\Delta u)(x), \end{aligned}$$

where the second line comes from [5, p. 3] and the third line comes from (3) with $u$ replaced by $\Delta u$. □

COROLLARY 8. *If $u \in C^2(\mathcal{A})$ and $\Delta u$ is a radial function, then $u$ equals a radial function plus a harmonic function.*

*Proof.* Lemma 7 gives

$$\Delta(\mathcal{R}(u)) = \mathcal{R}(\Delta u) = \Delta u,$$

so $\Delta(u - \mathcal{R}(u)) = 0$. In other words, $u - \mathcal{R}(u)$ is harmonic. Now,

$$u = \mathcal{R}(u) + (u - \mathcal{R}(u)),$$

which writes $u$ as a radial function plus a harmonic function. □

Applying Corollary 8 to our case $n = 2$ and since $\mathcal{R}(u)(z) = (u * L_0)(z)$ (which guarantees that $B(\mathcal{R}(u)) = \mathcal{R}(u)$), we arrive at the following theorem.

THEOREM 9. *If $u \in C(\bar{\mathcal{A}})$ satisfies $Bu = u$ in $\mathcal{A}$, then $u = \mathcal{R}(u) + f + \bar{g}$, where $f$ and $g$ are analytic on $\mathcal{A}$.*

## 2. Proof of Theorem 1

We need only prove the necessity. Proposition 3 yields that $u = f_1\bar{g}_2 - g_1\bar{f}_2$ must satisfy $Bu = u$ in $\mathfrak{A}$. We would like to apply the results from the previous section. However, our $u \notin C(\bar{\mathfrak{A}})$, but we will show that $u * L_n \in C(\bar{\mathfrak{A}})$. We denote

$$f_1(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \qquad g_2(z) = \sum_{m=-\infty}^{\infty} b_m z^m. \tag{4}$$

Because $f_i$ and $g_i$ belong to $H^2(\mathfrak{A})$ for $i = 1, 2$, it follows that

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty, \qquad \sum_{k=-\infty}^{\infty} |a_k|^2 R^{2k} < \infty, \\ &\sum_{m=-\infty}^{\infty} |b_m|^2 < \infty, \qquad \sum_{m=-\infty}^{\infty} |b_m|^2 R^{2m} < \infty. \end{aligned} \tag{5}$$

Now,

$$\begin{aligned} (f_1\bar{g}_2) * L_n(z) &= \sum_{k,m=-\infty}^{\infty} a_k \bar{b}_m \int_0^{2\pi} z^k \bar{z}^m e^{i(m+n-k)\phi} \frac{d\phi}{2\pi} \\ &= \sum_{k=-\infty}^{\infty} a_{k+n} \bar{b}_k z^{k+n} \bar{z}^k. \end{aligned}$$

The inequalities in (5) imply that $\sum_{k=-\infty}^{\infty} |a_{k+n}\bar{b}_k|$ and $\sum_{k=-\infty}^{\infty} |a_{k+n}\bar{b}_k R^{2k+n}|$ both converge, so that $(f_1\bar{g}_2) * L_n(z) \in C(\bar{\mathfrak{A}})$. Similarly, $(g_1\bar{f}_2) * L_n(z) \in C(\bar{\mathfrak{A}})$, and therefore $u * L_n$ belongs to $C(\bar{\mathfrak{A}})$. The averages of $u * L_n$ are equal to 0 on both boundary circles if $n \neq 0$, and also $B(u * L_n) = u * L_n$ in $\mathfrak{A}$, so it follows again that $\Delta u$ is a radial function on $\mathfrak{A}$. Because

$$\Delta = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z},$$

it follows that

$$f_1' \bar{g}_2' - g_1' \bar{f}_2' \text{ is a radial function on } \mathfrak{A}. \tag{6}$$

We represent

$$\begin{aligned} f_1'(z) &= \sum_{n=-\infty}^{\infty} a_n z^n, \qquad f_2'(z) = \sum_{m=-\infty}^{\infty} d_m z^m, \\ g_1'(z) &= \sum_{n=-\infty}^{\infty} c_n z^n, \qquad g_2'(z) = \sum_{m=-\infty}^{\infty} b_m z^m. \end{aligned}$$

(Of course, the coefficients $a_n$ and $b_m$ are not the same as $a_n$ and $b_m$ in (4), but we use the same letters for convenience.) Statement (6) now means that

$$\sum_{n,m=-\infty}^{\infty} (a_n \bar{b}_m - c_n \bar{d}_m) z^n \bar{z}^m \text{ is radial on } \mathfrak{A}.$$

Therefore

$$a_n \bar{b}_m - c_n \bar{d}_m = 0 \tag{7}$$

if $n \neq m$. The analysis of (7) requires a discussion of several cases. At first, we introduce the vectors $u_n, v_m \in \mathbf{C}^2$ by

$$u_n = (a_n, -c_n), \qquad v_m = (b_m, d_m).$$

Then (7) says that $u_n$ is orthogonal to $v_m$ if $n \neq m$.

*Case 1:* $u_n = 0$ for all $n$. This immediately implies (ii) in Theorem 1.

*Case 2:* $u_{n_0} \neq 0$ for some $n_0$, and all $u_n$ are multiples of $u_{n_0}$: $u_n = l_n u_{n_0}$ for all $n$. By (7), $v_m = k_m(\bar{c}_{n_0}, \bar{a}_{n_0})$ for all $m \neq n_0$ and for some numbers $k_m$.

*Subcase A:* If also $v_{n_0} = k_{n_0}(\bar{c}_{n_0}, \bar{a}_{n_0})$ for some number $k_{n_0}$, it then follows that

$$c_{n_0}(f_1' + \bar{f}_2') = a_{n_0}(g_1' + \bar{g}_2'),$$

which implies (iii).

*Subcase B:* $v_{n_0}$ is not orthogonal to $u_{n_0}$. Then $u_n = 0$ for all $n \neq n_0$. Thus we have

$$f_1'(z) = a_{n_0} z^{n_0},$$
$$g_1'(z) = c_{n_0} z^{n_0},$$
$$c_{n_0} \bar{f}_2' - a_{n_0} \bar{g}_2' = (c_{n_0} \bar{d}_{n_0} - a_{n_0} \bar{b}_{n_0}) \bar{z}^{n_0} \equiv \alpha \bar{z}^{n_0}, \quad \alpha \neq 0.$$

Notice that $n_0 = -1$ would imply that $f_1$ and $g_1$ are not single-valued analytic functions on $\mathfrak{A}$. If $[T_\varphi, T_\psi]$ denotes the commutator $T_\varphi T_\psi - T_\psi T_\varphi$, then a short computation gives

$$0 = [T_\varphi, T_\psi] = [T_\varphi, T_{\bar{z}^N}] = [T_{z^N}, T_{\bar{z}^N}],$$

where $N = n_0 + 1 \neq 0$. In other words, $T_{z^N}$ is a normal operator. But

$$\|T_{z^N} 1\| = \|z^N\| = \|\bar{z}^N\| > \|P(\bar{z}^N)\| = \|T_{\bar{z}^N} 1\|,$$

a contradiction.

*Case 3.* By symmetry, we can proceed in the same way as before if all $v_m = 0$, or if $v_m$ are not all zero but are all multiples of one another. This leads (respectively) to (i) and (iii) of Theorem 1.

*Case 4.* The only remaining case is when each of the families $\{u_n\}$, $\{v_m\}$ contains two linearly independent vectors. Let $u_{n_0}$ and $u_{n_1}$ be linearly independent. Then, by (7), $v_m = 0$ for all $m \neq n_0, n_1$ and so the two linearly independent $v_m$s must be $v_{n_0}$ and $v_{n_1}$. This in turn implies that $u_n = 0$ for all $n \neq n_0, n_1$. Thus we have

$$f_1'(z) = a_{n_0} z^{n_0} + a_{n_1} z^{n_1}, \qquad f_2'(z) = d_{n_0} z^{n_0} + d_{n_1} z^{n_1};$$
$$g_1'(z) = c_{n_0} z^{n_0} + c_{n_1} z^{n_1}, \qquad g_2'(z) = b_{n_0} z^{n_0} + b_{n_1} z^{n_1}.$$

Integrate and compute the commutator $0 = [T_\varphi, T_\psi] = [T_{f_1}, T_{\bar{g}_2}] + [T_{\bar{f}_2}, T_{g_1}]$. Using the equations

$$a_{n_1} \bar{b}_{n_0} - c_{n_1} \bar{d}_{n_0} = 0 \quad \text{and} \quad a_{n_0} \bar{b}_{n_1} - c_{n_0} \bar{d}_{n_1} = 0,$$

we cancel out the terms containing $[T_{z^{N_0}}, T_{\bar{z}^{N_1}}]$ and $[T_{z^{N_1}}, T_{\bar{z}^{N_0}}]$, with $N_j = n_j + 1$, $j = 0, 1$. This results in

$$0 = N_1^2(a_{n_0}\bar{b}_{n_0} - c_{n_0}\bar{d}_{n_0})[T_{z^{N_0}}, T_{\bar{z}^{N_0}}] + N_0^2(a_{n_1}\bar{b}_{n_1} - c_{n_1}\bar{d}_{n_1})[T_{z^{N_1}}, T_{\bar{z}^{N_1}}].$$

Since the pairs $u_{n_0}, u_{n_1}$ and $v_{n_0}, v_{n_1}$ are linearly independent, $a_{n_0}\bar{b}_{n_0} - c_{n_0}\bar{d}_{n_0}$ and $a_{n_1}\bar{b}_{n_1} - c_{n_1}\bar{d}_{n_1}$ are nonzero numbers and we have

$$[T_{z^{N_0}}, T_{\bar{z}^{N_0}}] = \alpha[T_{z^{N_1}}, T_{\bar{z}^{N_1}}]$$

for some constant $\alpha \neq 0$ with $N_0 \neq N_1$. Our goal is to show that this cannot happen.

LEMMA 10. *If* $[T_{z^{n_0}}, T_{\bar{z}^{n_0}}] = \alpha[T_{z^{n_1}}, T_{\bar{z}^{n_1}}]$ *for some constant* $\alpha \neq 0$ *and integers* $n_0$ *and* $n_1$, *then* $n_0 = n_1$.

*Proof.* Recall that $\mathfrak{A} = \{z \in \mathbf{C}: R < |z| < 1\}$. If the standard orthonormal basis for $L_a^2(\mathfrak{A})$ is denoted by $\{e_k(z)\}$, then for $m, n \in \mathbf{Z}$ we have

$$P(\bar{z}^m z^n) = \sum_k \langle P(\bar{z}^m z^n), e_k \rangle e_k = \frac{n-m+1}{n+1} \frac{1-R^{2(n+1)}}{1-R^{2(n-m+1)}} z^{n-m}$$

if $n \neq -1$ and $n-m \neq -1$. Suppose on the contrary that $n_0 \neq n_1$. Without loss of generality, we can assume that $n_0 < n_1$. We want to apply the commutators to a function $z^j$, yielding

$$\begin{aligned}
&(T_{z^{n_0}} T_{\bar{z}^{n_0}} - T_{\bar{z}^{n_0}} T_{z^{n_0}})(z^j) \\
&= \left[\frac{j-n_0+1}{j+1} \frac{1-R^{2(j+1)}}{1-R^{2(j-n_0+1)}} - \frac{j+1}{j+n_0+1} \frac{1-R^{2(j+n_0+1)}}{1-R^{2(j+1)}}\right] z^j \\
&= \left[\frac{(j+1)^2[R^{(j+n_0+1)} - R^{(j-n_0+1)}]^2 - n_0^2(1-R^{2(j+1)})^2}{(j+1)(j+n_0+1)(1-R^{2(j+1)})(1-R^{2(j-n_0+1)})}\right] z^j \qquad (8)
\end{aligned}$$

for $j \neq -1$ and $j \pm n_0 \neq -1$. For any positive integer $k$, we choose $j = -1+k$ and $j = -1-k$. If $j = -1+k$, using (8) we can write the equality $[T_{z^{n_0}}, T_{\bar{z}^{n_0}}] = \alpha[T_{z^{n_1}}, T_{\bar{z}^{n_1}}]$ as

$$\frac{k^2[R^{k+n_0} - R^{k-n_0}]^2 - n_0^2(1-R^{2k})^2}{k(k+n_0)(1-R^{2k})(1-R^{2(k-n_0)})} = \alpha \frac{k^2[R^{k+n_1} - R^{k-n_1}]^2 - n_1^2(1-R^{2k})^2}{k(k+n_1)(1-R^{2k})(1-R^{2(k-n_1)})};$$

after a simplification, this becomes

$$\frac{(n_1+k)(1-R^{2(k-n_1)})}{(n_0+k)(1-R^{2(k-n_0)})} = \alpha \frac{k^2[R^{k+n_1} - R^{k-n_1}]^2 - n_1^2(1-R^{2k})^2}{k^2[R^{k+n_0} - R^{k-n_0}]^2 - n_0^2(1-R^{2k})^2}. \qquad (9)$$

For $j = -1-k$, we obtain a similar equation with $k$ being replaced by $-k$. Notice that the right-hand side of the new equation is the same as the right-hand side of (9). Hence,

$$\frac{(n_1+k)(1-R^{2(k-n_1)})}{(n_0+k)(1-R^{2(k-n_0)})} = \frac{(n_1-k)(1-R^{-2(k+n_1)})}{(n_0-k)(1-R^{-2(k+n_0)})}$$

or

$$\frac{(n_0-k)(1-R^{-2(k+n_0)})}{(n_0+k)(1-R^{2(k-n_0)})} = \frac{(n_1-k)(1-R^{-2(k+n_1)})}{(n_1+k)(1-R^{2(k-n_1)})}$$

for $k \in \mathbf{N} \setminus \{n_0, n_1\}$. Let

$$f_k(x) = \frac{(x-k)(1-R^{-2(k+x)})}{(x+k)(1-R^{2(k-x)})}.$$

Then $f_k(n_0) = f_k(n_1)$ for $k \in \mathbf{N} \setminus \{n_0, n_1\}$. Our goal is to show that there is a $k$ such that $f_k(x)$ is one-to-one on an interval containing $(n_0, n_1)$. For that purpose it is enough to show that the numerator of $f_k'$ is positive on an interval containing $(n_0, n_1)$. A calculation and simplification gives that the numerator is equal to

$$2k(1-R^{-2(k+x)})(1-R^{2(k-x)}) + 2\ln R(x^2-k^2)(R^{-2(k+x)} - R^{2(k-x)}),$$

which we can rewrite as

$$2\ln R(x^2-k^2)(R^{-2(k+x)} - R^{2(k-x)}) \times \left[\frac{k(1-R^{-2(k+x)})(1-R^{2(k-x)})}{\ln R(x^2-k^2)(R^{-2(k+x)} - R^{2(k-x)})} + 1\right]. \tag{10}$$

For $\epsilon > 0$ small, let $x \in (n_0 - \epsilon, n_1 + \epsilon)$. For $k$ large enough,

$$\left|\frac{k(1-R^{-2(k+x)}((1-R^{2(k-x)})}{\ln R(x^2-k^2)(R^{-2(k+x)} - R^{2(k-x)})}\right| \leq \frac{k[R^{-2k} - R^{2(n_1+\epsilon)}][R^{2(n_0-\epsilon)} - R^{2k}]}{-\ln R[k^2-(n_1+\epsilon)^2]R^{2(n_1+\epsilon)}(R^{-2k} - R^{2k})}$$

which approaches 0 as $k \to \infty$. Thus we can find $k$ large enough such that, for all $x \in (n_0 - \epsilon, n_1 + \epsilon)$, the bracketed expression in (10) stays positive. In other words, there exists $k_0$ such that $f_{k_0}'(x) > 0$ for $x \in (n_0 - \epsilon, n_1 + \epsilon)$; that is, $f_k(x)$ is one-to-one on $(n_0 - \epsilon, n_1 + \epsilon)$. Therefore $f_k(n_0) = f_k(n_1)$ implies $n_0 = n_1$, a contradiction. Thus Lemma 10 is proved. □

Lemma 10, moreover, guarantees that Case 4 cannot occur. Thus all possible cases and their subcases lead to one of the statements (i), (ii), or (iii), and Theorem 1 is proved. □

A few questions remain open. What are the necessary and sufficient conditions that two Toeplitz operators commute if their symbols are harmonic functions with logarithmic terms? Theorem 9 suggests the following question: What radial functions, continuous on $\bar{\mathfrak{A}}$, satisfy $Bu = u$ in $\mathfrak{A}$? Is there an analog of Lemma 2 (and consequently of Proposition 4) for harmonic functions in $L^1(\mathfrak{A})$? And probably the most difficult problem is a characterization of functions integrable on $\mathfrak{A}$ that satisfy $Bu = u$ in $\mathfrak{A}$. We could also consider all these problems for regions other than an annulus. We hope this paper will stimulate further research in that direction.

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