The Effect of Boundary Regularity on the Singular Numbers of Friedrichs Operators on Bergman Spaces

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1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$, and let $L^2(\Omega)$ be the usual Lebesgue space over Ω. Let $A^2(\Omega)$ be the subspace of $L^2(\Omega)$ consisting of holomorphic functions. We set $\bar{A}^2(\Omega) = \{\bar{f} : f \in A^2(\Omega)\}$. Let $P: L^2(\Omega) \to A^2(\Omega)$ be the orthogonal projection and let $K(z, w)$ be the Bergman kernel. It is well known that, if Ω is the unit ball, then $A^2(\Omega)$ and $\bar{A}^2_0(\Omega)$ (all functions in $\bar{A}^2(\Omega)$ that vanish at 0) are orthogonal. In other words, the Friedrichs operator

$$T(f)(z) = \int_\Omega K(z, w)\bar{f}(w)\,dv(w). \tag{1.1}$$

is a rank-1 operator (the same result holds for any complete Reinhardt domain). It is of interest to determine for which domains in $\mathbb{C}^n$ the Friedrichs operator has finite rank. We say that $\Omega \subset \mathbb{C}^n$ is a *quadrature domain* if there are finitely many points $z_1, \dots, z_m \in \Omega$, nonnegative integers n_j, $j = 1, \dots, m$, and numbers $\lambda_{1,\alpha}, \dots, \lambda_{m,\alpha}$ such that

$$\int_\Omega f(z)\,dv(z) = \sum_{j=1}^{m} \sum_{|\alpha| \le n_j} \lambda_{j,\alpha} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z_j) \quad \forall f \in A^2(\Omega). \tag{1.2}$$

This definition was first introduced for $n = 1$ by Shapiro [24], who proved for a planar domain Ω that the operator T has finite rank if and only if Ω is a quadrature domain. In higher dimensions, it was proved by Janson, Peetre, and Rochberg [11] that: If Ω is a bounded Runge domain in $\mathbb{C}^n$ with C^1 boundary then T has finite rank if and only if Ω is a quadrature domain. It would be useful to find some characterization for the defining function of a domain with associated Friedrichs operator having finite rank. We note in passing that the Friedrichs operator is essentially a Hankel operator in the sense of [11].

We know from [23] that a simply connected domain D in $\mathbb{C}$ has operator T having finite rank if and only if D is the image of the unit disc Δ by a

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conformal (holomorphic) rational map ϕ. Moreover, it was proved in [23] and references therein that T has finite rank on D only if ∂D is an irreducible algebraic curve. A natural question arises concerning what we can say about the spectrum of T when ∂D is smooth or rough. Alternatively, one may ask how the boundary regularity affects the asymptotic behavior of the sequence of singular numbers of T. (See [22] or [6] for a discussion of singular numbers.) This problem has been treated by several authors. For example, it was shown by Friedrichs [9] and Norman [19] that there is a Lipschitz domain in the complex plane such that the operator T is not compact on $A^2(\Omega)$.

For $0 < p \leq \infty$, we say that a bounded linear operator L from a Hilbert space H_1 to another Hilbert space H_2 belongs to the *Schatten ideal* $S_p(H_1, H_2)$ if the singular numbers $\{s_j(L)\}_{j=0}^{\infty} \in l^p$, where

$$s_j(L) = \inf\{\|L - M_j\| : M_j : H_1 \to H_2 \text{ has rank at most } j\}, \quad j = 0, 1, \ldots.$$

In [17], Lin and Rochberg studied the above problem on a planar domain, and gave a characterization of S_p on a finitely connected domain in $\mathbb{C}^1$; as a consequence of their results, we know that if $\partial\Omega$ is $C^{2+\epsilon}$ with some $\epsilon > 0$, then $T \in S_p(A^2, \bar{A}^2)$ for all $p > 1$ and $T \in S_1$ if Ω has $C^{3+\epsilon}$ boundary. Their arguments are based on the conformal map (the Ahlfors map) from the unit disc to the domain Ω. However, for a general domain in $\mathbb{C}^n$, their method is not applicable because there are no biholomorphic mappings between generic domains—even domains of the same topological type. This leads us to try to understand the operator T itself. It is clear that the definition of T depends only on the Bergman kernel or Bergman projection. We shall produce a method, based on Bell's ideas concerning Condition R (see [3]), which will help us to handle many interesting problems concerning T.

According to Bell [3], we say that a smoothly bounded domain $\Omega \subset \mathbb{C}^n$ satisfies *Condition R* if $P(C^\infty(\bar{\Omega})) = A^2(\Omega) \cap C^\infty(\bar{\Omega})$. Now we can state the main results of the present paper as follows.

THEOREM 1.1. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ satisfying Condition R. Then $T \in S_p$ for all $p > 0$.*

We point out here that all pseudoconvex domains of finite type (in the sense of D'Angelo and Catlin) satisfy Condition R [5]. Ad hoc arguments show that all smoothly bounded domains in $\mathbb{C}$ satisfy Condition R.

THEOREM 1.2. *Let Ω be a bounded domain in $\mathbb{C}^1$ with $C^{1+\alpha}$ boundary.*

(a) *If $\alpha = 1$, then $T \in S_p$ for all $p > 1$.*
(b) *If $0 < \alpha < 1$, then $T \in S_p$ for all $p > 1/\alpha$.*

As noted earlier, any simply connected planar domain D with finite rank Friedrichs operator D must have algebraic defining function. We may show that if D is a smoothly bounded domain in $\mathbb{C}^n$ that is biholomorphically equivalent to the unit ball in $\mathbb{C}^n$ by a mapping ϕ and if the Friedrichs operator T associated to D has finite rank, then $\det(\phi'(z))$ is rational. But, unfortunately, we cannot conclude that ∂D is an algebraic surface (except possibly for a small exceptional set). For example, for any holomorphic function f on the disc Δ, $\phi(z_1, z_2) = (z_1 + f(z_2), z_2)$ is a biholomorphic mapping (a *shear*) with $\det[\phi'(z)] = 1$ on $\mathbb{C}^2$. Let $\Omega = \phi(B_2)$. Then the Bergman kernel for Ω is

$$K(z, w) = c_2(1 - z_2\bar{w}_2 - (z_1 - f(z_2))(\bar{w}_1 - \bar{f}(w_2))^{-3}$$

and, for $z \in B_2$, we have

$$\begin{aligned} T(h)(\phi(z)) &= \int_\Omega K(\phi(z), w)\bar{h}(w)\,dv(w) \\ &= c_2 \int_{B_2} c_2(1 - \langle z, w\rangle)^{-3}\bar{h}(\phi(w))\,dv(w) \\ &= c_2\bar{h}(\phi(0)) \\ &= c_2\bar{h}((f(0), 0)). \end{aligned}$$

Therefore T has rank 1. But the boundary of Ω can be very rough (depending on how rough f is near the boundary of Δ). Therefore, the above example gives a significant difference between the situation in one complex variable and that in several complex variables. It is obvious that not all bounded domains in $\mathbb{C}^n$ with real analytic boundary have Friedrichs operator with finite rank. (In fact, it is not even true in one dimension—by virtue of the "irreducible algebraic curve" criterion. A multi-dimensional example, with smooth boundary, may be obtained as follows. Let $D = \{z \in \mathbb{C}; r(z) < 0\}$ be a 1-dimensional example. Set $D_e = \{(z, w) \in \mathbb{C}^2 : z \in D, r(z) + |w|^2 < 0\}$.) However, we do have the following refinement.

THEOREM 1.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ with real analytic boundary. Then the singular numbers s_j of T satisfy*

$$s_j(T) = O(c^{-j^{1/n}}) \text{ as } j \to \infty$$

for some $0 < c < 1$.

This paper is organized as follows: In Section 2, we shall prove Theorem 1.1. Theorem 1.3 will be proved in Section 3. In Section 4, we consider the problem on strictly pseudoconvex domains in $\mathbb{C}^n$ with C^k boundary, and we give a sufficient condition so that $T \in S_p$. Finally, in Section 5, we give an estimate on the Bergman kernel of a domain in $\mathbb{C}$ with $C^{1+\alpha}$ boundary; as an application, we shall prove Theorem 1.2.

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2. Proof of Theorem 1.1

In this section, we shall complete the proof of Theorem 1.1. First we need the following theorem of Paraska [20].

LEMMA 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $L(x,y)$ be an integrable function in $L^2(\Omega\times\Omega)$. Let $s_1 \ge s_2 \ge \cdots \ge s_n \ge \cdots$ be the singular numbers of the integral operator T_L with kernel function L. If k is a nonnegative integer and if $L \in W^{2,k}(\Omega\times\Omega)$, then*

$$s_j(T) = o(j^{-r}), \quad r = \frac{n+2k}{2n}.$$

We also need the following theorem of Bell [3].

THEOREM 2.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with $C^{k,\alpha}$ boundary. Then there is a function $\Phi(z)$ such that*

$$|\nabla^l \Phi(z)| \le C_k \delta(z)^{k+\alpha-1-l}, \quad z \in \Omega,$$

and

$$\int_\Omega \bar{f}(w)\, dv(w) = \int_\Omega \bar{f}(w)\Phi(w)\, dv(w)$$

for all $f \in A^2(\Omega) \cap W^{1,1}(\Omega)$, where $\delta(z)$ is the distance from z to $\partial\Omega$.

We need the following result from Bell and Boas [4].

LEMMA 2.3. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Then Ω satisfy Condition R if and only if, for any multi-index*

$$\alpha = (\alpha_1, \dots, \alpha_n),$$

there is a number $m(\alpha) > 0$ such that

$$\left\| \frac{\partial^\alpha K(z,\cdot)}{\partial z^\alpha \partial \bar{w}^\beta} \right\|_{L^2(\Omega)} \le C_{\alpha,k}\delta(z)^{-m(\alpha)-m(\beta)}$$

for all $z \in \Omega$.

Now we are ready to prove Theorem 1.1. Let $f \in A^2(\Omega)$. Then

$$\begin{aligned} T(f)(z) &= \int_\Omega K(z,w)\bar{f}(w)\, dv(w) \\ &= \int_\Omega K(z,w)\bar{f}(w)\Phi(w)\, dv(w) \\ &\equiv \int_\Omega L(z,w)\bar{f}(w)\, dv(w), \end{aligned}$$

where

$$L(z,w) = K(z,w)\Phi(w).$$

Thus, if Ω satisfies Condition R and $\partial\Omega$ is C^∞ then, for any $l > 0$, there is a $\Phi \in C^\infty(\bar{\Omega})$ such that

$$|\Phi(z)| \le C_l \delta^{m(2l)+l},$$

where

$$m(2l) = 2\max_{|\alpha|\le l}\{m(\alpha)\}, \qquad C_l = \max_{|\alpha|\le l} C(\alpha).$$

Thus, by Lemma 2.3, we have

$$\begin{aligned}
\|L(\cdot,\cdot)\|_{W^{l,2}} &\le C \sum_{|\alpha|,|\beta|\le l} \int_\Omega \int_\Omega \left| \frac{\partial^{\alpha+\beta} L(z,w)}{\partial z^\alpha \partial \bar{w}^\beta} \right| dv(w)\, dv(z) \\
&\le C_l \sum_{|\alpha|,|\beta|\le l} \int_\Omega \int_\Omega \left| \frac{\partial^{\alpha+\beta} K(z,w)}{\partial z^\alpha \bar{w}^\beta} \right| \delta(w)^{m(2l)+l-l}\, dv(w)\, dv(z) \\
&\le C_l \sum_{|\alpha|,|\beta|\le l} \int_\Omega \int_\Omega \left| \frac{\partial^{\alpha+\beta} K(z,w)}{\partial z^\alpha \bar{w}^\beta} \right| dv(z)\, \delta(w)^{m(2l)}\, dv(w) \\
&\le C_l \sum_{|\beta|\le l} \int_\Omega \delta(w)^{-m(|\alpha|)-m(|\beta|)} \delta(w)^{m(2l)}\, dv(w) \\
&\le C \cdot C_l |\Omega|.
\end{aligned}$$

Of course we have used Lemma 2.3. Therefore $L(z,w) \in W^{l,2}(\Omega \times \Omega)$ and, by Lemma 2.1, we have

$$s_j = o(j^{-r}) \text{ as } j \to \infty, \quad r = \frac{n+2l}{2n}.$$

Since $l > 0$ is an arbitrary integer, the proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. We need the following.

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $L(x,y) \in L^2(\Omega \times \Omega)$, and assume that the support of L is contained in Ω_{δ^0} for some $\delta^0 > 0$; here $\Omega_{\delta^0} = \{z \in \Omega : \delta(z) \ge \delta^0\}$. Let T_L be the integral operator defined on $A^2(\Omega)$ using the kernel L. Then the singular numbers s_j of T_L have the asymptotic behavior*

$$s_j = O(c^{-j^{1/n}}) \text{ as } j \to \infty$$

for some $0 < c < 1$ (depending only on Ω and δ^0).

Proof. For a positive integer k, we choose N_k points in Ω_{δ^0} such that $\Omega_{\delta^0} \subset \bigcup_{i=1}^{N_k} B(z_i, r_k) \subset \Omega_{\delta^0/2}$. For each positive integer l, we let $P_{k,l}$ be the projection of $A^2(\Omega)$ to the subspace M, where $(I - P_{k,l})(f)$ and its derivatives up to order l are zero at z_i. (Note that $M = A^2(\Omega) \ominus (I - P_{k,l}(A^2(\Omega)))$; N_k, r_k, and l will be chosen later.) Thus $P_{k,l}$ is an operator whose rank is at most $l^n N_k$. Hence, by choosing $r_k > 0$ such that $|\Omega|/2 \le N_k r_k^{2n} \le$ (diameter of $\Omega)^n$, we may conclude that

$$\begin{aligned}
\|(T-TP)(f)\|_{A^2}^2 &= \|T(I-P)(f)\|_{A^2}^2 \\
&\le C \int_{\Omega_{\delta^0}} |(I-P)(f)|^2\, dv(z) \\
&\le C(\delta^0)^{-l-(n+1)/2} \|f\|_{A^2} \sum_{i=1}^{N_k} r_k^l \\
&\le C(\delta^0)^{-l-(n+1)/2} \|f\|_{A^2} N_k r_k^l r_k^{2n} \\
&\le C(\delta^0)^{-l-(n+1)/2} \|f\|_{A^2} r_k^l.
\end{aligned}$$

We set $l^n N_k = k$, so $r_k = (c/N_k)^{1/2n} = (cl^n/k)^{1/2n}$. Therefore

$$(\delta^0)^{-l-(n+1)/2} r_k^l \le C(\delta^0)^{-l-(n+1)/2}(cl^n/k)^l = C(c_1 l^n/k)^l,$$

where $c_1 = C(\delta^0)^{-2}$. Now we let $l^n = k/2c_1$; then

$$(c_1 l/k)^l = (1/2)^l = (1/2)^{k^{1/n}/(2c_1)^{1/n}} = c_2^k,$$

where

$$c_2 = (1/2)^{1/(2c_1)^{1/n}} < 1.$$

Combining the above arguments, we see that the proof of Theorem 3.1 is complete. □

We are now ready to prove Theorem 1.3. Since Ω has real analytic boundary, there is a positve $\delta^0 > 0$ such that, for all $0 \le t \le \delta^0$, the domain Ω_t has real analytic boundary. Let h be a smooth function on Ω with support in Ω_{δ^0}.

For each $0 < t < \delta^0$ we write

$$dv(z) = g_t(z)\, d\sigma_t(z)\, dt, \quad \delta(z) > t.$$

Then $g_t(z)$ is real analytic near the boundary $\partial\Omega_t$. We consider the following Cauchy problem:

$$\begin{aligned} \Delta u^t &= h && \text{near } \partial\Omega_t, \\ u^t &= 0 && \text{on } \partial\Omega_t, \\ D_\nu u^t(z) &= g_t(z) && \text{on } \partial\Omega_t. \end{aligned}$$

It is known that this Cauchy problem has a C^∞ solution u^t, and that

$$\|u^\delta\|_{C^k(\bar\Omega_t)} \le C_k$$

(see [10] or [8]). Here C_k is a constant depending only on k and Ω. Now we consider the operator T. Since $\partial\Omega$ is real analytic, it follows that Ω is of finite type and satisfies Condition R. Note that, for each $f \in A^2(\Omega)$, we have

$$\begin{aligned} T(f)(z) &= \int_\Omega K(z,w)\bar f(w)\, dv(w) \\ &= \int_{\Omega_{\delta^0}} K(z,w)\bar f(w)\, dv(w) + \int_0^{\delta^0}\int_{\partial\Omega_t} K(z,w)\bar f(w) g_t(w)\, d\sigma_t(w)\, dt \\ &= \int_{\Omega_{\delta^0}} K(z,w)\bar f(w)\, dv(w) + \int_0^{\delta^0}\int_{\partial\Omega_t} K(z,w)\bar f(w) D_{\nu_t} u^t(w)\, d\sigma_t(w)\, dt \\ &= \int_{\Omega_{\delta^0}} K(z,w)\bar f(w)\, dv(w) - \int_0^{\delta^0}\int_{\Omega_t} K(z,w)\bar f(w)\Delta u^t(w)\, dv(w)\, dt \\ &= \int_{\Omega_{\delta^0}} K(z,w)\bar f(w)\, dv(w) - \int_0^{\delta^0}\int_{\Omega_t} K(z,w)\bar f(w) g(w)\, dv(w)\, dt \\ &= \int_{\Omega_{\delta^0}} K(z,w)\bar f(w)\, dv(w) - \int_\Omega \delta^0 K(z,w)\bar f(w) h(w)\, dv(w) \end{aligned}$$

$$= \int_\Omega K(z,w)(\Phi_{\delta^0}(w) - h(w))\bar{f}(w)\,dv(w)$$

$$= \int_\Omega L(z,w)\bar{f}(w)\,dv(w).$$

Here Φ_{δ^0} is given by Theorem 2.2 for the domain Ω_{δ^0}. Therefore our kernel L satisfies all the conditions of Lemma 2.1. The proof of Theorem 1.3 is complete. □

4. Finite Smoothness of ∂Q

In this section, we shall assume that Ω has $C^{k,\alpha}$ boundary.

THEOREM 4.1. *Let Ω be a bounded domain with $C^{k,\alpha}$ boundary. If*

$$\int_\Omega \int_\Omega |K(z,\xi)||K(\xi,\eta)||K(z,\eta)||\Phi(\eta)||\Phi(\xi)|\,dv(\eta)\,dv(\xi)$$
$$\leq CK(z,z)^{1-(2k+2\alpha-2)/(n+1)},$$

then $T \in S_p(A^2, \bar{A}(\Omega))$ for all $2 \geq p > \max\{n/(k+\alpha-1), 2n/(n+1)\}$.

Proof. We know that $T \in S_p(A^2, \bar{A}^2)$ if and only if $T^*T \in S_{p/2}(A^2(\Omega), A^2(\Omega))$. Let $H = T^*T$. Then H is a positive operator on $A^2(\Omega)$. Thus $\bar{P} \in S_p(A^2, \bar{A}^2)$ if and only if $H^{p/2} \in S_p(A^2)$. By Lemma 2.1, we have

$$T(K_z)(w) = \int_\Omega K_z(\xi)K(\xi,w)\Phi(w)\,dv(\xi) = \int_\Omega K(\xi,z)K(\xi,w)\Phi(w)\,dv(\xi).$$

Also

$$\int_\Omega |TK_z(w)|^2\,dv(w)$$

$$= \int \int_\Omega K(z,\xi)K(w,w\xi)\,dv(\xi) \int_\Omega K(\eta,z)K(\eta,w)\overline{\Phi(\eta)}\,dv(\eta)\,dv(w)$$

$$= \int \int \int K(z,\xi)K(w,\xi)K(\eta,z)K(\eta,w)\,dv(w)\,\Phi(\xi)\overline{\Phi(\eta)}\,dv(\eta)\,dv(\xi)$$

$$= \int \int K(z,\xi)K(\eta,z)K(\eta,\xi)\Phi(\xi)\overline{\Phi(\eta)}\,dv(\eta)\,dv(\xi).$$

In addition,

$$\operatorname{tr}(H^{p/2}) \leq \int_\Omega \langle Hk_z \rangle^{p/2} K(z,z)\,dv(z)$$

$$= \int_\Omega \langle Tk_z, Tk_z \rangle^{p/2} K(z,z)\,dv(z)$$

$$= \int_\Omega \langle TK_z, TK_z \rangle^{p/2} K(z,z)^{1-p/2}\,dv(z)$$

$$\le C\int_\Omega K(z,z)^{1-p/2}K(z,z)^{p/2-p(k+\alpha-1)/(n+1)}\,dv(z)$$

$$\le C\int_\Omega K(z,z)^{1-p(k+\alpha-1)/(n+1)}\,dv(z)$$

$$\le C_p$$

if $p > n/(k+\alpha-1)$. Thus $H\in S_{p/2}$ and so $T\in S_p$. Therefore the proof of the theorem is complete. □

COROLLARY 4.2. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^{k,\alpha}$ boundary, $k\ge 4$. If $2\ge p>\max\{n/(k+\alpha-1),2n/(n+1)\}$ then $T\in S_p$.*

Proof. By using the asymptotic expansion of the Bergman kernel given by Fefferman [7] (or Ligocka [16]), one can verify the hypotheses of Theorem 4.1. The proof is complete. □

5. Proof of Theorem 1.2

In this section, we shall consider a bounded domain in the complex plane and prove Theorem 1.2. In order to achieve this goal, we need to estimate the Bergman kernel for domains with $C^{1,\alpha}$ boundary.

THEOREM 5.1. *Let Ω be a bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary, $\alpha>0$. Then*

$$|K(z,w)|\le C(|z-w|^2+\delta(z)^2+\delta(w)^2)^{-1}.$$

The proof of Theorem 5.1 can be effected using classical techniques (conformal mapping or estimates on the Green's function); alternatively, the techniques in Fefferman [7] can be specialized down to one complex variable. As a third method, one can prove Theorem 5.1 by using the technique of Kerzman and Stein [12], which was presented also in Ligocka [16].

By Theorem 5.1, one can easily check that the following holds:

$$\int_{\Omega^2}|K(z,\xi)||K(\xi,\eta)||K(z,\eta)|\delta(\eta)\delta(\xi)\,dv(\eta)\,dv(\xi)\le C.$$

By Theorem 4.1, we have that $T\in S_p(\Omega)$ when $p>1$ and $\alpha=1$. Therefore the proof of (a) of Theorem 1.3 follows.

Now we prove part (b) of Theorem 1.2. We first need the following sufficient condition of Russo [see [2]).

LEMMA 5.2. *Let $(X,\mathfrak{A},\mu)$ be a σ-finite measure space, and let $k(x,y)$ be a measurable function on $X\times X$ satisfying*

$$\|k\|_{p',p}=\left(\int\left(\int|k(x,y)|^{p'}d\mu(x)\right)^{p/p'}d\mu(y)\right)^{1/p}<\infty$$

and

$$\|k^*\|_{p',p} = \left(\int \left(\int |k(y,x)|^{p'} d\mu(x)\right)^{p/p'} d\mu(y)\right)^{1/p} < \infty$$

for some $2 \le p < \infty$. Then the associated integral operator I_k, defined on a dense subspace of $L^2(\mu)$ by

$$I_k f(x) = \int f(y) k(x,y)\, d\mu(y),$$

is in the trace ideal S_p; moreover

$$\|I_k\|_{S_p} \le (\|k\|_{p',p} \|k^*\|_{p',p})^{1/2}.$$

Now we return to the proof of part (b) of Theorem 1.2. We first consider the case $p > 2$. We know that T is an integral operator with kernel function

$$k(z,w) = K(z,w)\Phi(w) = K(z,w)\, O(\delta(w)^\alpha).$$

By Theorem 4.1 we have, since $p\alpha > 1$, that

$$\begin{aligned}
&\int_\Omega \left(\int_\Omega |k(z,w)|^{p'} dv(w)\right)^{p/p'} dv(z) \\
&\quad \le C \int_\Omega \left[\int_\Omega \left(\frac{\delta(w)^\alpha}{|z-w|^2 + \delta(z)^2 + \delta(w)^2}\right)^{p'} dv(w)\right]^{p/p'} dv(z) \\
&\quad \le C \int_\Omega \delta(z)^{(\alpha-2)p + 2p/p'} dv(z) \\
&\quad \le C \int_\Omega \delta(z)^{(\alpha p - 2) +} dv(z) \le C_p.
\end{aligned}$$

Similarly,

$$\int_\Omega \left(\int_\Omega |k(z,w)|^{p'} dv(z)\right)^{p/p'} dv(w) \le C_p.$$

Therefore, by Lemma 5.2, the proof of part (b) of Theorem 1.2 is complete for the case $p > 2$.

Next we prove the case $1 < p \le 2$. One could verify directly the hypotheses of Theorem 4.1 and then prove the theorem. Instead, we give a slightly different proof. First we need the following lemma, which is similar to lemmas in [2; 14; 15].

LEMMA 5.3. *Let Ω be a bounded domain in $\mathbb{C}^n$. If*

$$\int_\Omega \|T(K_z)\|_{A^2}^p K(z,z)^{1-p/2}\, dv(z) \le C,$$

then $T \in S_p(A^2, \bar{A}^2)$ for all $0 < p \le 2$.

Proof. For completeness, we present the proof in full. Since

$$\begin{aligned}
\|T\|_{S_p} &= \operatorname{tr}((T^*T)^{p/2}) \\
&= \int_\Omega \langle (T^*T)^{p/2}(K_z), K_z\rangle \, dv(z) \\
&= \int_\Omega \langle (T^*T)^{p/2}(k_z), k_z\rangle K(z,z)\, dv(z) \\
&\le \int_\Omega \langle T^*T(k_z), k_z\rangle^{p/2} K(z,z)\, dv(z) \\
&= \int_\Omega \|T(k_z)\|_{L^2}^p K(z,z)\, dv(z) \\
&= \int_\Omega \|T(K_z)\|_{A^2}^p K(z,z)^{1-p/2}\, dv(z),
\end{aligned}$$

we see that the proof of the lemma is complete. □

We are now ready to prove the case $1 < p \le 2$ of part (b) in Theorem 1.2.

Since Ω has $C^{1,\alpha}$ boundary, we have $|\Phi(w)| \le C\delta(z)^\alpha$. Thus

$$\begin{aligned}
\|T(K_z)\|_{A^2} &= \left\| \int_\Omega K(\cdot,\xi)K(z,\xi)\Phi(\xi)\, dv(\xi) \right\|_{A^2} \\
&\le \int_\Omega |K(z,\xi)|\,|\Phi(\xi)|\,\|K(\cdot,\xi)\|_{A^2}\, dv(\xi) \\
&\le \int_\Omega |K(z,\xi)|\,|\Phi(\xi)|\,K(\xi,\xi)^{1/2}\, dv(\xi) \\
&\le CK(z,z)^{1/2-\alpha/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Omega \|T(K_z)\|_{A^2}^p K(z,z)^{1-p/2}\, dv(z) \\
&\quad\le C\int_\Omega (|K(z,z)|^{1/2-\alpha/2})^p K(z,z)^{1-p/2}\, dv(z) \\
&\quad\le \int_\Omega K(z,z)^{1-\alpha p/2}\, dv(z) \\
&\quad\le C\int_\Omega \delta(z)^{-2+\alpha p/2}\, dv(z) \\
&\quad\le C_p.
\end{aligned}$$

Here we have used the fact that $1 - p\alpha/2 < 1/2$, that is, $p\alpha > 1$. The proof of Theorem 1.2 is complete. □

Closing Remarks. It is clear from the statements and the proofs of Theorems 1.1 and 1.2 that one could sharpen the statement of Theorem 1.2 in the following fashion:

Let Ω be a bounded domain in the complex plane $C^{k,\alpha}$ boundary. If $p > 1/(k+\alpha-1)$, then $T \in S_p$.

The version of Theorem 1.2 that we have already proved gives the case $p > 1$. For the case $p \leq 1$, one can establish this sharper statement by improving Lemma 2.1, and then applying the argument of the proof of Theorem 1.1. We shall investigate these matters in future papers.

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