A Class of Operators and Similarity to Contractions

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1. Introduction

Let H^2 denote the usual Hardy space on the disk in the complex plane. Let S denote the shift operator on H^2, S^* the adjoint of S and let Q be an element in $\mathcal{L}(H^2)$, the set of all bounded linear operators on H^2. Let R denote the corresponding operator on $H^2 \oplus H^2$ defined by the matrix of operators $\begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}$.

Let $\Gamma_\varphi f = P(\varphi \tilde{f})$, where $\tilde{f}(z) = f(\bar{z})$ for $|z| = 1$ and P is the orthogonal projection of L^2 onto H^2. (Γ_φ is defined precisely in Section 2.)

In the case where $Q = \Gamma_\varphi$, Peller [Pel] shows that if $\varphi' \in$ BMOA then R is polynomially bounded, and Bourgain [Bo] shows that with the same assumption R is similar to a contraction. The Bourgain result contains the Peller result because a bounded operator on a Hilbert space which is similar to a contraction is also polynomially bounded.

A major objective of this paper is to obtain a stronger conclusion than the one in the Bourgain theorem from the same hypothesis, and then show that the converse of the stronger implication holds. These results are Theorems 3.1 and 3.2.

Bourgain obtains the conclusion of his result by showing that R is completely polynomially bounded; he then uses a theorem of Paulsen [Pa]. Our approach is to decompose R and show that R is similar to $S^* \oplus S$. This quite simple decomposition approach can be used to obtain the same conclusion for quite general Q, which is the content of Theorem 5.4.

Roughly speaking, our converse result, Theorem 3.2, states that if R (with $Q = \Gamma_\varphi$) is similar to $S^* \oplus S$ and $\varphi' \in H^2$, then $\varphi' \in$ BMOA. The key to the proof of the converse result is Theorem 4.1, which is of independent interest. That theorem states that if $\varphi' \in H^2$, then the domain of the operator $Gf = \Gamma_\varphi f'$ is a space of the form H^2_W if and only if G is bounded. The space H^2_W ($W \geq C > 0$) is the space of functions f in H^2 such that $|f|^2 W$ is integrable on the circle.

Section 6 is devoted to some key lemmas that are used in Sections 3 and 4.

Operators of the form R seem to have first arisen in [Fo] (see also [H1]) where it is shown that, for a certain Q, R is power bounded ($\|R^n\| \leq C$, $n \geq 1$)

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but R is not similar to a contraction. An operator T on a Hilbert space is polynomially bounded if

$$\|p(T)\| \leq C \max\{|p(z)| : |z| \leq 1\}$$

for all complex polynomials p. Lebow [Le] showed that the operator defined by Foquel [Fo] is not polynomially bounded. The question (see [H2, Prob. 6]) of whether every polynomially bounded operator on Hilbert space is similar to a contraction is well-known and unanswered.

The above references and our results suggest a similar question. Is every polynomially bounded operator of the form R, for a general Q or for Q equal to some Γ_φ, similar to a contraction?

2. Notation

Normalized Lebesgue measure on the circle $\{z : |z| = 1\}$ in the complex plane will be denoted dm. For $1 \leq p \leq \infty$, L^p will denote $L^p(m)$ and $\| \; \|_p$ its usual norm. The set $\{z : |z| < 1\}$ in the complex plane will be referred to as the disc and H^p, for $1 \leq p \leq \infty$, will denote the usual Hardy spaces over the disc. The functions in H^p will frequently be identified with their boundary values and, correspondingly, H^p will be identified with a subspace of L^p. The norm of H^p will also be denoted by $\| \; \|_p$. If W is a real measurable function on the circle and $W \geq C$ where C is a positive constant, then H^2_W is the space of functions f such that $|f|^2 W$ is integrable with respect to m on the circle and the norm of f is $(\int |f|^2 \, dm)^{1/2}$.

We denote by BMOA the space of functions g in H^1 for which

$$\|g\|_{1^*} \equiv \sup\left\{\left|\int f\bar{g}\, dm\right| : f \in H^1, \|f\|_1 \leq 1\right\}$$

is finite. The rather deep fact that the functions in BMOA can be characterized in terms of "bounded mean oscillation" will not be used in this paper. The orthogonal projection of L^2 onto H^2 will be denoted by P. If f is analytic on the disc, then f' denotes its derivative and $\int f$ denotes its antiderivative that vanishes at $z = 0$.

If X is a B space, $\mathcal{L}(X)$ will denote the space of all bounded linear operators on X and $B(X)$ will denote the unit ball of X. If K_1 and K_2 are subspaces of a Hilbert space K, then $K = K_1 \oplus K_2$ will denote a direct sum for which the corresponding projections are bounded; in particular, it need not indicate an orthogonal direct sum. Similarity of operators will be denoted by $\simeq$.

The following special notation defines the main objects of interest in the paper. The Hilbert space $H^2 \oplus H^2$, the usual orthogonal direct sum of H^2 with itself, will be denoted by H. The shift operator S on H^2 is defined by $Sf(z) = zf(z)$ and its adjoint is denoted by S^*. For any Q in $\mathcal{L}(H^2)$, the corresponding operator R on $H = H^2 \oplus H^2$ is defined by the operator matrix

$$R=\begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}.$$

The subspaces H_1 and H_2 of $H^2 \oplus H^2$ are defined as follows: $H_1 = \{(f, 0)\colon f \in H^2\}$ and H_2 is the closed linear span of the set $\{R^n v\colon n \geq 0\}$ in $H^2 \oplus H^2$ where $v = (0, 1)$. The operator M is defined to be $R \mid H_2$, the restriction of R to H_2. For $k = 0, 1, 2, \ldots$, the operator Q_k is defined by the equation

$$R^k=\begin{pmatrix} S^{*k} & Q_k \\ 0 & S^k \end{pmatrix}.$$

For each polynomial $p(z) = \sum c_k z^k$, we define G_0 by $G_0 p = \sum c_k Q_k 1$ and regard G_0 as an operator with domain in H^2 and values in H^2. Note that

$$\sum c_k R^k v = \begin{pmatrix} G_0 p \\ p \end{pmatrix}.$$

When G_0 is closable, its closure will be denoted by G, $\mathfrak{D}(G)$ will denote the domain of G, and the norm of $\mathfrak{D}(G)$ will be the usual domain norm

$$\|f\|_{\mathfrak{D}(G)} = \|f\| + \|Gf\|.$$

For $\varphi \in \mathrm{BMOA}$, Γ_φ is defined by $\Gamma_\varphi f = P(\varphi \tilde{f})$ for $f \in H^2$, where $\tilde{f}(z) = f(\bar{z})$ for $|z| = 1$.

PROPOSITION. $\|\Gamma_\varphi\| = \|\varphi\|_{1^*}$.

This is easily proved by considering $(\Gamma_\varphi f, g)$ with f, g in H^2 and using the factorization theorem.

3. $Q = \Gamma_\varphi$

This section is devoted to the proofs of the following two theorems with converse implications.

THEOREM 3.1. *If $Q = \Gamma_\varphi$ and $\varphi' \in \mathrm{BMOA}$, then $H = H_1 \oplus H_2$, where the corresponding projections are bounded, $M \simeq S$ and $R \simeq S^* \oplus S$, where $S^* \oplus S$ is the orthogonal direct sum of the operator S^* and S defined on $H = H^2 \oplus H^2$.*

THEOREM 3.2. *If $Q = \Gamma_\varphi$, $\varphi' \in H^2$ and $M \simeq S$, then $\varphi' \in \mathrm{BMOA}$.*

For convenience, we state the six results from Sections 4, 5, and 6 which we use in the proofs of Theorems 3.1 and 3.2.

THEOREM 4.1. *Suppose that $Q = \Gamma_\varphi$ and $\varphi' \in H^2$. Then $\mathfrak{D}(G) = H^2_W$ for some $W \geq C > 0$, and the norms of these two spaces are equivalent if and only if G is bounded.*

LEMMA 5.2. *Suppose G_0 is closable. Then $\mathfrak{D}(G)$ is invariant under multiplication by the independent variable z and M is unitarily equivalent to multiplication by z on $\mathfrak{D}(G)$.*

THEOREM 5.4. *If G_0 is bounded, then $H = H_1 \oplus H_2$ (in particular, the projections are bounded), M is similar to S, and R is similar to $S^* \oplus S$.*

LEMMA 6C.1. *There is a constant C such that*

$$\|\Gamma_\varphi f'\|_2 \leq C(|\varphi(0)| + \|\varphi'\|_{1^*})\|f\|_2$$

for any $\varphi' \in$ BMOA and f a polynomial.

LEMMA 6C.2. *If $Q = \Gamma_\varphi$ and G_0 is bounded, then $\varphi' \in$ BMOA and*

$$|\varphi(0)| + \|\varphi'\|_{1^*} \leq 5\|G\|.$$

LEMMA 6C.4. *If φ is analytic on the open disk and $\varphi' \in H^2$, then (in case $Q = \Gamma_\varphi$) G_0 is closable.*

Proof of Theorem 3.1. If $\varphi' \in$ BMOA, then Lemma 6C.1 asserts that

$$\|\Gamma_\varphi f'\| \leq C(|\varphi(0)| + \|\varphi'\|_{1^*})\|f\|_2$$

for each polynomial f. Thus, G_0 is bounded and the conclusion of the theorem follows from Theorem 5.4. □

Proof of Theorem 3.2. By Lemma 6C.4, the operator G_0 is closable and therefore G is defined and closed. Let $Tf(z) = zf(z)$ for $f \in \mathfrak{D}(G)$. By Lemma 5.2, T is unitarily equivalent to M, which by hypothesis is similar to S. Hence there is a map $V: \mathfrak{D}(G) \to H^2$ which is linear, one-to-one, onto, and bicontinuous such that $VT = SV$. Let $f_0 = V^{-1}(1)$. Then

(1) $V(pf_0) = p$ and
(2) $C_1\|p\|_{H^2} \leq \|pf_0\|_{\mathfrak{D}(G)} \leq C_2\|p\|_{H^2}$, where p is any polynomial and C_1, C_2 are positive constants independent of p.

Let $W = |1/f_0|^2$. We will show that from (1) and (2) the following obtain:

(3) $W \geq C$ for some $C > 0$, and
(4) $\mathfrak{D}(G) = H^2_W$ and the norms of these two spaces are equivalent.

Since $\varphi' \in H^2$ and (3) and (4) hold, it follows from Theorem 4.1 that G is bounded. Thus, by Lemma 6C.2, $\varphi' \in$ BMOA and the proof of Theorem 3.2 is complete once we establish (3) and (4).

From the inequalities (recall $V(pf_0) = p$)

$$\|pf_0\|_{H^2} \leq \|pf_0\|_{\mathfrak{D}(G)} \leq C\|p\|_{H^2},$$

it follows that

(5) f_0 is bounded.

We also need

(6) f_0 is an outer function.

Since $V\colon \mathfrak{D}(G) \to H^2$ is onto and bicontinuous, $V(pf_0) = p$ for any polynomial p, and the polynomials are dense in H^2, it follows that the set $Y = \{pf_0\colon p$ a polynomimal$\}$ is dense in $\mathfrak{D}(G)$. Since G is the closure of G_0 and the domain of G_0 contains the polynomials, it follows that Y is dense in H^2. Thus, f_0 is an outer function (see e.g. [Ho, Chap. 5]) and (6) is established.

We now prove (3) and (4). Clearly, (3) follows from (5). Let $f \in \mathfrak{D}(G)$. Let p_n be a sequence of polynomials such that p_n converges to $Vf = g$ in H^2. Then $p_n f_0 = V^{-1}(p_n)$ converges to $V^{-1}(Vf) = f$ in $\mathfrak{D}(G)$. In particular, $p_n f_0$ converges to f in H^2 by definition of the norm of $\mathfrak{D}(G)$. Since f_0 is bounded, $p_n f_0$ also converges to gf_0 in H^2. Thus, $gf_0 = f$ as functions in H^2 and hence as functions in $\mathfrak{D}(G)$. Furthermore, from the properties of V,

$$C_1 \| p_n \|_{H^2} \leq \| p_n f_0 \|_{\mathfrak{D}(G)} \leq C_2 \| p_n \|_{H^2}$$

for some $C_1, C_2 > 0$. Also, $\| p_n f_0 \|_{\mathfrak{D}(G)} \to \| g f_0 \|_{\mathfrak{D}(G)}$ and $\| p_n \|_{H^2} \to \| g \|_{H^2} = (\int |gf_0|^2 W\, dm)^{1/2}$. Thus, $f \in H^2_W$ and

$$C_1 \left(\int |f|^2 W\, dm \right)^{1/2} \leq \| f \|_{\mathfrak{D}(G)} \leq C_2 \left(\int |f|^2 W\, dm \right)^{1/2}.$$

So to prove (4) it remains to show that $H^2_W \subset \mathfrak{D}(G)$.

Let $f \in H^2_W$. Then (i) $f \in H^2$; (ii) $\int |f/f_0|^2\, dm < \infty$; and (iii) f_0 is an outer function. Note that (i) follows from the fact that $W \geq C$ for some $C > 0$, (ii) from the fact that $W = |1/f_0|^2$, and (iii) from (6). Therefore, $g = f/f_0 \in H^2$ (see e.g. [Ho, Chap. 5]). Thus, $f = gf_0$ for some g in H^2. Choose polynomials p_n such that $p_n \to g$ in H^2. Then $p_n f_0 = V^{-1}(p_n) \to V^{-1}(g)$ in $\mathfrak{D}(G)$. Thus, $p_n f_0 \to V^{-1}(g)$ in H^2 and $G(p_n f_0) \to G(V^{-1}(g))$ in H^2. Since f_0 is bounded, $p_n f_0 \to g f_0$ in H^2. Thus, since G is closed, $f = gf_0 \in \mathfrak{D}(G)$, which completes the proof of (4) and therefore the proof of Theorem 3.2. □

4. Comparison of $\mathfrak{D}(G)$ and H^2_W

In this section the following theorem is proved.

THEOREM 4.1. *Suppose that $Q = \Gamma_\varphi$ and $\varphi' \in H^2$. Then $\mathfrak{D}(G) = H^2_W$ for some $W \geq C > 0$, and the norms of these two spaces are equivalent if and only if G is bounded.*

Proof. If G is bounded, then $\mathfrak{D}(G) = H^2 = H^2_W$ for $W \equiv 1$. Now consider the converse. Suppose G is not bounded. We will show it is not possible for the spaces $\mathfrak{D}(G)$ and H^2_W to be the same and have equivalent norms. Since $\mathfrak{D}(G) \neq H^2$, and $H^2_W = H^2$ if W is bounded, we may assume that W is unbounded. So it suffices to show that given $M > 0$ there is an f in $\mathfrak{D}(G)$ such that

(1) $\|f\|^2 + \|Gf\|^2 \leq 1$ and
(2) $\int |f|^2 W\, dm > M$.

Because W is bounded, we can choose $C>0$ and a complex with $|a|>1$ such that the function $F(z)=C/(z-a)$ satisfies

(3) $\int |F|^2\,dm=1$ and

(4) $\int |F|^2 W\,dm>M.$

Since $\varphi'\in H^2$, we can choose a function φ_0 of the form

$$\varphi_0(z)=\sum \frac{a_k}{b_k-z},$$

where the sum is finite and $|b_k|>1$, such that

(5) $|\varphi(0)-\varphi_0(0)|+\|(\varphi-\varphi_0)'\|_{H^2}<1/C\|F\|_{H^\infty}$, where C is the constant in Lemma 6C.3.

Let B denote the finite Blaschke product

$$B(z)=\prod \beta_k(z)$$

where $\beta_k(z)=(z-(1/b_k))/(1-(1/\bar{b}_k)z)$ and where b_k is as in the definition of φ_0. Let $f=B^2F$.

In view of (3) and (4), to complete the proof of (1) and (2) it suffices to show, with appropriate adjustments of constants, that

(6) $f\in\mathfrak{D}(G)$ and

(7) $\|Gf\|\le 1$.

For (7), we use

$$Gf=P((\varphi-\varphi_0)\tilde{f}')+P(\varphi_0\tilde{f}').$$

We first show that

(8) $P(\varphi_0\tilde{f}')=0.$

Now $f'=2BB'F+B^2F'=BF_1$ and F_1 is analytic on the closed disc. Hence $\varphi_0(z)\tilde{f}'(z)=\varphi_0(z)B(1/z)F_1(1/z)$. A simple computation shows that

$$\frac{a_k}{b_k-z}\beta_k\left(\frac{1}{z}\right)=\frac{1}{z}\frac{a_k}{b_k}\frac{1}{1-(1/\bar{b}_k)(1/z)}.$$

Thus, $\varphi_0\tilde{f}'$ has the form

$$\varphi_0(z)\tilde{f}'(z)=\frac{1}{z}F_2\left(\frac{1}{z}\right),$$

where F_2 is an analytic function on the closed disc, and (8) follows.

From Lemma 6C.3, we conclude that

(9) $\|P((\varphi-\varphi_0)\tilde{f}')\|\le C(|\varphi(0)-\varphi_0(0)|+\|(\varphi-\varphi_0)'\|_{H^2})\|f\|_{H^\infty}.$

Now (7) follows from (5), (8), and (9).

Recall that G is the closure of G_0 and that $G_0p=\Gamma_\varphi p'$ for p a polynomial. Since f is analytic on the closed disc, a routine closure argument establishes (6).

5. Case of General Q

This section is devoted to the proof of Theorem 5.4. We consider the case of a general operator Q in $\mathcal{L}(H^2)$. The corresponding operators G_0, G, R, and M are defined in Section 2.

LEMMA 5.1. *If G_0 is closable, then*

$$H_2 = \left\{ \begin{pmatrix} Gf \\ f \end{pmatrix} : f \in \mathfrak{D}(G) \right\}.$$

Proof. Let $\binom{g}{h} \in H_2$. Then there is a sequence of polynomials p_n such that

$$\begin{pmatrix} G_0 p_n \\ p_n \end{pmatrix} = p_n(R)v \to \begin{pmatrix} g \\ h \end{pmatrix}$$

in $H = H^2 \oplus H^2$. Since G_0 is closed and G is the closure of G_0, it follows that $h \in \mathfrak{D}(G)$ and $g = Gh$. Thus,

$$H_2 \subset \left\{ \begin{pmatrix} Gf \\ f \end{pmatrix} : f \in \mathfrak{D}(G) \right\}.$$

The opposite inclusion is proved in a similar manner. □

LEMMA 5.2. *Suppose G_0 is closable. Then $\mathfrak{D}(G)$ is invariant under multiplication by the independent variable z and M is unitarily equivalent to multiplication by z on $\mathfrak{D}(G)$.*

Proof. Since G_0 is closable, G is defined and $\mathfrak{D}(G)$ is a Hilbert space. Define V by

$$Vf = \begin{pmatrix} Gf \\ f \end{pmatrix}, \quad f \in \mathfrak{D}(G).$$

From the definition of $\mathfrak{D}(G)$ and Lemma 5.1, it follows that the map V: $\mathfrak{D}(G) \to H_2$ is linear, isometric, and onto. To complete the proof we will show

(1) $z\mathfrak{D}(G) \subset \mathfrak{D}(G)$ and
(2) $Vzf = MVf$ when $f \in \mathfrak{D}(G)$.

Let $p(z) = \sum c_k z^k$ be a polynomial. By the definition of G_0, $G_0 p = \sum c_k Q_k 1$ where Q_k is defined by the equation

$$R^k = \begin{pmatrix} S^{*k} & Q_k \\ 0 & S^k \end{pmatrix}, \quad k = 0, 1, \ldots .$$

Recall that $v = \binom{0}{1}$. A simple computation shows that

$$p(R)v = \begin{pmatrix} Gp \\ p \end{pmatrix}.$$

Let $f \in \mathfrak{D}(G)$. Choose a sequence of polynomials p_n such that $p_n \to f$ and $Gp_n \to Gf$ in H^2. Note that

$$Vzp_n = \begin{pmatrix} Gzp_n \\ zp_n \end{pmatrix} = Rp_n(R)v = R\begin{pmatrix} Gp_n \\ p_n \end{pmatrix}$$

and

$$R\begin{pmatrix} Gp_n \\ p_n \end{pmatrix} \to R\begin{pmatrix} Gf \\ f \end{pmatrix} = RVf \text{ in } H.$$

In particular, $zp_n \to zf$ in H^2 and Gzp_n converges in H^2; consequently, $zf \in \mathfrak{D}(G)$. Furthermore, $Vzp_n \to Vzf$ and therefore $Vzf = RVf = MVf$ (recall that $M = R \mid H_2$), which completes the proof. □

LEMMA 5.3. *If G_0 is bounded, then $H = H_1 \oplus H_2$ (with bounded projection).*

Proof. Since G_0 is bounded, $\mathfrak{D}(G) = H^2$ and G_0 is closable. Thus, by Lemma 5.1,

$$H_2 = \left\{\begin{pmatrix} Gf \\ f \end{pmatrix} : f \in H^2\right\}.$$

Let $\binom{g}{f} \in H$. Then

$$\begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} g - Gf \\ 0 \end{pmatrix} + \begin{pmatrix} Gf \\ f \end{pmatrix}.$$

Suppose $\binom{g}{f} \in H_1 \cap H_2$. Then $f = 0$, and therefore $g = 0$. Thus, $H = H_1 \oplus H_2$ and since G is bounded, the corresponding projections onto H_1 and H_2 are bounded. □

THEOREM 5.4. *If G_0 is bounded, then $H = H_1 \oplus H_2$ (in particular, the projections are bounded), M is similar to S, and R is similar to $S^* \oplus S$.*

Proof. From Lemma 5.3, we have $H = H_1 \oplus H_2$. Since G_0 is bounded, G_0 is closable, $\mathfrak{D}(G) = H^2$ and these two spaces have equivalent norms; thus, by Lemma 5.2, $M = R \mid H_2$ is similar to S. Clearly, $R \mid H_1$ is similar to S^*. Thus, R is similar to $S^* \oplus S$, completing the proof. □

6. The Lemmas

For purposes of keeping track of the interconnections of the several lemmas, we have divided this section into three subsections. In Section 6A we state three known results with references. The lemmas in 6B follow from those in 6A, and Section 6C contains the lemmas used in Sections 3 and 4. The lemmas in 6C follow from those in 6B.

The idea for the proof of (i) in Lemma 6B.1 is quite ingenious; we learned this idea from [Pel, p. 202] and [Bo, Lemma 1].

6A. Known Results

Fix α, $0 < \alpha < 1$. Let $\Omega(\theta)$ denote the interior of the smallest convex set in the plane containing the disc $|z| \leq \alpha$ and the point $e^{i\theta}$. For an analytic function f on the disc $(|z|<1)$, the nontangential maximal function Nf is defined by

$$Nf(e^{i\theta}) = \sup\{|f(z)| : z \in \Omega(\theta)\}$$

and the square function sf is defined by

$$sf(e^{i\theta}) = \left(\int_{\Omega(\theta)} |f'(z)|^2 \, dA \right)^{1/2},$$

where dA denotes integration with respect to area.

The following results are known. In each lemma, C or C_0 denotes a constant which depends only on α.

LEMMA 6A.1. $\|f\|_1 \le C(|f(0)| + \|s(f)\|_1)$, $f \in H^1$.

LEMMA 6A.2. $\|Nf\|_2 \le C\|f\|_2$, $f \in H^2$.

LEMMA 6A.3. $C_0\|f\|_2 \le |f(0)| + \|s(f)\|_2 \le C\|f\|_2$, $f \in H^2$.

Lemma 6A.1 is proved in [FS]. A closely related result is proved in [BGS]. These issues are discussed in [Fe] and [Pet]. For a proof of Lemma 6A.2, see [Zy, Vol. 1, Chap. 7, Thm. 7.36]. For a proof of Lemma 6A.3, see [Zy, Vol. 2, Chap. 14, Lemma 2.3].

6B. Consequences

The following three lemmas follow from those in Section 6A, and are used in Section 6C.

LEMMA 6B.1. *There is a constant $C > 0$ such that for any polynomials f, g, h,*

(i) $\|\int f'g\|_1 \le C\|f\|_2\|g\|_2$ *and*
(ii) $|\int (\int f'g)\bar{h}\, dm| \le C\|f\|_2\|g\|_2\|h\|_{1^*}$.

Proof. Clearly, (ii) follows from (i) and the definition of the norm $\|\ \|_{1^*}$, and the constant C is the same in (i) and (ii).

The first assertion follows from Lemma 6A.1, Schwarz's inequality, and Lemmas 6A.2 and 6A.3 as follows:

$$\begin{aligned} C_1 \left\| \int f'g \right\|_1 &\le \left\| s\left(\int f'g \right) \right\|_1 \\ &= \int \left(\int_{\Omega(\theta)} |f'g|^2 \, dA \right)^{1/2} dm \\ &\le \int N(g) s(f) \, dm \le \|N(g)\|_2 \|s(f)\|_2 \\ &\le C_2 \|g\|_2 \|f\|_2. \end{aligned}$$

□

The next lemma gives the converse of the previous lemma.

LEMMA 6B.2. *If h is a polynomial and $h(0)=0$, then*

$$\|h\|_{1^*} \leq 4 \sup\left\{\left|\int\left(\int f'g\right)\bar{h}\,dm\right| : f, g \in B(H^2)\right\},$$

where $B(H^2)$ denotes the 1-ball of H^2.

Proof. Let $E = \{\int f'g : f, g \in B(H^2)\}$. The set E is bounded in H^1 by Lemma 6B.1. Let $k \in B(H^1)$ and $k_1 = k - k(0)$. Since $h(0)=0$, $\int k\bar{h}\,dm = \int k_1\bar{h}\,dm$. Since $k_1 \in 2B(H^1)$, $k_1/2 = fg$ for $f, g \in B(H^2)$ by the factorization theorem. Thus, since $k_1'(0)=0$,

$$k_1 = \int k_1' = 2\int f'g = 2\left(\int f'g + \int fg'\right) \in 2(E+E).$$

Therefore $k_1 = 2(k_2+k_3)$ for some k_2, k_3 in E. Thus,

$$\begin{aligned}\left|\int k\bar{h}\,dm\right| &= \left|2\int (k_2+k_3)\bar{h}\,dm\right| \\ &\leq 4\sup\left\{\left|\int k\bar{h}\,dm\right| : k \in E\right\}\end{aligned}$$

and the conclusion follows. □

LEMMA 6B.3. $\|\int f'g\|_2 \leq C\|f\|_\infty\|g\|_2$ *for any polynomials f and g.*

Proof. Let $F = \int f'g$ and $G = \int fg'$. Note that $F(0) = 0 = G(0)$. Thus,

$$F(e^{i\theta}) = \int_0^1 \frac{d}{dr} f(re^{i\theta})g(re^{i\theta})\,dr.$$

Integrating by parts gives (for $z = e^{i\theta}$)

$$F(z) = f(z)g(z) - f(0)g(0) - G(z).$$

From Lemma 6A.3,

$$\begin{aligned}C_1\|G\|_2^2 &\leq \int\left(s\left(\int fg'\right)\right)^2 dm \\ &= \int\int |f(z)g'(z)|^2\,dA\,dm \\ &\leq (\|f\|_\infty)^2\|s(g)\|_2^2 \leq (C_1\|f\|_\infty\|g\|_2)^2,\end{aligned}$$

and the conclusion follows. □

6C. Key Lemmas

The following lemmas are used in Sections 3 and 4.

LEMMA 6C.1. *There is a constant C such that*

$$\|\Gamma_\varphi f'\|_2 \le C(|\varphi(0)| + \|\varphi'\|_{1^*})\|f\|_2$$

for any $\varphi' \in$ BMOA *and f a polynomial.*

Proof. Let g be any polynomial. It suffices to show that

(1) $|(\Gamma_\varphi f', g)| \le C(|\varphi(0)| + \|\varphi'\|_{1^*})\|f\|_2\|g\|_2$.

Let $g_1(z) = g(z) - g(0)$; hence $g = g_1 + g(0)$. A computation shows that

$$(\Gamma_\varphi f', g_1) = \int z\varphi' \overline{\int f^{*\prime} g_2}\, dm,$$

where $f^*(z) = \overline{f(\bar{z})}$, and $g_2(z) = g_1(z)/z$. From Lemma 6B.1, it follows that

(2) $|(\Gamma_\varphi f', g_1)| \le C\|\varphi'\|_{1^*}\|f\|_2\|g\|_2$.

Next consider $(\Gamma_\varphi f', g(0))$. Using ordinary integration by parts with respect to the variable θ, it follows that

$$(3)\quad \overline{(\Gamma_\varphi f', g(0))} = g(0)\left[-\int f^*(e^{i\theta})e^{i\theta}\overline{\varphi(e^{i\theta})}\frac{d\theta}{2\pi} + \int f^*(e^{i\theta})\overline{\varphi'(e^{i\theta})}\frac{d\theta}{2\pi}\right]$$

A routine argument shows that

(4) $\|\varphi\|_{1^*} \le |\varphi(0)| + 2\|\varphi'\|_{1^*}$.

From (3) and (4) it follows that

(5) $|(\Gamma_\varphi f', g(0))| \le (|\varphi(0)| + 3\|\varphi'\|_{1^*})\|f\|_2\|g\|_2$.

Finally, (1) follows from (2) and (5), which completes the proof. □

The following lemma shows that the reverse of the inequality in Lemma 6C.1 is also true. Recall that when $Q = \Gamma_\varphi$, $G_0 f' = \Gamma_\varphi f'$ for any polynomial f and G is the closure of G_0.

LEMMA 6C.2. *If* $Q = \Gamma_\varphi$ *and* G_0 *is bounded, then* $\varphi' \in$ BMOA *and*

$$|\varphi(0)| + \|\varphi'\|_{1^*} \le 5\|G\|.$$

Proof. Let f, g be polynomials in the 1-ball of H^2 with $g(0) = 0$. Then

$$\|G\| \ge |(\Gamma_\varphi f', g)| \ge \left|\int z\varphi' \overline{\left(\int f^{*\prime}\frac{g}{z}\right)}\, dm\right|.$$

From Lemma 6B.2 we conclude that

$$\|G\| \ge \tfrac{1}{4}\|z\varphi'\|_{1^*} \ge \tfrac{1}{4}\|\varphi'\|_{1^*}.$$

If $f(z) = z$, then $G_0 f = \varphi$. Thus, $\|G\| \ge |\varphi(0)|$. □

The proof of the following lemma is similar to the proof of Lemma 6C.1, except that Lemma 6B.3 is used instead of Lemma 6B.1; therefore, the proof is omitted.

LEMMA 6C.3. *If φ is analytic on the open disk and $\varphi' \in H^2$, then*

$$\|\Gamma_\varphi f'\|_2 \leq C(|\varphi(0)| + \|\varphi'\|_2)\|f\|_\infty$$

for any polynomial f.

LEMMA 6C.4. *If φ is analytic on the open disk and $\varphi' \in H^2$, then (in case $Q = \Gamma_\varphi$) G_0 is closable.*

Proof. In case $Q = \Gamma_\varphi$, the domain of G_0 is the polynomials and $G_0 f = \Gamma_\varphi f'$ for f a polynomial. To show that G_0 is closable it suffices to show that if f_n is a sequence of polynomials such that $f_n \to 0$ and $G_0 f_n \to g$ in H^2, then $g = 0$.

Let h be any polynomial. An easy computation shows that $(G_0 f_n, h) = \int f_n(e^{i\theta}) w(e^{i\theta})\, d\theta/2\pi$, where

$$w(e^{i\theta}) = \frac{d}{d\theta} ie^{i\theta}\varphi(e^{i\theta})\overline{h(e^{i\theta})}.$$

Since $\varphi' \in H^2$, it follows that $(G_0 f_n, h) \to 0$. Hence $(g, h) = 0$. Thus $g = 0$, and so G_0 is closable. □

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