Inductive Limits of Algebras of Generalized Analytic Functions

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We study inductive limits of algebras of generalized analytic functions generated by compact abelian groups with ordered duals. In particular, we answer a question raised in [2] for inductive limits of spaces of type H^∞ on a compact abelian group with ordered dual.

1. Introduction

Let Γ be a subgroup of the group $\mathbb{R}$ of real numbers. We assume that Γ is equipped with the discrete topology. Denote by G the dual group of Γ, that is, G is the group of characters of Γ. Note that G is a compact abelian group with unit e.

In what follows we make use of the terminology and notation from [7]. By the Pontryagin duality theorem, the dual group $\hat{G}$ of G is isomorphic to Γ. For a given $a \in \Gamma$ let $\chi^a \in \hat{G}$ be the character $\chi^a(g) = g(a)$, $g \in G$. Let σ be the normalized Haar measure on G. Every function f in $L^1(G, \sigma)$ relative to σ has a formal Fourier series

$$f(g) \sim \sum_{a \in \Gamma} c_a^f \chi^a(g),$$

where

$$c_a^f = \int_G f(g) \bar{\chi}^a(g)\, d\sigma(g)$$

are the Fourier coefficients of f. The set $S(f)$ of numbers a in Γ for which $c_a^f \neq 0$ is the *spectrum* of f. A function $f \in L^1(G, \sigma)$ is called a *generalized analytic function on G* if $S(f)$ is contained in the semigroup $\Gamma_+ = \{a \in \Gamma \mid a \geq 0\}$.

Let Δ_G be the set of semi-characters (i.e., homomorphisms from Γ_+ into the unit disc in $\mathbb{C}$) of the semigroup Γ_+. Δ_G is called the *big disc* over G. It is well known that Δ_G is a compact set and can be obtained from the Cartesian product $[0,1] \times G$ by identifying the points in the fiber $\{0\} \times G$. Every point $m \in \Delta_G$ can be expressed in the *polar* form $m = rg$ for some $r \in [0,1]$ and $g \in G$. Observe that $G \equiv \{1\} \times G \subset \Delta_G$ since the characters on Γ are semi-

Received July 7, 1995.

Research of the second author was partially supported by a grant from the National Science Foundation.

Michigan Math. J. 42 (1995).

characters on Γ_+. Actually, G is the topological boundary of Δ_G. It is well known that the big disc Δ_G is the maximal ideal space of the algebra A_G of continuous generalized analytic functions on G, while G is its Shilov boundary. Every function $f \in A_G$ gives rise to a continuous function $\hat{f}$ on Δ_G by the rule

$$\hat{f}(m) = m(f), \quad m \in \Delta_G.$$

It is well known that A_G is a Dirichlet algebra on G. Therefore, given a point $m \in \Delta_G^0 = \Delta_G \backslash G$, there is a *unique* positive measure μ_m on G, the *representing* measure of m, such that $\operatorname{supp}(\mu_m) = G$ and

$$\hat{f}(m) = \int_G f(g)\, d\mu_m(g) \tag{1}$$

for every $f \in A_G$.

For a given $m \in \Delta_G$ and $g \in G$ the function $m_g(a) = m(a)g^{-1}(a)$ is a character on Γ_+, i.e. $m_g \in \Delta_G$. As it follows from (1),

$$\hat{f}(m_h) = \int_G f(gh)\, d\mu_m(g) = \int_G f(g)\, d\mu_m(gh^{-1}), \tag{2}$$

i.e., the representing measure for the point m_h can be obtained from the representing measure μ_m of m by a translation with h^{-1}. Henceforth the representing measure μ_{rg} of the point $m = rg$ in Δ_G^0 coincides with the representing measure μ_{re} of re translated by g.

Given an $r \in (0,1)$, we denote by $f^{(r)}$ the function $f^{(r)}(g) = \hat{f}(rg)$. Clearly $f^{(r)} \in A_G$ and

$$\sup_{g\in G} |f^{(r_1)}(g)| \geq \sup_{g \in G} |f^{(r_2)}(g)| \tag{3}$$

whenever $r_1 \geq r_2$.

Let H^∞ be the algebra of bounded functions on the interior Δ_G^0 of the big disc that can be approximated on compact subsets of Δ_G^0 by functions $\hat{f}$, $f \in A_G$. Given an $f \in H^\infty$, the limits

$$f^*(g) = \lim_{r \to 1} f^{(r)}(g)$$

exist for σ-almost all $g \in G$. The boundary value function f^* belongs to $H^\infty(G, \sigma)$. We identify f with its boundary value function f^*. The space of functions f^*, $f \in H^\infty$, we denote again by H^∞. Thus, the algebra H^∞ we view as the space of functions in $L^\infty(G, \sigma)$ that are boundary values of continuous functions on Δ_G^0. Note that H^∞ is a closed subalgebra of $L^\infty(G, \sigma)$ with respect to the norm $\|\cdot\|_\infty$ (see e.g. [5]), and

$$\|f\|_\infty = \lim_{r\to 1} \sup_{g \in G} |f^{(r)}(g)|.$$

The algebra $\mathcal{H}^\infty$ of generalized analytic functions in $L^\infty(G, \sigma)$ is the weak*-closure of A_G in $L^\infty(G, \sigma)$ [4]. Clearly H^∞ is a closed subalgebra of $\mathcal{H}^\infty$.

Let I be a directed set; that is, let I be a partially ordered set such that for every pair i_1 and i_2 in I there is an $i_3 \in I$ such that $i_1 < i_3$ and $i_2 < i_3$. We consider a family $\{\Gamma_i\}_{i\in I}$ of subgroups of Γ indexed by I such that $\Gamma_{i_1}\subset\Gamma_{i_2}$ whenever $i_1 < i_2$. Under the natural inclusions, $\{\Gamma_i\}_{i\in I}$ becomes an inductive system of groups. Suppose that Γ coincides with the inductive limit of the system $\{\Gamma_i\}_{i\in I}$, that is, $\Gamma = \varinjlim_{i\in I}\Gamma_i$. Let H_i^∞ denote the space of functions $f\in H^\infty$ with $S(f)\subset\Gamma_i$. Clearly, H_i^∞ is a closed subalgebra of H^∞, and $H_i^\infty\subset H_j^\infty$ if and only if $\Gamma_i\subset\Gamma_j$. Therefore the family $\{H_i^\infty\}_{i\in I}$ of subalgebras of H^∞ is ordered by the inclusion. Denote by H_I^∞ the closure with respect to the norm $\|\cdot\|_\infty$ of the set $\bigcup_{i\in I}H_i^\infty$, that is, of the inductive limit $\varinjlim_{i\in I}H_i^\infty$. Clearly H_I^∞ is a commutative Banach algebra.

In a similar way we define the algebra $\mathcal{H}_I^\infty$ as the $\|\cdot\|_\infty$-closure of the inductive limit $\varinjlim_{i\in I}\mathcal{H}_i^\infty$, where $\mathcal{H}_i^\infty = \{f\in\mathcal{H}^\infty \mid S(f)\subset\Gamma_i\}$.

Algebras of type H_I^∞ were introduced in [6] (see also [7]) in connection with the corona problem for algebras of generalized analytic functions. Curto, Muhly, and Xia [2] have introduced other algebras of this type in connection with their study of Wiener–Hopf operators with almost-periodic symbols. They have raised the question of whether the algebras of type H_I^∞ coincide with H^∞.

The next theorem gives a criteria for the coincidence of these algebras.

THEOREM 1. *Let G be a compact abelian group whose dual group $\Gamma = \hat{G}$ is a subset of $\mathbb{R}$ and such that $\Gamma = \varinjlim_{i\in I}\Gamma_i$, where $\{\Gamma_i\}_{i\in I}$ is a family of subgroups of Γ. Let H_i^∞ (resp. $\mathcal{H}_i^\infty$) be the space of functions in H^∞ (resp. $\mathcal{H}^\infty$) whose spectrum is in Γ_i, $i\in I$. Then the following are equivalent.*

(a) *$H^\infty = \bigcup_{i\in I}H_i^\infty$ and $\mathcal{H}^\infty = \bigcup_{i\in I}\mathcal{H}_i^\infty$.*
(b) *$H^\infty = H_I^\infty$ and $\mathcal{H}^\infty = \mathcal{H}_I^\infty$.*
(c) *Every countable subgroup Γ_0 in Γ is contained in some group from the family $\{\Gamma_i\}_{i\in I}$.*

2. Proof of Theorem 1

The proof is based on the following lemma.

LEMMA 1. *Let $r\in(0,1)$ and let μ_r be the representing measure on G of the point $re\in\Delta_G^0$. Then*

$$\lim_{j\to\infty}\sup_{g\in G}\mu_r(gV_j) = 0 \tag{4}$$

for every nested family $V_1\supset V_2\supset\cdots\supset V_n\supset\cdots$ of neighborhoods of the identity e with $\bigcap_{j=1}^\infty V_j = \{e\}$.

Proof. Assume on the contrary that

$$\lim_{j\to\infty}\sup_{g\in G}\mu_r(gV_j) > 0,$$

and let $\{g_j\}_1^\infty$ be a sequence in G with $\mu_r(g_jV_j)\to c>0$. By the compactness of G there is a subsequence of $\{g_j\}_1^\infty$, say $\{h_n\}_1^\infty$, that converges to a point

$h \in G$. Note that $\mu_r(hV_n) \geq \lim_{k\to\infty} \mu_r(h_k V_k) = c$ for every integer n, since $hV_n \supset h_k V_k$ for k big enough. Consequently, $\mu_r(h) = \lim_{k\to\infty} \mu_r(h_k V_k) = c$. Consider the Lebesgue decomposition of $\mu_r - \delta_{re}$ with respect to the Dirac measure δ_h at h, namely

$$\mu_r - \delta_{re} = \mu_r(h)\delta_h + \nu,$$

where the measure ν is singular with respect to δ_h. By Ahern's theorem (cf. [3, Chap. II, Cor. 7.8]), the measure δ_h (as well as ν) is orthogonal to the algebra A. This is impossible because there certainly is a function in A that is nonvanishing at $h \neq re$. □

Proof of Theorem 1. We give the proof of the theorem for the space H^∞ only. The proof for the corresponding space $\mathcal{H}^\infty$ is virtually the same.

The implication (a) $\Rightarrow$ (b) is trivial.

(c) $\Rightarrow$ (a). Let $f \in H^\infty$. By Parseval's identity the spectrum $S(f)$ of f is countable. Therefore the group Γ_0 generated by the set $S(f)$ is countable as well. By the supposition there is a group $\Gamma_{i_0} \in \{\Gamma_i\}_{i\in I}$ that contains Γ_0. Hence $f \in H^\infty_{i_0} \subset \bigcup_I H^\infty_i$.

(b) $\Rightarrow$ (c). Let Γ_0 be a countable subgroup of Γ. Without loss of generality we can assume from the beginning that Γ_0 coincides with the group Γ (i.e., that Γ is a countable group). Then $G = \hat{\Gamma}$ is a metric and separable space. Let $\{h_n\}_1^\infty \to e$ be a sequence of different points in G, and let $\{B_n\}$ be a collection of disjoint (metric) balls centered at h_n and not containing e, such that for every neighborhood V of e there is a natural number N such that $B_n \subset V$ for all $n \geq N$.

Consider a function $f_n \in A_G$ such that $\|f_n\|_\infty = f_n(h_n) = 1$, $f_n(e) = 0$, and $|f_n| < 1/2^n$ on $G \backslash B_n$. Such a function exists because the points of G are peak points for A_G. Identities (1) and (2) imply

$$\begin{aligned} |f_n^{(r)}(g)| &= \left|\int_G f_n(gh)\,d\mu_r(h)\right| \leq \int_G |f_n(gh)|\,d\mu_r(h) \\ &= \int_{g^{-1}B_n} |f_n(gh)|\,d\mu_r(h) + \int_{G\backslash g^{-1}B_n} |f_n(gh)|\,d\mu_r(h) \\ &\leq \mu_r(g^{-1}B_n) + \frac{1}{2^n}. \end{aligned}$$

The specific requirements for the balls B_n guarantee that

$$\sum_{n=k}^{\infty} |f_n^{(r)}(g)| \leq \mu_r(g^{-1}V_k) + \frac{1}{2^{k-1}} < 2,$$

where $V_k = \bigcup_{n=k}^\infty B_n$. As follows from (4), for every $\epsilon > 0$ there is a k such that $\mu_r(g^{-1}V_k) < \epsilon$ for all $g \in G$. Consequently, the series $\sum_{n=1}^\infty f_n^{(r)}$ converges uniformly on G to a function $\tilde{f}^{(r)} \in A_G$. Clearly $\|\tilde{f}^{(r)}\|_\infty < 2$. Therefore the function $\tilde{f} = \sum_{n=1}^\infty f_n$ belongs to the algebra H^∞. Since (by hypothesis) $H^\infty = H_I^\infty$, the function $\tilde{f}$ is in H_I^∞. Then there is an f in one of the spaces H_i^∞ such that

$$\|\tilde{f}-f\|_\infty < \tfrac{1}{16}.$$

By the well known result from group theory, the group Γ_i is the dual group of the quotient group G/G_i, where $G_i = \{g \in G \mid \chi^a(g) = 1 \text{ for all } a \in \Gamma_i\}$. Therefore the space H_i^∞ coincides with the space of G_i-invariant functions in H^∞; that is, $u \in H_i^\infty$ if and only if $u \in H^\infty$ and $u(h) = u(gh) = u_g(h)$ for all $g \in G_i$ and $h \in G$. Consequently, $f = f_g$ for $g \in G_i$, and

$$\|\tilde{f}-\tilde{f}_g\|_\infty \le \|\tilde{f}-f\|_\infty + \|\tilde{f}_g - f_g\|_\infty < \tfrac{1}{8} \tag{5}$$

for every $g \in G_i$. Suppose that $\Gamma_i \ne \Gamma$, that is, suppose $G_i \ne \{e\}$. Fix a $g_0 \in G_i \backslash \{e\}$. By the continuity of $\tilde{f}$ on $G\backslash\{e\}$ the set

$$V = \{h \in G\backslash\{e\} \mid |\tilde{f}(h) - \tilde{f}(g_0)| < \tfrac{1}{16}\}$$

is an open neighborhood of $g_0 \ne e$. By the construction of $\tilde{f}$ there are g_1 and g_2 in $g_0^{-1}V\backslash\{e\}$ such that $|\tilde{f}(g_1)| > \frac{15}{16}$ and $|\tilde{f}(g_2)| < \frac{1}{16}$. Now

$$|\tilde{f}(g_i) - \tilde{f}_{g_0}(g_i)| \le \|\tilde{f} - \tilde{f}_{g_0}\|_\infty \le \tfrac{1}{8} \quad \text{for } i = 1, 2$$

implies

$$|\tilde{f}_{g_0}(g_1)| > \tfrac{13}{16} \quad \text{and} \quad |f_{g_0}(g_2)| < \tfrac{3}{16}.$$

Consequently,

$$|\tilde{f}_{g_0}(g_1) - \tilde{f}_{g_0}(g_2)| > \tfrac{5}{8},$$

which is impossible since $g_0 g_1$ and $g_0 g_2$ belong to V. Thus, $G_i = \{e\}$; that is, $\Gamma_0 = \Gamma_i \in \{\Gamma_i\}_{i \in I}$. □

3. Examples and Consequences

EXAMPLE 1. Let $\Gamma = \mathbb{Q}$ be the group of discrete rational numbers. Assume that $\{\Gamma_i\}_{i\in I}$ is an inductive system of subgroups of $\mathbb{Q}$ such that $\mathbb{Q} = \varinjlim_{i \in I} \Gamma_i$. If $\mathbb{Q}$ is not a member of $\{\Gamma_i\}_{i \in I}$, then by Theorem 1 we obtain that $H_I^\infty \ne H^\infty$.

The algebra H_I^∞ (of so-called hyper-analytic functions) was introduced and studied in [6] (see also [7]) for the case when the subgroups Γ_i are isomorphic to $\mathbb{Z}$, the group of integers. As shown in [6], this algebra does not have a corona and its maximal ideal space resembles the maximal ideal space of the algebra H^∞ related to the unit circle. As we have seen, in this case $H_I^\infty \ne H^\infty$.

The properties of algebras of type H^∞ related with general compact groups G are less known. In particular it is not known if they possess corona; their maximal ideal spaces and Shilov boundaries lack a satisfactory description.

EXAMPLE 2. The algebras introduced in [2] are another example of algebras of type H_I^∞. Let $\Gamma = \mathbb{R}$ and $\Lambda \subset \mathbb{R}_+$ be a basis in $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. Consider the family J of pairs $\{(\gamma, n)\}$, where γ is a finite subset in Λ and n is a natural number. We equip J with the following ordering:

$$(\gamma, n) \prec (\delta, k) \quad \text{if and only if} \quad \gamma \subset \delta \text{ and } n \le k.$$

For a $(\gamma, n) \in J$ with $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, we define the group

$$\Gamma_{(\gamma,n)} = \left\{ \frac{1}{n!}(m_1\gamma_1 + m_2\gamma_2 + \cdots + m_k\gamma_k) \mid m_j \in \mathbb{Z}, j = 1, \ldots, k \right\}.$$

Clearly $\Gamma_{(\gamma,n)}$ is isomorphic to the group $\mathbb{Z}^k = \bigoplus_{i=1}^k \mathbb{Z}$. Moreover, $\Gamma_{(\gamma,n)} \subset \Gamma_{(\delta,k)}$ whenever $(\gamma, n) \prec (\delta, k)$, and

$$\mathbb{R} = \varinjlim_{(\gamma,n)\in J} \Gamma_{(\gamma,n)}.$$

For a given j, $1 \le j \le k$, consider an increasing (resp. decreasing) sequence $\{\alpha_l^j\}_{l=1}^\infty$ (resp. $\{\beta_l^j\}_{l=1}^\infty$) of positive irrational numbers converging to $\gamma_j \in \gamma$:

$$\lim_{l\to\infty} \alpha_l^j = \gamma_j = \lim_{l\to\infty} \beta_l^j.$$

Denote

$$P_l = \bigcap_{j=1}^k \{(m_1, \ldots, m_k) \in \mathbb{Z}^k \mid m_1\gamma_1 + \cdots + m_j\lambda_l^j + \cdots + m_k\gamma_k \ge 0\},$$

where λ_l^j is either α_l^j or β_l^j, and let

$$P_l^\gamma(n) = \left\{ \frac{1}{n!}(m_1\gamma_1 + \cdots + m_k\gamma_k) \mid (m_1, \ldots, m_k) \in P_l \right\}.$$

Clearly $P_l^\gamma(n) \subset \mathbb{R}_+$. Moreover, the group generated by the semigroup $P_l^\gamma(n)$ coincides with the group $\Gamma_{(\gamma,n)}$. Denote by $H^\infty(P_l^\gamma(n))$ the set of functions in $L^\infty(G, \sigma)$ whose spectrum is contained in the set $P_l^\gamma(n)$. Clearly $H^\infty(P_l^\gamma(n)) \subset H^\infty(P_d^\delta(m))$ if $\gamma \subset \delta$, $n \le m$, and $l \le d$. It is easy to check that $H^\infty(P_l^\gamma(n)) \subset H^\infty_{(\gamma,n)}$, where

$$H^\infty_{(\gamma,n)} = \{f \in H^\infty \mid S(f) \subset \Gamma_{(\gamma,n)}\}.$$

The closure H under the $\|\cdot\|_\infty$-norm of the set $\bigcup_{((\gamma,n),l)\in J\times\mathbb{N}} H^\infty(P_l^\gamma(n))$, or (equivalently) of the inductive limit $\varinjlim_{((\gamma,n),l)\in J\times\mathbb{N}} H^\infty(P_l^\gamma(n))$, is isomorphic to the algebra considered in [2].

There arises the question of whether the algebra H from Example 2 coincides with H^∞ or not. The answer to this question is negative, as follows.

THEOREM 2. *The set* $H = \varinjlim_{((\gamma,n),l)\in J\times\mathbb{N}} H^\infty(P_l^\gamma(n))$ *is a proper closed subalgebra of* H^∞.

Proof. The inclusion $H \subset H^\infty$ is proved essentially in [2]. Assume that $H = H^\infty$. Then, by Theorem 1, $\mathbb{Q} \subset \mathbb{R}$ belongs to the family $\{\Gamma_{(\gamma,n)}\}_{(\gamma,n)\in J}$. However, this is impossible since, unlike $\mathbb{Q}$, the group $\Gamma_{(\gamma,n)}$ is isomorphic to $\mathbb{Z}^k$ for some natural k. □

The algebra H^∞ is isometrically isomorphic to the algebra $H^\infty_{AAP}(\mathbb{R}) \subset H^\infty(\mathbb{R})$ consisting of boundary values of Γ-almost periodic functions on $\mathbb{R}$ that are analytic in the upper half plane. In a similar way the algebra H is isomorphic to a subalgebra $H(\mathbb{R})$ of $H^\infty_{AAP}(\mathbb{R})$. As an immediate corollary from Theorem 2 we obtain that these two algebras are distinct as well.

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