Interpolating Sequences for the Bergman Space

KEHE ZHU

1. Introduction

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$, and let dA denote the normalized area measure on $\mathbf{D}$. The Bergman space $L_a^2(\mathbf{D})$ consists of analytic functions f in $\mathbf{D}$ such that

$$\|f\|^2 = \int_{\mathbf{D}} |f(z)|^2 \, dA(z) < +\infty.$$

$L_a^2(\mathbf{D})$ is clearly a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbf{D}} f(z)\overline{g(z)} \, dA(z), \quad f, g \in L_a^2(\mathbf{D}).$$

Throughout the paper, $\| \ \|$ and $\langle \ , \ \rangle$ will always denote the above norm and inner product in $L_a^2(\mathbf{D})$.

Let $A = \{a_n\}$ be a sequence of points in $\mathbf{D}$. We say that A is a *sequence of interpolation* for $L_a^2(\mathbf{D})$ if, for every sequence $\{w_n\}$ of complex numbers satisfying

$$\sum_{n=1}^{\infty} (1-|a_n|^2)^2 |w_n|^2 < +\infty,$$

there exists a function f in $L_a^2(\mathbf{D})$ such that $f(a_n) = w_n$ for all $n \geq 1$. Sequences of interpolation for $L_a^2(\mathbf{D})$ are studied and characterized in [2; 6; 7]. In particular, it is well known that every sequence of interpolation for $L_a^2(\mathbf{D})$ must be separated in the pseudohyperbolic metric. Thus we assume throughout the paper that the points in the sequence $A = \{a_n\}$ are all distinct. It is clear that if A is a sequence of interpolation for $L_a^2(\mathbf{D})$, then so is every subsequence of A. It is also easy to see that every sequence of interpolation for $L_a^2(\mathbf{D})$ is a zero set for $L_a^2(\mathbf{D})$; that is, there exists a nontrivial function f in $L_a^2(\mathbf{D})$ which vanishes on the sequence. Zero sets in this paper will always be assumed to be simple; namely, each zero set consists of distinct points in $\mathbf{D}$.

Suppose $A = \{a_n\}$ is a zero set for $L_a^2(\mathbf{D})$. The space

$$H_A = \{f \in L_a^2(\mathbf{D}) : f(a_n) = 0, \ n \geq 1\}$$

Received June 15, 1992. Revision received March 29, 1993.
Research supported by the National Science Foundation.
Michigan Math. J. 41 (1994).

is a nontrivial closed subspace of $L_a^2(\mathbf{D})$. Thus H_A is a Hilbert space with the inner product inherited from $L_a^2(\mathbf{D})$. We let $K_A(z, w)$ denote the reproducing kernel for the space H_A. We can now state the main results of the paper.

THEOREM A. *Suppose $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$. Then there exists a unique sequence $\{\psi_n\}$ in $L_a^2(\mathbf{D})$ such that the kernel function $K_A(z, w)$ admits the following partial fraction expansion:*

$$K_A(z, w)=\frac{1}{(1-z\bar{w})^2}-\sum_{n=1}^{\infty}\frac{\overline{\psi_n(w)}}{(1-\bar{a}_n z)^2}, \quad z, w\in\mathbf{D}.$$

Moreover, the functions ψ_n have the following additional properties:

(1) *$\psi_n(a_n)=1$ and $\psi_n(a_m)=0$ for all $n, m\geq 1$ and $n\neq m$.*
(2) *For each compact set K in $\mathbf{D}$ there exists a positive constant C_K such that*

$$|\psi_n(z)|\leq C_K(1-|a_n|^2)^{3/2}, \quad z\in K, \ n\geq 1.$$

(3) *There is a constant $C>0$ such that $1-|a_n|^2\leq\|\psi_n\|\leq C(1-|a_n|^2)$ for all $n\geq 1$.*

THEOREM B. *Suppose that $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and that $\{\psi_n\}$ is the sequence from Theorem A. If $\{w_n\}$ is a sequence of complex numbers satisfying*

$$\sum_{n=1}^{\infty}(1-|a_n|^2)^2|w_n|^2<+\infty,$$

then the series $\sum_{n=1}^{\infty} w_n\psi_n(z)$ converges to a function in $L_a^2(\mathbf{D})$ which uniquely solves the minimal interpolation problem $\inf\{\|f\|: f(a_n)=w_n, n\geq 1\}$.

We note that the sequence $\{\psi_n\}$ is given by

$$\psi_n(z)=\frac{K_{A_n}(z, a_n)}{K_{A_n}(a_n, a_n)}, \quad z\in\mathbf{D}, \ n\geq 1,$$

where $A_n=A-\{a_n\}$ for all $n\geq 1$. It is then clear that

$$\|\psi_n\|=\frac{1}{\sqrt{K_{A_n}(a_n, a_n)}}, \quad n\geq 1,$$

and property (1) in Theorem A is obvious.

When $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$, we call the operator T defined by

$$T_A(\{w_n\})(z)=\sum_{n=1}^{\infty} w_n\psi_n(z), \quad z\in\mathbf{D},$$

the *minimal interpolation operator* for $L_a^2(\mathbf{D})$. For each $\{w_n\}$ satisfying

$$\sum_{n=1}^{\infty}(1-|a_n|^2)^2|w_n|^2<+\infty$$

we have

$$\|T_A(\{w_n\})\| = \min\{\|f\| : f(a_n) = w_n,\ n \geq 1\}.$$

The main results of the paper are proved in [9] in the special case when $A = \{a_n\}$ is a classical interpolating sequence, namely, when A satisfies

$$\delta = \inf_n \prod_{k \neq n} \left| \frac{a_k - a_n}{1 - \bar{a}_k a_n} \right| > 0.$$

It follows from results in [7] that every classical interpolating sequence is a sequence of interpolation for $L_a^2(\mathbf{D})$. The corresponding results for the Hardy space $H^2(\mathbf{D})$ are also proved in [9].

The author wishes to thank Kristian Seip for useful conversations and suggestions.

2. Minimal Interpolation

Recall that for each $L_a^2(\mathbf{D})$-zero set $A = \{a_n\}$ we denote by $K_A(z, w)$ the reproducing kernel of

$$H_A = \{f \in L_a^2(\mathbf{D}) : f(a_n) = 0,\ n \geq 1\}.$$

$K_A(z, w)$ is the unique function on $\mathbf{D} \times \mathbf{D}$ satisfying the following two conditions:

(1) $K_A(\cdot, w)$ is in H_A for every $w \in \mathbf{D}$.
(2) $\langle f, K_A(\cdot, w)\rangle = f(w)$ for all f in H_A and $w \in \mathbf{D}$.

The kernel function K_A has the following additional properties:

(3) $\overline{K_A(z, w)} = K_A(w, z)$ for all z and w in $\mathbf{D}$.
(4) $K_A(z, w) = 0$ if and only if z or w is in A. Moreover, if w is not in A, then $K_A(z, w)$ has simple zeros at $z = a_n$ for each $n \geq 1$.
(5) If $\{e_n\}$ is any orthonormal basis for H_A, then

$$K_A(z, w) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}, \quad z, w \in \mathbf{D}.$$

(6) If $\varphi : \mathbf{D} \to \mathbf{D}$ is a Möbius map, then

$$\varphi'(z) K_{\varphi(A)}(\varphi(z), \varphi(w)) \overline{\varphi'(w)} = K_A(z, w)$$

for all z and w in $\mathbf{D}$. In particular,

$$K_A(z, w) = \frac{1}{(1 - z\bar{w})^2} K_{\varphi_w(A)}(\varphi_w(z), 0), \quad z, w \in \mathbf{D},$$

where φ_w is the involutive Möbius map given by

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}, \quad z \in \mathbf{D}.$$

We call this property the *transformation law* for the reproducing kernel K_A.

(7) For each $w \in \mathbf{D}$ the function $K_A(\cdot, w)$ is the unique solution to the extremal problem $\sup\{\operatorname{Re} f(w) : \|f\| \le 1, f \in H_A\}$.

If $A = \{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$, then A is necessarily a zero set for $L_a^2(\mathbf{D})$, and the points in A are necessarily distinct. For each $n \ge 1$ let $A_n = A - \{a_n\}$ and define

$$\psi_n(z) = \frac{K_{A_n}(z, a_n)}{K_{A_n}(a_n, a_n)}, \quad z \in \mathbf{D}.$$

Note that by property (4) of the reproducing kernels the denominator $K_{A_n}(a_n, a_n)$ above is always nonzero. By properties (1) and (2) of the kernel functions we also have

$$\begin{aligned} \|\psi_n\|^2 &= \frac{1}{K_{A_n}(a_n, a_n)^2} \langle K_{A_n}(\cdot, a_n), K_{A_n}(\cdot, a_n)\rangle \\ &= \frac{K_{A_n}(a_n, a_n)}{K_{A_n}(a_n, a_n)^2} = \frac{1}{K_{A_n}(a_n, a_n)} \end{aligned}$$

for all $n \ge 1$.

LEMMA 1. *Suppose that $A = \{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and that $\{w_n\}_{n=1}^N$ is a finite sequence of complex numbers. Then the function*

$$h(z) = \sum_{n=1}^{N} w_n \psi_n(z), \quad z \in \mathbf{D},$$

is the unique solution to the following extremal problem:

$$\inf\{\|f\| : f(a_n) = w_n, 1 \le n \le N, f(a_n) = 0, n > N\}.$$

Proof. Since A is a sequence of interpolation for $L_a^2(\mathbf{D})$, there exist functions f in $L_a^2(\mathbf{D})$ such that $f(a_n) = w_n$ for $1 \le n \le N$ and $f(a_n) = 0$ for $n > N$. Let f be such a function. It is clear that $f - h$ is a function in H_A. Fix a point w in $\mathbf{D} - A$. Then there exists an analytic function g in $\mathbf{D}$ such that

$$f(z) - h(z) = K_A(z, w) g(z), \quad z \in \mathbf{D}.$$

This implies that

$$\begin{aligned} \|f\|^2 &= \|h + K_A(\cdot, w) g\|^2 \\ &= \|h\|^2 + \|K_A(\cdot, w) g\|^2 + 2 \operatorname{Re}\langle K_A(\cdot, w) g, h\rangle. \end{aligned}$$

Note that

$$\langle K_A(\cdot, w) g, h\rangle = \sum_{n=1}^{N} \frac{\bar{w}_n}{K_{A_n}(a_n, a_n)} \langle K_A(\cdot, w) g, K_{A_n}(\cdot, a_n)\rangle$$

and $K_A(\cdot, w) g$ is in H_{A_n}. Thus, by the reproducing property of K_{A_n}, we have

$$\langle K_A(\cdot, w) g, h\rangle = \sum_{n=1}^{N} \frac{\bar{w}_n}{K_{A_n}(a_n, a_n)} K_A(a_n, w) g(a_n) = 0.$$

Therefore,

$$\|f\|^2 = \|h\|^2 + \|K_A(\cdot, w) g\|^2,$$

and hence h is the unique solution to the extremal problem

$$\inf\{\|f\|: f(a_n)=w_n,\ 1\le n\le N,\ f(a_n)=0,\ n>N\}. \qquad \square$$

LEMMA 2. *Suppose $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$. Then there exists a constant $C>0$ such that*

$$\inf\{\|f\|^2: f(a_n)=w_n,\ n\ge 1\}\le C\sum_{n=1}^{\infty}(1-|a_n|^2)^2|w_n|^2$$

for every sequence $\{w_n\}$ with $\sum_{n=1}^{\infty}(1-|a_n|^2)^2|w_n|^2<+\infty$.

Proof. Suppose A is a sequence of interpolation for $L_a^2(\mathbf{D})$. It is well known (see [6; 7]) that the operator

$$f\mapsto\{(1-|a_n|^2)f(a_n)\}$$

is a bounded linear operator from $L_a^2(\mathbf{D})$ onto l^2. The desired result then follows from the open mapping theorem. $\square$

We can now prove the main result of this section.

THEOREM 3. *Suppose that $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and that $A_n=A-\{a_n\}$ for $n\ge 1$. Let*

$$\psi_n(z)=\frac{K_{A_n}(z,a_n)}{K_{A_n}(a_n,a_n)},\quad z\in\mathbf{D},\ n\ge 1.$$

Then, for each sequence $\{w_n\}$ with $\sum_{n=1}^{\infty}(1-|a_n|^2)^2|w_n|^2<+\infty$, the series

$$h(z)=\sum_{n=1}^{\infty}w_n\psi_n(z)$$

converges in $L_a^2(\mathbf{D})$, and h is the unique solution to the minimal interpolation problem

$$\inf\{\|f\|: f(a_n)=w_n,\ n\ge 1\}.$$

Proof. Given $\epsilon>0$, there exists a positive integer N_0 such that

$$\sum_{n=N}^{N+p}(1-|a_n|^2)^2|w_n|^2<\frac{\epsilon}{C},\quad N\ge N_0,\ p\ge 1,$$

where C is the constant from Lemma 2. Let $w_n'=w_n$ for $N\le n\le N+p$ and $w_n'=0$ otherwise. It follows from Lemma 1 that

$$\left\|\sum_{n=N}^{N+p}w_n\psi_n\right\|=\inf\{\|f\|: f(a_n)=w_n',\ n\ge 1\}.$$

By Lemma 2,

$$\begin{aligned}\left\|\sum_{n=N}^{N+p}w_n\psi_n\right\|^2&\le C\sum_{n=1}^{\infty}(1-|a_n|^2)^2|w_n'|^2\\&=C\sum_{n=N}^{N+p}(1-|a_n|^2)^2|w_n|^2\\&<\epsilon\end{aligned}$$

for all $N \geq N_0$ and $p \geq 1$. Thus

$$h(z) = \sum_{n=1}^{\infty} w_n \psi_n(z)$$

converges in $L_a^2(\mathbf{D})$. It is clear that h has the property $h(a_n) = w_n$ for $n \geq 1$. Suppose f is another function in $L_a^2(\mathbf{D})$ with the property $f(a_n) = w_n$ for $n \geq 1$. Then

$$f(z) - h(z) = K_A(z, w) g(z), \quad z \in \mathbf{D},$$

for some analytic function g in $\mathbf{D}$, where w is any point in $\mathbf{D} - A$ (g depends on w). Using exactly the same arguments as in the proof of Lemma 1, we can prove that

$$\|f\|^2 = \|h\|^2 + \|K_A(\cdot, w) g\|^2.$$

Thus $\|f\| \geq \|h\|$, and so h solves the following minimal interpolation problem:

$$\inf\{\|f\| : f(a_n) = w_n, \; n \geq 1\}.$$

The uniqueness of the solution to the above problem follows from general functional analysis. In fact, the set of functions f in $L_a^2(\mathbf{D})$ satisfying $f(a_n) = w_n$ ($n \geq 1$) is closed and convex, and so must have a unique element of minimal norm. □

COROLLARY 4. *Suppose that A is a sequence of interpolation for $L_a^2(\mathbf{D})$ and that*

$$\psi_n(z) = \frac{K_{A_n}(z, a_n)}{K_{A_n}(a_n, a_n)}, \quad n \geq 1, \; z \in \mathbf{D},$$

where $A_n = A - \{a_n\}$ for $n \geq 1$. Then the series

$$\sum_{n=1}^{\infty} \frac{|\psi_n(z)|^2}{(1 - |a_n|^2)^2}$$

converges uniformly on every compact subset of $\mathbf{D}$.

Proof. Note that convergence in $L_a^2(\mathbf{D})$ implies uniform convergence on compact subsets of $\mathbf{D}$. By Theorem 3, the series

$$\sum_{n=1}^{\infty} (1 - |a_n|^2) w_n \frac{\psi_n(z)}{1 - |a_n|^2}$$

converges uniformly on compact subsets of $\mathbf{D}$ for all $\{w_n\}$ with

$$\sum_{n=1}^{\infty} (1 - |a_n|^2)^2 |w_n|^2 < +\infty.$$

In other words,

$$\sum_{n=1}^{\infty} b_n \frac{\psi_n(z)}{1 - |a_n|^2}$$

converges uniformly on compact subsets of $\mathbf{D}$ for all $\{b_n\}$ in l^2. Thus the series

$$\sum_{n=1}^{\infty} \frac{|\psi_n(z)|^2}{(1-|a_n|^2)^2}$$

converges uniformly on compact subsets of $\mathbf{D}$. □

3. Further Estimates on $\{\psi_n\}$

It follows from Corollary 4 that if $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and if K is a compact subset of $\mathbf{D}$ then there exists a constant $C_K>0$ such that $|\psi_n(z)| \le C_K(1-|a_n|^2)$ for all z in K and $n \ge 1$. The purpose of this section is to improve this estimate; we show that $|\psi_n(z)| \le C_K(1-|a_n|^2)^{3/2}$ for $n \ge 1$ and $z \in K$. To achieve this estimate we need to introduce certain notions of uniform density for sequences in $\mathbf{D}$.

Let $A=\{a_n\}$ be a sequence of distinct points in $\mathbf{D}$. We say that A is *separated* if there exists a constant $\delta>0$ such that $\rho(a_n, a_m)>\delta$ for all $n, m \ge 1$ with $n \ne m$, where ρ is the pseudohyperbolic metric on $\mathbf{D}$ defined by

$$\rho(z,w)=|\varphi_z(w)|, \quad \varphi_z(w)=\frac{z-w}{1-\bar{z}w}, \quad z, w \in \mathbf{D}.$$

Note that φ_z is the involutive Möbius map of $\mathbf{D}$ which interchanges 0 and z.

Suppose $A=\{a_n\}$ is a separated sequence in $\mathbf{D}$. For each $1/2<r<1$ let

$$D(A,r)=\sum\left\{\log\frac{1}{|a_k|} : \frac{1}{2}<|a_k|<r\right\}\Big/\log\frac{1}{1-r}.$$

The lower and upper uniform densities of A are then defined, respectively, as

$$D^-(A)=\liminf_{r\to 1^-}\inf_{z\in\mathbf{D}} D(\varphi_z(A), r)$$

and

$$D^+(A)=\limsup_{r\to 1^-}\sup_{z\in\mathbf{D}} D(\varphi_z(A), r).$$

By Theorem 6.2 of [7], $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ if and only if A is separated and $D^+(A)<\frac{1}{2}$. Note that if A is a classical interpolating sequence then $D^+(A)=0$. Clearly such a sequence is also separated. Thus every classical interpolating sequence is a sequence of interpolation for $L_a^2(\mathbf{D})$.

It is clear that both the upper and lower densities are Möbius invariant; that is,

$$D^-(\varphi(A))=D^-(A) \quad\text{and}\quad D^+(\varphi(A))=D^+(A)$$

whenever φ is a Möbius map of the disk. It is also clear that

$$D^+(A)\ge D^+(B) \quad\text{and}\quad D^-(A)\ge D^-(B)$$

whenever A contains B. It follows from these observations that

$$D^+(A)\ge D^+(A_n') \quad\text{and}\quad D^-(A)\ge D^-(A_n')$$

for all $n \geq 1$, where

$$A'_n = \varphi_{a_n}(A_n) = \{\varphi_{a_n}(a_k) : k \geq 1, k \neq n\}, \quad n \geq 1.$$

LEMMA 5. *Suppose that $A = \{a_n\}$ is a sequence of interpolation for $L^2_a(\mathbf{D})$ and that $A'_n = \varphi_{a_n}(A_n)$ (see the previous paragraph). Then there exists a constant $\sigma > 0$ such that $\sigma \leq K_{A'_n}(0,0) \leq 1$ for all $n \geq 1$.*

Proof. By the reproducing property of the kernel functions we have

$$K_{A'_n}(0,0) = \sup\{|f(0)|^2 : \|f\| \leq 1, f \in H_{A'_n}\}, \quad n \geq 1.$$

Since $|f(0)| \leq \|f\|$ for all $f \in L^2_a(\mathbf{D})$, we see that $K_{A'_n}(0,0) \leq 1$ for all $n \geq 1$. Using standard Hilbert space arguments we can write

$$K_{A'_n}(0,0) = \frac{1}{I_n}, \quad n \geq 1,$$

where

$$I_n = \inf\{\|f\|^2 : f(0) = 1, f \in H_{A'_n}\}.$$

Thus we need to show that the sequence $\{I_n\}$ is bounded.

Since A is a sequence of interpolation for $L^2_a(\mathbf{D})$, by Theorem 6.2 of [7] we have

$$\delta(A) = \inf\{\rho(a,b) : a, b \in A, a \neq b\} > 0$$

and $D^+(A) < \frac{1}{2}$. Using Möbius invariance, we also have

$$\delta(A'_n) \geq \delta(A) > 0, \quad n \geq 1,$$

and

$$D^+(A'_n) \leq D^+(A) < \tfrac{1}{2}, \quad n \geq 1.$$

By Lemma 5.7 of [7] there exists $\epsilon > 0$ such that for each $n \geq 1$ there is an analytic function g_n in $\mathbf{D}$ with the following properties:

(1) $g_n(0) = 1$ and $g_n(z) = 0$ for all $z \in A'_n$, and
(2) $|g_n(z)| \leq C(1-|z|^2)^{-1/2+\epsilon}$ for all $z \in \mathbf{D}$,

where $C > 0$ is a constant depending only on $\delta(A)$, $D^+(A)$, and ϵ (but not on n). It follows that

$$I_n \leq \|g_n\|^2 \leq C^2 \int_{\mathbf{D}} (1-|z|^2)^{-1+2\epsilon}\, dA(z), \quad n \geq 1.$$

This finishes the proof of Lemma 5. □

Recall that $\|\psi_n\| = K_{A_n}(a_n, a_n)^{-1/2}$ for all $n \geq 1$. By the transformation law for the kernel functions K_A and the estimate in Lemma 5, there exists a constant $C > 0$ such that

$$1 - |a_n|^2 \leq \|\psi_n\| \leq C(1 - |a_n|^2)$$

for all $n \geq 1$.

THEOREM 6. *Suppose that $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and that*

$$\psi_n(z)=\frac{K_{A_n}(z,a_n)}{K_{A_n}(a_n,a_n)},\quad z\in\mathbf{D},\ n\geq 1,$$

where $A_n=A-\{a_n\}$ $(n\geq 1)$. There exists a constant $C>0$ such that

$$|\psi_n(z)|\leq\frac{C(1-|a_n|^2)^{3/2}}{|1-\bar{a}_n z|\sqrt{1-|z|^2}}$$

for all $z\in\mathbf{D}$ and $n\geq 1$.

Proof. By the transformation law for the kernel function K_A we can write

$$\psi_n(z)=\left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)^2\frac{K_{A_n'}(\varphi_{a_n}(z),0)}{K_{A_n'}(0,0)},\quad z\in\mathbf{D},\ n\geq 1,$$

where $A_n'=\varphi_{a_n}(A_n)=\{\varphi_{a_n}(a_k):k\geq 1,k\neq n\}$ and $\varphi_a(z)=(a-z)/(1-\bar{a}z)$. Since the function

$$G_{A_n'}(z)=\frac{K_{A_n'}(z,0)}{\sqrt{K_{A_n'}(0,0)}},\quad z\in\mathbf{D},$$

solves the extremal problem

$$\sup\{\operatorname{Re} f(0): f\in H_{A_n'},\ \|f\|\leq 1\},$$

Corollary 4.5 of [3] implies that

$$|\psi_n(z)|\leq\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^2\frac{1}{\sqrt{1-|\varphi_{a_n}(z)|^2}\sqrt{K_{A_n'}(0,0)}}$$

or

$$|\psi_n(z)|\leq\frac{(1-|a_n|^2)^{3/2}}{|1-\bar{a}_n z|\sqrt{1-|z|^2}\sqrt{K_{A_n'}(0,0)}},\quad z\in\mathbf{D},\ n\geq 1.$$

The desired result now follows from Lemma 5. □

COROLLARY 7. *If $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and K is a compact set in $\mathbf{D}$, there exists a constant $C_K>0$ such that $|\psi_n(z)|\leq C_K(1-|a_n|^2)^{3/2}$ for all $z\in K$ and $n\geq 1$.*

COROLLARY 8. *If $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$, then the series $\sum_{n=1}^{\infty}|\psi_n(z)|$ converges uniformly on each compact subset of $\mathbf{D}$.*

Proof. By Corollary 7, it suffices to show that

$$\sum_{n=1}^{\infty}(1-|a_n|^2)^{3/2}<+\infty.$$

But this is true for every $L_a^2(\mathbf{D})$-zero set A; see [4] and [5]. □

Note that Corollary 8 also follows from Corollary 4, the Cauchy–Schwarz inequality, and the fact that

$$\sum_{n=1}^{\infty}(1-|a_n|^2)^2<+\infty.$$

The convergence of $\sum_{n=1}^{\infty}(1-|a_n|^2)^p$ $(p>1)$ for sequences of interpolation for $L_a^2(\mathbf{D})$ can be proved elementarily without appealing to the theory of zero sets. In fact, if

$$r=\tfrac{1}{2}\inf\{\rho(a_n,a_m): n\neq m\}$$

then $D(a_n,r)=\{z\in\mathbf{D}: \rho(z,a_n)<r\}$ are disjoint disks in $\mathbf{D}$. By 4.3 of [10] there exists a constant $C>0$ such that

$$(1-|a_n|^2)^p\leq C\int_{D(a_n,r)}(1-|z|^2)^{p-2}\,dA(z),\quad n\geq 1,$$

and hence

$$\sum_{n=1}^{\infty}(1-|a_n|^2)^p\leq C\int_{\mathbf{D}}(1-|z|^2)^{p-2}\,dA(z)<+\infty.$$

4. Partial Fraction Expansion for Reproducing Kernels

Let A be a sequence of interpolation for $L_a^2(\mathbf{D})$. In this section we derive a partial fraction representation for the kernel function $K_A(z,w)$. The expansion will be in terms of the sequence of functions $\{\psi_n\}$.

THEOREM 9. *Suppose $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$. Then*

$$K_A(z,w)=\frac{1}{(1-z\bar{w})^2}-\sum_{n=1}^{\infty}\frac{\overline{\psi_n(w)}}{(1-\bar{a}_n z)^2}$$

for all z and w in $\mathbf{D}$.

Proof. Define a function K_1 on $\mathbf{D}\times\mathbf{D}$ as follows:

$$K_1(z,w)=\frac{1}{(1-z\bar{w})^2}-\sum_{n=1}^{\infty}\frac{\psi_n(z)}{(1-a_n\bar{w})^2},\quad z,w\in\mathbf{D}.$$

By Corollary 8, the above series converges absolutely and uniformly if both z and w are restricted to a compact subset of $\mathbf{D}$. In particular, $K_1(z,w)$ is analytic in z and conjugate analytic in w. Fix $w\in\mathbf{D}$ and let

$$w_n=\frac{1}{(1-a_n\bar{w})^2},\quad n\geq 1.$$

It is clear that $\{w_n\}$ is a bounded sequence. In particular,

$$\sum_{n=1}^{\infty}(1-|a_n|^2)^2|w_n|^2<+\infty$$

(recall that $\sum_{n=1}^{\infty}(1-|a_n|^2)^2<+\infty$ for every $L_a^2(\mathbf{D})$-zero set A; see the last paragraph in the previous section). By Theorem 3, the series

$$h(z)=\sum_{n=1}^{\infty}\frac{\psi_n(z)}{(1-a_n\bar{w})^2},\quad z\in\mathbf{D},$$

converges to a function in $L_a^2(\mathbf{D})$. Thus $K_1(\cdot,w)$ belongs to $L_a^2(\mathbf{D})$ for each w in $\mathbf{D}$. Since $\psi_n(a_n)=1$ and $\psi_n(a_m)=0$ for all $n,m\geq 1$ and $n\neq m$, we see that $K_1(\cdot,w)$ belongs to H_A for every w in $\mathbf{D}$. Furthermore, if f is in H_A then f is in each H_{A_n}, and hence

$$\begin{aligned}\int_{\mathbf{D}} f(z)\overline{\psi_n(z)}\,dA(z)&=\frac{1}{K_{A_n}(a_n,a_n)}\int_{\mathbf{D}} f(z)K_{A_n}(a_n,z)\,dA(z)\\&=\frac{f(a_n)}{K_{A_n}(a_n,a_n)}=0.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f,K_1(\cdot,w)\rangle&=\int_{\mathbf{D}} f(z)\overline{K_1(z,w)}\,dA(z)\\&=\int_{\mathbf{D}}\frac{f(z)\,dA(z)}{(1-w\bar{z})^2}-\sum_{n=1}^{\infty}\frac{1}{(1-\bar{a}_n w)^2}\int_{\mathbf{D}} f(z)\overline{\psi_n(z)}\,dA(z)\\&=f(w)\end{aligned}$$

for every w in $\mathbf{D}$. By uniqueness of the reproducing kernel, we must have

$$K_A(z,w)=K_1(z,w),\quad z,w\in\mathbf{D}.$$

Since K_A is symmetric, that is, $\overline{K_A(z,w)}=K_A(w,z)$, we conclude that

$$K_A(z,w)=\overline{K_1(w,z)}=\frac{1}{(1-z\bar{w})^2}-\sum_{n=1}^{\infty}\frac{\overline{\psi_n(w)}}{(1-\bar{a}_n z)^2}$$

for all z and w in $\mathbf{D}$. □

COROLLARY 10. *Suppose $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$. For every w in $\mathbf{D}$, the kernel function $K_A(\cdot,w)$ extends analytically across each arc of $\partial\mathbf{D}$ which does not contain any accumulation points of $\{a_n\}$.*

Note that the above corollary is established in [3] for every $L_a^2(\mathbf{D})$-zero set A. Related extension theorems for extremal functions can be found in [1] and [8].

COROLLARY 11. *Suppose that $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and that $H_A^{\perp}$ is the orthogonal complement of H_A in $L_a^2(\mathbf{D})$. The reproducing kernel for $H_A^{\perp}$ is given by*

$$K_{A^{\perp}}(z,w)=\sum_{n=1}^{\infty}\frac{\overline{\psi_n(w)}}{(1-\bar{a}_n z)^2},\quad z,w\in\mathbf{D}.$$

Proof. Let $\{e_n\}$ be an orthonormal basis for H_A and let $\{\sigma_n\}$ be an orthonormal basis for $H_A^{\perp}$. Then $\{e_n\}\cup\{\sigma_n\}$ is an orthonormal basis for $L_a^2(\mathbf{D})$. Recall that $(1-z\bar{w})^{-2}$ is the reproducing kernel for $L_a^2(\mathbf{D})$. Thus

$$\frac{1}{(1-z\bar{w})^2} = \sum e_n(z)\overline{e_n(w)} + \sum \sigma_n(z)\overline{\sigma_n(w)}$$
$$= K_A(z, w) + K_{A^\perp}(z, w),$$

and the desired result follows from the expansion for K_A given in Theorem 9. □

REMARK. It is difficult to see directly that the function

$$K_{A^\perp}(z, w) = \sum_{n=1}^{\infty} \frac{\overline{\psi_n(w)}}{(1-\bar{a}_n z)^2}$$

is symmetric. As a consequence of the known symmetry we obtain the following interesting identity:

$$\sum_{n=1}^{\infty} \frac{\psi_n(z)}{(1-a_n\bar{w})^2} = \sum_{n=1}^{\infty} \frac{\overline{\psi_n(w)}}{(1-\bar{a}_n z)^2}, \quad z, w \in \mathbf{D}.$$

COROLLARY 12. *Suppose $A = \{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$. For each function f in $H_A^\perp$, we have*

$$f(z) = \sum_{n=1}^{\infty} f(a_n)\psi_n(z), \quad z \in \mathbf{D}.$$

Proof. By the proof of Theorem 10, each function ψ_n belongs to $H_A^\perp$. If f is in $H_A^\perp$ then the function

$$F(z) = f(z) - \sum_{n=1}^{\infty} f(a_n)\psi_n(z), \quad z \in \mathbf{D},$$

belongs to both H_A and $H_A^\perp$, and hence it must be the zero function. □

It is clear that the sequence $\{\psi_n\}$ is linearly independent in $H_A^\perp$. Thus the above corollary shows that $\psi_1, \psi_2, \ldots, \psi_n, \ldots$ form a basis for $H_A^\perp$. Note that the functions in $\{\psi_n\}$ are not mutually orthogonal.

The following is an atomic decomposition theorem for functions in $H_A^\perp$ when A is a sequence of interpolation for $L_a^2(\mathbf{D})$. See [10] for more information on the atomic decomposition of general functions in the Bergman space.

THEOREM 13. *Suppose $A = \{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$.*

(1) *For every sequence $\{c_n\}$ in l^2, the series*

$$\sum_{n=1}^{\infty} c_n \frac{1-|a_n|^2}{(1-\bar{a}_n z)^2}$$

converges in norm to a function in $H_A^\perp$.

(2) *For every function f in $H_A^\perp$, there exists a unique sequence $\{c_n\}$ in l^2 such that*

$$f(z)=\sum_{n=1}^{\infty} c_n \frac{1-|a_n|^2}{(1-\bar{a}_n z)^2}, \quad z\in \mathbf{D},$$

and the convergence is in norm.

(3) *There is a constant* $C>0$ *such that*

$$C^{-1}\sum_{n=1}^{\infty}|c_n|^2 \le \int_{\mathbf{D}} \left|\sum_{n=1}^{\infty} c_n \frac{1-|a_n|^2}{(1-\bar{a}_n z)^2}\right|^2 dA(z) \le C\sum_{n=1}^{\infty}|c_n|^2$$

for all sequences $\{c_n\}$ *in* l^2.

Proof. Define a linear operator $T\colon H_A^{\perp}\to l^2$ by

$$Tf=\{(1-|a_n|^2)f(a_n)\}.$$

We first show that the operator T is bounded, one-to-one, and onto.

That T is bounded is well known; it follows from the fact that $\{a_n\}$ is separated (see e.g. [10, Lemma 4.4.2]). The operator T is one-to-one because the intersection of H_A and $H_A^{\perp}$ consists of the zero function only. To see that T is onto, let $\{c_n\}$ be a sequence in l^2 and write $c_n=(1-|a_n|^2)w_n$. Since $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$, there exists a function F in $L_a^2(\mathbf{D})$ such that $F(a_n)=w_n$ for $n\ge 1$. Let f be the orthogonal projection of F in $H_A^{\perp}$; then we have $Tf=\{c_n\}$.

Let $T^*\colon l^2\to H_A^{\perp}$ be the adjoint of T. It is easy to check that T^* is given by the following formula:

$$T^*\{c_n\}(z)=\sum_{n=1}^{\infty} c_n \frac{1-|a_n|^2}{(1-\bar{a}_n z)^2}, \quad z\in\mathbf{D}.$$

The desired results now follow from the fact that T^* is bounded, one-to-one, and onto (and hence its inverse is also bounded by the open mapping theorem). This completes the proof of Theorem 13. □

REMARK. Suppose that $A=\{a_n\}$ is a sequence of interpolation for $L_a^2(\mathbf{D})$ and that $T\colon H_A^{\perp}\to l^2$ is the linear operator defined in the proof of the above theorem. Then the inverse of T is given by the following formula:

$$T^{-1}\{c_n\}(z)=\sum_{n=1}^{\infty}\frac{c_n}{1-|a_n|^2}\psi_n(z), \quad z\in\mathbf{D}.$$

This is simply the minimal interpolation operator.

COROLLARY 14. *Suppose* $A=\{a_n\}$ *is a sequence of interpolation for* $L_a^2(\mathbf{D})$. *Then*

$$f(z)=\sum_{n=1}^{\infty}\frac{\langle f,\psi_n\rangle}{(1-\bar{a}_n z)^2}, \quad z\in\mathbf{D},$$

for every function f *in* $H_A^{\perp}$. *Moreover, the above series converges in norm in* $L_a^2(\mathbf{D})$.

Proof. Suppose f is in H_A^1. By Theorem 13, there exists a sequence $\{c_n\}$ such that

$$f(z)=\sum_{n=1}^{\infty} c_n \frac{1-|a_n|^2}{(1-\bar{a}_n z)^2}, \quad z\in \mathbf{D},$$

with the series converging in norm in $L_a^2(\mathbf{D})$. It follows that

$$\begin{aligned}\langle f, \psi_k\rangle &= \sum_{n=1}^{\infty} c_n(1-|a_n|^2)\int_{\mathbf{D}} \frac{\overline{\psi_k(z)}\,dA(z)}{(1-\bar{a}_n z)^2}\\ &= \sum_{n=1}^{\infty} c_n(1-|a_n|^2)\overline{\psi_k(a_n)}\\ &= c_k(1-|a_k|^2)\end{aligned}$$

for all $k\geq 1$, or

$$c_k=\frac{\langle f, \psi_k\rangle}{1-|a_k|^2}, \quad k=1,2,\ldots. \qquad \square$$

References

[1] E. Akutowicz and L. Carleson, *The analytic continuation of interpolatory functions,* J. Analyse Math. 7 (1960), 223–247.

[2] E. Amar, *Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de* $\mathbf{C}^n$, Canad. J. Math. 30 (1978), 711–737.

[3] H. Hedenmalm, *A factorization theorem for square area-integrable analytic functions,* J. Reine Angew. Math. 442 (1991), 45–68.

[4] C. Horowitz, *Zeros of functions in Bergman spaces,* Duke Math. J. 41 (1974), 693–710.

[5] B. Korenblum, *An extension of the Nevanlinna theory,* Acta Math. 135 (1975), 187–219.

[6] R. Rochberg, *Interpolation by functions in Bergman spaces,* Michigan Math. J. 29 (1982), 229–236.

[7] K. Seip, *Beurling type density theorems in the unit disk,* preprint, 1991.

[8] H. Shapiro, *Overconvergence of sequences of rational functions with sparse poles,* Ark. Mat. 7 (1968), 343–349.

[9] K. Zhu, *Reproducing kernels generated by interpolating sequences,* preprint, 1991.

[10] ———, *Operator theory in function spaces,* Marcel Dekker, New York, 1990.

Department of Mathematics
State University of New York
Albany, NY 12222